Signature

Printed Name

Math 202
October 22, 2009

Exam 1

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9:00 — 10:20

Directions: Part A has 4 shorter problems (5 points each) while Part B has 6 traditional problems (10 points each). To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3 ×5 with notes on both sides.

Part A: Four shorter Problems, 5 points each.

A-1. Determine the image of the function $f(x) := \dfrac{2x^2}{1+x^2}$.

Score	
A-1	
A-2	
A-3	
A-4	
B-1	
B-2	
B-3	
B-4	
B-5	
B-6	
Total	

A-2. If a and b are rational numbers, consider the set S of real numbers of the form $a+b\sqrt{7}$. Show that the elements in S have multiplicative inverses in S. [This is the key step in showing that S is a field.]

A-3. Determine if the set $S=\{x \in \mathbb{R}: \ 4x^2 > x^3+3x\}$ is bounded above and/or below, and if so, find $\inf(S)$ and $\sup(S)$ – if they exist.

A-4. Let $f,\ g:\mathbb{R}\to\mathbb{R}$ be bounded functions such that $f(x)\le g(x)$ for all x. Let F denote the image of f and G the image of g. Give an example (a picture) of pairs of such functions with $\sup(F) > \inf(G)$.

PART B: Six traditional problems, 10 points each.

B-1. Let n be a positive integer. For any integers a, b we say that a *equals* b *mod* n if a and b have the same remainders when divided by n (equivalently, if $b-a$ is divisible by n). We write: $a \equiv b \pmod{n}$. So modulo n the possible remainders are $0,\ 1,\ 2,\ \ldots,\ (n-1)$ and every integer is equivalent to one of these.

If $a \equiv r \pmod{n}$ and $b \equiv s \pmod{n}$, show that $ab \equiv rs \pmod{n}$.

As a special case, since $3^4 \equiv 1 \pmod{5}$, then $3^8 \equiv ? \pmod{5}$ and $3^9 \equiv ? \pmod{5}$.

B-2. If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$ for $n \geq 1$, show that $a_n < 4$ for all $n \geq 1$.

B-3. Let c be a complex number with $|c| < 1$. Show that $2nc^n \to 0$.

B-4. For each condition below, give an example of an *unbounded* sequence such that $a_{n+1} - a_n > 0$ for all $n \in \mathbb{N}$ and the specified condition holds.

a) $\lim\,(a_{n+1} - a_n) = L$, where $L > 0$.

b) $\lim\,(a_{n+1} - a_n) = 0$.

B-5. Let a_n and b_n be sequences of real numbers. If $a_n \to A$ and $b_n \to B$, show that $a_n b_n$ converges to AB.

B-6. Let b_n be a sequence of real numbers with the properties $b_n > 0$ and $b_n \to B$. Show that $B \geq 0$.

Also, give an example where $B = 0$.