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Math 202 **Exam 2** Jerry L. Kazdan
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DIRECTIONS: Part A has 4 shorter problems (5 points each) while Part B has 6 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3×5 card with notes on both sides.

PART A: Four shorter Problems, 5 points each.

A-1. Show that there is an $x > 0$ such that $x^5 = 17$.

SOLUTION: Let $f(x) := x^5 - 17$. Note that $f(0) = -17 < 0$ while $f(2) = 32 - 17 > 0$. Thus by the Intermediate Value Theorem there is a point $x \in (0, 2)$ where $f(x) = 0$.

A-2. Give an example of a bounded continuous function $f(x)$, for $x \in \mathbb{R}$ that does *not* attain its supremum.

SOLUTION: Three examples:

$$f(x) = \frac{x^2}{1+x^2}, \qquad f(x) = \frac{x}{1+|x|}, \qquad \arctan x$$

A-3. Let f be a differentible function with the property that $f'(x) = 0$ for all $x \in [0, 3]$. Show that $f(x) = \text{CONSTANT}$ on the interval $[0,\ 3]$.

SOLUTION: For any $t \in [0, 3]$, by the Mean Value Theorem there is a $c \in (0, t)$ such that $f(t) - f(0) = f'(c)(t - 0) = 0$ so $f(t) = f(0)$.

A-4. Let $f(x)$ be continuous with $\displaystyle\int_1^x f(t)\,dt = C + e^{(x-1)^2}$. Find f and the constant C.

SOLUTION: Let $x = 1$. Then $0 = C + e^0 = C + 1$ so $C = -1$. To find $f(x)$ take the derivative of both sides and use the Fundamental Theorem of Calculus to conclude that $f(x) = 2(x-1)e^{(x-1)^2}$.

PART B: Six traditional problems, 10 points each.

B-1. Let $f(x)$ be a differentiable function that in never zero. Use the definition of the derivative as the limit of a difference quotient to derive the usual formula for the derivative of $1/f(x)$.

SOLUTION: Let $g(x) := 1/f(x)$. Then as $h \to 0$

$$\begin{aligned}\frac{g(x+h) - g(x)}{h} &= \left(\frac{1}{f(x+h)} - \frac{1}{f(x)}\right)\frac{1}{h} \\ &= \left(\frac{f(x) - f(x+h)}{h}\right)\frac{1}{f(x+h)f(x)} \\ &\to \frac{-f'(x)}{f^2(x)}.\end{aligned}$$

B-2. Let a smooth function f have the properties:

$$f(0) = 4, \quad f(1) = 0, \quad f(3) = 6.$$

Show that at some point $0 < c < 3$ one has $f''(c) > 0$.

SOLUTION: By the Mean Value Theorem there are points $a \in (0,1)$ and $b \in (1,3)$ such that

$$f'(a) = \frac{f(1) - f(0)}{1 - 0} = -4 \qquad \text{and} \qquad f'(b) = \frac{f(3) - f(1)}{3 - 1} = \frac{6}{2} = 3.$$

Therefore applying the Mean Value Theorem, this time to f', we find there is a point $c \in (a,b)$ such that

$$f''(c) = \frac{f'(b) - f'(a)}{b - a} = \frac{7}{b - a} > \frac{7}{3} > 0.$$

B-3. Use the definition of the integral as the limit of a Riemann sum to compute $\int_0^c x^2\, dx$ (here $c > 0$). [The formula $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ may be useful.]

SOLUTION: Partition the interval $[0,c]$ into n equal segments, each having length $h := c/n$. Then

$$\begin{aligned}\int_0^c x^2\, dx &\approx \left[(h^2)h + (2h)^2h + (3h)^2h + \cdots + (nh)^2h\right] \\ &= \left[1^2 + 2^2 + 3^2 + \cdots + n^2\right] h^3 \\ &= \frac{n(n+1)(2n+1)}{6}\left(\frac{c^3}{n^3}\right) \to \frac{c^3}{3}.\end{aligned}$$

B-4. Determine the disk of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2(x-2)^{2n}}{4^n}$. [Only find the center and radius of this disk. You need not determine the convergence on the boundary of the disk.]

SOLUTION: We use the ratio test with $a_n = \frac{n^2(x-2)^{2n}}{4^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2(x-2)^{2(n+1)}}{4^{n+1}}}{\frac{n^2(x-2)^{2n}}{4^n}} = \frac{(n+1)^2(x-2)^2}{4n^2}$$

so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|^2}{4}.$$

To find the radius of the disk of convergence, this ratio should be less than 1, so the disk of convergence is $|x - 2| < 2$. It has center at $x = 2$ and has radius 2.

[To determine convergence on the boundary of this disk (where the ratio is 1) the ratio test gives no information. In this case it is simple to see directly that the series becomes $\sum n^2$ which obviously diverges.]

B-5. Suppose u is a twice differentiable function on $\mathbb{R}$ which satisfies the differential equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} - c(x)u = 0, \tag{1}$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.

a) Show that u cannot have a positive local maximum at an interior point of the interval $(0,1)$. Also show that u cannot have a negative local minimum in $(0,1)$.

SOLUTION: If u has a positive local maximum at an interior point $x = p \in (0,1)$ of the interval, then $u(p) > 0$, $u'(p) = 0$, and $u''(p) \le 0$. Thus at p we have $u'' + bu' - cu < 0$, which contradicts (1).

Similarly, u cannot have a negative local minimum at an interior point.

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.

SOLUTION: Since $u(x)$ is a continuous function on the closed and bounded interval $[0,1]$, it takes on its maximum value somewhere in the interval $[0,1]$. if u is positive somewhere, since $u(0) = u(1) = 0$, it takes on its (positive) maximum at an *interior* point of the interval. But by part a) this can't happen.

Similarly, if u is negative somewhere it takes on its (negative) minimum at an *interior* point of the interval. Again by part a) this can't happen.

Thus u is zero everywhere.

B-6. Let I_k be closed bounded nested intervals, so $I_{k+1} \subseteq I_k$.

a) Use the completeness property of the real numbers ("bounded monotone sequences converge") to show that there is at least one point in the intersection, $\cap I_k$.

SOLUTION: Let $I_k = \{a_k \le x \le b_k\}$. Since the intervals are nested, then $a_1 \le a_2 \le a_3 \le \cdots$ and $b_1 \ge b_2 \ge \cdots$. Thus a_k is a bounded monotone increasing sequence that converges to some point A that is in all of the intervals (since these intervals are closed) so the intersection of all of these intervals has at least this one point.

[Although not needed for this part of the problem, by the same reasoning the right-hand endpoints b_k converge to some limit B. If the lengths of the intervals, $b_k - a_k$, converges to 0, then $A = B$. Otherwise the whole interval, $A \le x \le B$ is in the intersection of all the intervals I_k.]

b) Give an example where the intersection is the interval $\{-1 \le x \le 1\}$.

SOLUTION: Three typical examples:

$$I_k = \{ -\tfrac{1}{n} - 1 \le x \le 1 + \tfrac{1}{n}\}, \qquad I_k = \{ -1 \le x \le 1 + \tfrac{1}{n}\}, \qquad I_k = \{-1 \le x \le 1\}.$$

The third example above is completely valid, but very special and not so interesting.