Math 202 Jerry L. Kazdan

Homework Set 2 (Due in class on Thursday, Sept. 25)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo - West text.

1. [# 1.8] In the morning section of a calculus course, 2 of the 9 women and 2 of the 10 men received the grade of A. In the afternoon section 6 of the 9 womem and 9 of the 14 men received an A. Verify that in each section, a higher proportion of womrn than men received an A – but that in the combined course a lower portion of women than men received a grade of A. Explain!

2. [# 1.20] Suppose that r and s are distinct real solutions of the quadratic equation $ax^2+bx+c=0$. Express $r+s$ and rs in terms of a, b, and c.

3. Suppose that r, s, and t are distinct real solutions of the cubic equation $ax^3+bx^2+cx+d=0$. Express $r+s+t$ and rst in terms of a, b, c. and d. [For this you don't need the formula for the roots of cubic].

4. [# 1.22] You have two identical glasses. The first has with a cup of wine, the second has a cup of water. You remove 1 teaspoon of wine from the first glass and add it to the second. Stir carefully. Then take one teaspoon of the mixture from the second glass and add to the first. Prove that the amount of wine in the first glass now equals the amount of water in the second glass.

5. [# 1.30] Let x, y, u, and v be real numbers.
   a) Prove that $(xu+yv)^2 \le (x^2+y^2)(u^2+v^2)$.
   b) Determine precisely when equality holds in this.

6. [# 1.46] Determine the images of the functions $f:\mathbb{R}\to\mathbb{R}$ defined as follows:

$$a).\ f(x) := \frac{x^2}{1+x^2} \qquad\qquad b).\ f(x) := \frac{x}{1+|x|}$$

7. [# 1.49] Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For their sum and product, determine which statements below are True. If True, please give a proof. If False, give a counterexample.
   a) If f and g are bounded, then $f+g$ is bounded.

b) If f and g are bounded, then fg is bounded.

c) If $f+g$ is bounded, then f and g are bounded.

d) If fg is bounded, then f and g are bounded.

e) If both $f+g$ and fg are bounded, then f and g are bounded.

8. [# 1.50] NOTATION: If S is a subset of the domain of a function f, write $f(S) := \{f(x) : x \in S\}$.

   Let C and D be subsets of the domain of f.

   a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

   b) Give an example where equality does not hold in the above.

9. [# 1.52] Let M and N be non-negative real numbers. Suppose that $|x+y| \le M$ and $|xy| \le N$. Determine the maximum possible value of x as a function of M and N.

10. [# 1.55] Let $\mathbb{F}$ be a field consisting of exactly three elements: 0, 1, and x. Prove that $1+1=x$ and that $x \cdot x = 1$. Obtain the addition and multiplication table for $\mathbb{F}$.

[The following problem used to be the last part of #12]

11. If 3^{1025} is divided by 5, what is the remainder?

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[The following two problems are now **deleted** from this assignment.]

A. [# 1.56] Is there a field with exactly four elements? Is there a field with exactly six elements?

B. Let n be a positive integer. For any integers a, b we say that a *equals* b *mod* n if a and b have the same remainders when divided by n (equivalently, if $b-a$ is divisible by n). We write: $a \equiv b \pmod{n}$. So modulo n the possible remainders are $0, 1, 2, \ldots, (n-1)$ and every integer is equivalent to one of these. We tell time modulo 24 hours.

a) If $a \equiv r \pmod{n}$ and $b \equiv s \pmod{n}$, show that $a+b \equiv r+s \pmod{n}$ and $ab \equiv rs \pmod{n}$.

b) Show that the integers mod 5 form a field.

c) If p is a prime number, show that the integers mod p form a field.

   [The only place cleverness is needed is showing that each non-zero element has a multiplicative inverse. There is no simple formula for this. One approach to show that k $(k \neq 0)$ has a multiplicative inverse is to first show that the numbers k, $2k$, $3k, \ldots, (p-1)k$ are all distinct mod p. Then, why must they be the integers 1, 2,..., $p-1$ only rearranged? Why does this show that k has a multiplicative inverse?]

d) If 3^{1025} is divided by 5, what is the remainder?

[Last revised: September 22, 2009]