Math 202, Fall 2009 Jerry L. Kazdan

# Homework Set 1 (Due in class on Thursday, Sep. 17)

(late papers accepted until 1:00 Friday)

1. a) Describe (and sketch) the real numbers x that satisfy $|x-2|<3$.

   b) Sketch the points $(x, y)$ in the plane where $|x-y|>1$.

   Solution: (a) $|x-2|<3$ means the distance between $x$ and $2$ is smaller than $3$. Thus $-3<x-2<3$, that is, $-1<x<5$.

2. a) How many real roots does $x^4+x^2-2x+2=0$ have?

   b) Find all points $(x, y)$ in the plane that satisfy $x^2-2xy+5y^2=0$.

   Solution: (a) Since $x^4+x^2-2x+2=x^4+(x-1)^2+1>0$, there are no real roots. (b) $0=x^2-2xy+5y^2=(x-y)^2+4y^2$. Here both terms are non-negative and their sum is $0$. So they are both $0$, that is, $x-y=0$ and $y=0$. Hence $x=y=0$.

3. Show that $\sqrt{7+2\sqrt{6}}-\sqrt{7-2\sqrt{6}}=2$.

   Solution: Compute the square of the left sides. We get

$$\begin{aligned}\left(\sqrt{7+2\sqrt{6}}-\sqrt{7-2\sqrt{6}}\right)^2 &= 7+2\sqrt{6}-2\sqrt{7^2-(2\sqrt{6})^2}+7-2\sqrt{6}\\ &= 14-2\sqrt{49-24}\\ &= 4=2^2.\end{aligned}$$

   Since $\sqrt{7+2\sqrt{6}}-\sqrt{7-2\sqrt{6}}>0$, $\sqrt{7+2\sqrt{6}}-\sqrt{7-2\sqrt{6}}=2$.

4. Solve $\log_9(5-3x)=-1/2$ for $x$.

   Solution: $\log_9(5-3x)=-1/2$ is equivalent to $5-3x=9^{-1/2}$. Hence $x=\frac{5-9^{-1/2}}{3}=14/9$.

5. Let $A=(-6,3)$, $B=(2,7)$, and $C$ be the vertices of a triangle. Say the altitudes through the vertices $A$ and $B$ intersect at $Q=(2,-1)$. Find the coordinates of $C$.

   [ The *altitude* through a vertex of a triangle is a straight line through the vertex that is perpendicular to the opposite side – or an extension of the opposite side.]

   Solution: We have $\overrightarrow{AQ}=(2-(-6),(-1-3))=(8,-4)$ and $\overrightarrow{BQ}=(0,-8)$. Since $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{AQ}$, $\overrightarrow{BC}=(s,2s)$ for some $s$, and hence $C=(2+s,7+2s)$ for some $s$. Similarly, $C=(-6+t,3)$ for some $t$. Now we have $(2+s,7+2s)=(-7+t,3)$. Solving this gives $s=-2$ and $t=7$. Thus $C=(0,3)$.

6. Let $y = f(x)$ describe a smooth curve in the plane $(-\infty < x < \infty)$ that does not pass through the origin. Say the point $P = (a, b)$ on the curve is closest to the origin. Show that the straight line from the origin to P is perpendicular to the curve.

SOLUTION: Consider the circle centered at the origin which goes through $P$. Since $P$ is closest to the origin, the curve does not go inside the circle. Hence the curve is tangent to the circle at $P$. Notice that the straight line from the origin to $P$ is perpendicular to the circle at $P$, so it is also perpendicular to the curve at $P$.

sc Alternate: The distance from the origin to a point $(x, y)$ on the curve is $D(x) = \sqrt{x^2 + f^2(x)}$ Use calculus to minimize $D(x)$ oo or, simpler, minimize $h(x) = D^2(x) = x^2 + f^2(x)$. At a point $x = a$ where $h$ has a minimum (or maximum), $0 = h'(a) = 2[a + f(a)f'(a)]$. This means that the straight line from the origin to $P = (a,\ f(a))$ is perpendicular to the tangent line to the curve at $P$.

[This alternate approach is easier to see using vectors and the dot product. Let $\mathbf{V(t)} = (x(t),\ y(t))$ describe the curve, where $t$ is a real parameter. Then using the inner product the square of the distance from the the point $\mathbf{V(t)}$ on the curve to the origin is $g(t) := |\mathbf{V}(t)|^2 = \mathbf{V}(t) \cdot \mathbf{V}(t)$. At a point $t = c$ where $g(t)$ has a minimum (or maximum), $0 = g'(t) = 2\mathbf{V}(c) \cdot \mathbf{V}'(c)$. This means the vector $\mathbf{V}(c)$ is perpendicular to the tangent line to the curve.]

7. Let $f(x)$ be a continuous function that satisfies $\displaystyle\int_0^x f(t)dt = c - \cos(x^2)$. Find the function $f(t)$ and the constant $c$.

SOLUTION: By the Fundamental Theorem of Calculus, taking a derivative ($\frac{d}{dx}$) on both sides, we get $f(x) = 2x \sin(x^2)$. When $x = 0$, the original equation becomes $0 = c - 1$, so $c = 1$.

8. a) Prove that the product of two odd integers is also odd.

   b) If $k > 0$ is an integer, is $k(k+1)(k+2)$ always divisible by $6$?

SOLUTION: (a) Write the two odd interges as $2k + 1$ and $2l + 1$ where $k$ and $l$ are integers. Then their product is $(2k+1)(2l+1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$, which is also odd.

(b) Either $k$ or $k + 1$ is divisible by $2$, so $k(k + 1)(k + 2)$ is divisible by $2$. One of $k$, $k + 1$ and $k + 2$ is divisible by $3$, so $k(k + 1)(k + 2)$ is divisible by $3$. Hence $k(k + 1)(k + 2)$ is divisible by $6$.

In other words, if you have three consecutive integers, at least one musts be divisible by $2$ and one must be divisible by $3$. Therefore their product must be divisible by $6$.

9. List these numbers from smallest to largest:

$$2^{121} \qquad 9^{55} \qquad 7^{88} \qquad u_0,$$

where $u_0 :=$ number of seconds since the birth of our universe.

SOLUTION: $2^{121} = (2^{11})^{11} = 2048^11$. $9^{55} = (9^5)^{11} = 59049^{11}$. $7^{88} = (7^8)^{11} = 5764801^{11}$. So $7^{88} > 9^{55} > 2^{121}$. There are many other ways to do this, for instance, either estimating cleverly or taking logarithms.

To estimate $u_0$ There are 86400 seconds a day, 365 days a years, so less then $10^8$ seconds a year. The age of the universe is less than $10^{11}$ years. Hence $u_0 < 10^{19} < 1000^{19/3} < 1024^{19/3} = (2^{10})^{19/3} = 2^{190/3} < 2^{121}$. This is the *smallest* number in the list.

[Last revised: September 10, 2013]