Homework Set 10 (Due in class on Tuesday, Dec. 1)

(late papers accepted until 1:00 Wednesday)

The problem numbers refer to the D'Angelo-West text.

1. Using the definition of the integral as a Riemann sum, compute $\int_0^b t^2\,dt$. The formula you found earlier this semester for $1^2 + 2^2 + \cdots + n^2$ might be useful.

2. Find a formula for $\cos x + \cos 2x + \cdots + \cos nx$. The simplest approach is probably to follow the procedure we used for $\sin x + \cdots + \sin nx$ in `http://www.math.upenn.edu/~kazdan/202F09/sum-sin_kx.pdf`.

3. Using the definition of the integral as a Riemann sum, compute $\int_0^x \cos t\,dt$. The formula you found in the previous problem might be useful.

4. Assume that f is Riemann integrable in the interval $[a, b]$.
   a) If $f \geq 0$, use the definition of the integral as a Riemann sum to show that
   $$\int_a^b f(x)\,dx \geq 0.$$
   [This takes just one or two sentences.]
   b) Use the previous part to conclude that if $m \leq f(x) \leq M$ on this interval, then
   $$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$
   [Consider $g(x) := f(x) - m$ and $h(x) := M - f(x)$.]

5. [#17.15] Let f be continuous on the interval $[a, b]$ and assume that $f(x) \geq 0$ for all $a \leq x \leq b$. Use the definition of the integral as a Riemann sum to show that if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ everywhere. [You will need to use that since f is continuous, if it is positive at some point, then it is positive in some interval containing the point.]

6. For $x > 1$ define the function
   $$H(x) = \int_1^x \frac{1}{t}\,dt.$$

Since the integrand, $1/t$ is a monotonic function on the interval $[1, x]$, this is Riemann integrable. Use the definition of the Riemann integral directly to show that for any $y > 0$,

$$H(x) + H(y) = H(xy), \tag{1}$$

thus establishing that $H(x)$ has the basic property of the logarithm.
SUGGESTION: Rewrite (1) in the form $H(x) = H(xy) - H(y)$, that is,

$$\int_1^x \frac{1}{t}\,dt = \int_y^{xy} \frac{1}{s}\,ds$$

and use a geometric argument that relates a Riemann sum for the integral on the left to a corresponding Riemann sum on the right. [First try the special case $x = 2$, $y = 2$.]

REMARK: It is not difficult to extend this result to $0 < x < 1$, but that is not part of this problem.

7. [#17.13] Let $f(x)$ be continuous for $x \in [a, b]$. Show there is some point $c \in [a, b]$ where f has its average value, that is,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

[SUGGESTION: First do the case where $\int_a^b f(x)\,dx = 0$. Then reduce the general case to the special case by using $g(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt$.]

[Last revised: November 23, 2009]