

# Sheaves, Co-Sheaves, and Verdier Duality

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## Abstract

This note demonstrates that the derived category of cellular sheaves is covariantly equivalent to the derived category of cellular co-sheaves. First we prove the equivalence directly, then, for finitely generated stalks, we show how Verdier duality falls out of the construction.

## 1 Introduction

Constructible sheaves enjoyed considerable popularity during the 70s-90s under the heading of holonomic D-modules, intersection cohomology and perverse sheaves. More recently, constructible sheaves have entered the arena of ideas surrounding mirror symmetry and the Fukaya category, see David Nadler's work, in particular [6]. Finally, the combinatorial nature of constructible sheaves makes it an excellent candidate for applications in science and engineering requiring local-to-global ideas, see [2] and references therein. This note, however, is concerned more with the technical apparatus itself and was inspired by a recent conjecture of Robert MacPherson.

## 2 Background

Let us begin with a resumé of the theory of cellular sheaves and cosheaves. This theory has the advantage that it is more concrete than the traditional theory of sheaves that one might find in the classic [3]. However, the reader will benefit from a pre-existing knowledge of derived categories.

Given a Whitney stratified space  $X$ , MacPherson first observed that the category of constructible sheaves is equivalent to representations of the so-called **exit-path category**  $\mathbf{Exit}(X)$  whose objects are the points of  $X$  and morphisms are homotopy classes of paths whose ambient dimension is always non-decreasing. Thus a constructible sheaf can be viewed as a functor  $F : \mathbf{Exit}(X) \rightarrow \mathbf{Vect}$  and sheaf morphisms are natural transformations of these functors.

In the case where  $X$  is stratified as a regular cell-complex and the strata are simply-connected the category  $\mathbf{Exit}(X)$  is equivalent to  $\mathbf{Cell}(X)$ , which is the poset category of  $X$ ,

i.e. objects are cells and morphisms are inclusions. Sometimes, when this is understood, authors will write  $X$  for  $\mathbf{Cell}(X)$  and speak of diagrams indexed by  $X$  or representations of quivers with relations. We will define the category of **cellular sheaves** to be the functor category

$$S(X) := \mathbf{Fun}(\mathbf{Cell}(X), \mathbf{Vect}) \supset \mathbf{Fun}(\mathbf{Cell}(X), \mathbf{vect}) =: S_f(X)$$

A related category is that of cellular cosheaves, which are defined similarly:

$$C(X) := \mathbf{Fun}(\mathbf{Cell}(X)^{\text{op}}, \mathbf{Vect}) \supset \mathbf{Fun}(\mathbf{Cell}(X)^{\text{op}}, \mathbf{vect}) =: C_f(X).$$

Here  $\mathbf{vect}$  denotes the full sub-category of finite-dimensional vector spaces. When Verdier duality is discussed, we will have need to restrict to the finitely generated case.

The fact that constructible sheaves can be viewed as covariant functors out of poset category seemed to be independently known by Masaki Kashiwara in 1984 [4] and Allen Shepard [7] (working under MacPherson) in his 1985 thesis.

This formulation may seem odd. First of all the maps follow the direction of inclusion, contrary to the way presheaves on the open set category  $\mathbf{Open}(X)$  are defined. For this reason the maps internal to a cellular sheaf  $F$  are called **co-restriction maps**, which are linear maps  $\rho_{\sigma, \tau}^F : F(\sigma) \rightarrow F(\tau)$  whenever  $\sigma \leq \tau$ . Furthermore there is no sheafification step, which the reader may be accustomed to seeing. The reason for this is the following: finite posets  $(X, \leq)$  can be equipped with a finite, non-Hausdorff topology, called the **Alexandrov topology**. Here a subset  $Y \subset X$  is closed if  $x \in Y$  and  $y \leq x$  then  $y \in Y$ .

It is not difficult to check that a functor  $F : \mathbf{Cell}(X) \rightarrow \mathbf{Vect}$  is a sheaf with respect to the Alexandrov topology on  $(X, \leq)$ . This topology makes sense of sheaves on finite posets. See [5] for a concise, transparent review. Sheaves on finite posets have the advantage that the basic functors of sheaf theory can be described very explicitly.

Suppose  $f : (X, \leq_X) \rightarrow (Y, \leq_Y)$  is a map of posets that is order-preserving. Such a map is continuous with respect to the Alexandrov topology, alternatively said,  $f : \mathbf{Cell}(X) \rightarrow \mathbf{Cell}(Y)$  is a functor. We can use this to define other sheaves as well, i.e.

$$\begin{array}{ccc} \mathbf{Cell}(X) & \xrightarrow{F} & \mathbf{Vect} \\ f \downarrow & \nearrow_{f_*F} & \\ \mathbf{Cell}(Y) & & \end{array} \quad \begin{array}{c} \\ \\ \xrightarrow{f_!F} \end{array}$$

Where we define

$$\begin{aligned} f_*F(y) &= \lim_{\leftarrow} \{F(x) : f(x) \geq y\} \\ f_!F(y) &= \lim_{\rightarrow} \{F(x) : f(x) \leq y\} \end{aligned}$$

as the right and left Kan extensions of  $F$ . Alternatively, if we had a sheaf  $G$  on  $Y$ , then

$$f^*G(x) = G(f(x)).$$

In general lots of the pre-sheaf descriptions for classical functors hold more or less on the nose for sheaves on finite posets.

Another attractive property of cellular sheaves is that there are enough injectives and projectives and they are easily described. We write  $[\sigma]^V$  for an **elementary injective sheaf**, whose value on a cell is determined as follows

$$[\sigma]^V(\tau) = \begin{cases} V & \sigma \geq \tau \\ 0 & \text{o.w.} \end{cases}$$

The maps between cells being the identity when  $V$  is assigned. Notice that  $[\sigma]^V$  is supported on smallest closed set containing  $\sigma$ . Similarly, the **elementary projective sheaves** are written  $\{\sigma\}^V$  and are supported on the smallest open neighborhoods.

$$\{\sigma\}^V(\tau) = \begin{cases} V & \sigma \leq \tau \\ 0 & \text{o.w.} \end{cases}$$

An important fact about injective sheaves is the following:

**Theorem 2.1** ([7] 1.3.2). *A sheaf  $I$  is injective if and only if it is isomorphic to one of the form  $\bigoplus_{\sigma \in X} [\sigma]^{V_\sigma}$ .*

Finally, a comment about **cellular co-sheaves**. These are functors  $cF : \mathbf{Cell}(X)^{\text{op}} \rightarrow \mathbf{Vect}$  and have **restriction maps**  $r_{\tau, \sigma}^{cF}$  that go from higher dimensional cells to lower dimensional ones. Alternatively they are sheaves on the Alexandrov topology of  $(X, \leq_{\text{op}})$ . Note that for any finite poset there are two Alexandrov topologies given by exchanging open sets for closed sets. The dualities relating sheaves and cosheaves are abundant. In particular injectives and projectives are swapped between the two categories.

If we just turn around the internal co-restriction maps we obtain the **elementary projective co-sheaves** written  $c[\sigma]^V$  whose support is the same as  $[\sigma]^V$  and the **elementary injective co-sheaves**  $c\{\sigma\}^V$  whose support is the same as  $\{\sigma\}^V$ . We will not always write a  $c$  prefix to remember that the objects at hand are co-sheaves. Hopefully the surrounding text will be enough to infer what is what.

### 3 The Equivalence

We want to exploit the fact that the category of cellular sheaves possesses both enough injectives and enough projectives. This means that the derived category can be modeled via the homotopy category of complexes of either injectives or projectives. The preference for either of these models then boils down to whether the functors of interest are either left-exact or right-exact. In the case of sheaves, the global section functor is left-exact and so the category of injectives is preferred. For co-sheaves, however, the opposite is the case.

**Lemma 3.1.** *The derived category of co-sheaves is equivalent to the homotopy category of projectives, i.e.  $D^b(\mathbf{C}(X)) \cong K^b(\mathbf{Proj} - \mathbf{C})$*

This is a corollary of the following more general algebraic fact.

**Theorem 3.2** ([1] Thm 6.7). *Suppose  $\mathcal{A}$  is an abelian category with enough projectives, then  $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$  where  $\mathcal{P}$  denotes projective objects of  $\mathcal{A}$ . Similarly, if  $\mathcal{A}$  has enough injectives then  $D^+(\mathcal{A}) \cong K^+(\mathcal{I})$  where  $\mathcal{I}$  denotes injective objects of  $\mathcal{A}$ .*

Here we explicitly define a covariant functor that associates to a sheaf (or more generally a complex of sheaves) a complex of projective co-sheaves. This functor establishes the desired equivalence.

**Definition 3.3.** Let  $P : D^b(S(X)) \rightarrow D^b(C(X))$  be a functor that takes a sheaf and uses its co-restriction maps to define the differentials in a complex of projective co-sheaves. For a complex of sheaves, this is performed on the individual sheaves to produce a double complex, the totalization of which will be a complex of projectives, i.e. to a sheaf  $F \in S(X)$  we associate the following complex of projective co-sheaves

$$\cdots \longrightarrow \bigoplus_{\sigma^i \in X} \mathbf{c}[\sigma^i]^{F(\sigma^i)} \xrightarrow{[\sigma:\gamma]\rho^F} \bigoplus_{\gamma^{i+1} \in X} \mathbf{c}[\gamma^{i+1}]^{F(\gamma^{i+1})} \xrightarrow{[\gamma:\tau]\rho^F} \bigoplus_{\tau^{i+2} \in X} \mathbf{c}[\tau^{i+2}]^{F(\tau^{i+2})} \longrightarrow \cdots$$

Here  $\sigma^i$  denotes the  $i$ -cells and  $[\sigma^i : \gamma^{i+1}] = \{0, \pm 1\}$  records whether the cells are incident and whether orientations agree or disagree. The maps in between are to be understood as the block diagonal matrix  $\bigoplus[\sigma^i : \gamma^{i+1}]\rho_{\sigma,\gamma}^F$ .

For a complex of sheaves

$$\begin{array}{ccccccc} F^i & \rightsquigarrow & \cdots \longrightarrow & \bigoplus_{\gamma^{j+1} \in X} \mathbf{c}[\gamma^{j+1}]^{F^i(\gamma^{j+1})} & \xrightarrow{[\gamma:\tau]\rho^{F^i}} & \bigoplus_{\tau^{j+2} \in X} \mathbf{c}[\tau^{j+2}]^{F^i(\tau^{j+2})} & \longrightarrow \cdots \\ \downarrow & & & \downarrow & & \downarrow & \\ F^{i+1} & \rightsquigarrow & \cdots \longrightarrow & \bigoplus_{\gamma^{j+1} \in X} \mathbf{c}[\gamma^{j+1}]^{F^{i+1}(\gamma^{j+1})} & \xrightarrow{[\gamma:\tau]\rho^{F^{i+1}}} & \bigoplus_{\tau^{j+2} \in X} \mathbf{c}[\tau^{j+2}]^{F^{i+1}(\tau^{j+2})} & \longrightarrow \cdots \\ \downarrow & & & \downarrow & & \downarrow & \\ F^{i+2} & \rightsquigarrow & \cdots \longrightarrow & \bigoplus_{\gamma^{j+1} \in X} \mathbf{c}[\gamma^{j+1}]^{F^{i+2}(\gamma^{j+1})} & \xrightarrow{[\gamma:\tau]\rho^{F^{i+2}}} & \bigoplus_{\tau^{j+2} \in X} \mathbf{c}[\tau^{j+2}]^{F^{i+2}(\tau^{j+2})} & \longrightarrow \cdots \end{array}$$

where we then pass to the totalization.

**Theorem 3.4** (Equivalence).  $P : D^b(S(X)) \rightarrow D^b(C(X))$  is an equivalence.

*Proof of the Main Theorem.* First let us point out that the functor  $P$  really is a functor. Indeed if  $\alpha : F \rightarrow G$  is a map of sheaves then we have maps  $\alpha(\sigma) : F(\sigma) \rightarrow G(\sigma)$  that commute with the respective co-restriction maps  $\rho^F$  and  $\rho^G$ . As a result, we get maps  $\mathbf{c}[\sigma]^{F(\sigma)} \rightarrow \mathbf{c}[\sigma]^{G(\sigma)}$ . Moreover, these maps respect the differentials in  $P(F)$  and  $P(G)$ , so we get a chain map. It is clearly additive, i.e. for maps  $\alpha, \beta : F \rightarrow G$   $P(\alpha + \beta) = P(\alpha) + P(\beta)$ . This implies that  $P$  preserves homotopies.

It is also clear that  $P$  preserves quasi-isomorphisms. Note that a sequence of cellular sheaves  $A^\bullet$  is exact if and only if  $A^\bullet(\sigma)$  is an exact sequence of vector spaces for every

$\sigma \in X$ . This implies that  $P(A^\bullet)$  is a double-complex with exact rows. By standard results surrounding the theory of spectral sequences this implies that the totalization is exact.

Let us understand what this functor does to an elementary injective sheaf  $[\sigma]^\vee$ . Applying the definition we can see that

$$P : [\sigma]^\vee \rightsquigarrow \bigoplus_{\tau^0 \subset \sigma} c[\tau^0]^\vee \longrightarrow \dots \longrightarrow \bigoplus_{\tau^i \subset \sigma} c[\tau^i]^\vee \longrightarrow \dots \longrightarrow c[\sigma]^\vee$$

which is nothing other than the projective cosheaf resolution of the skyscraper (or simple) (co)sheaf  $S_\sigma^\vee$  supported on  $\sigma$ , i.e.

$$S_\sigma^\vee(\tau) = \begin{cases} V & \sigma = \tau \\ 0 & \text{o.w.} \end{cases}$$

Consequently, there is a quasi-isomorphism  $q : P([\sigma]^\vee) \rightarrow S_\sigma^\vee$  where  $S_\sigma^\vee$  is placed in degree equal to the dimension of  $\sigma$  assuming that  $[\sigma]^\vee$  is initially in degree 0. By abusing notation and letting  $P$  send cosheaves to sheaves, we see that

$$P(q) : P^2([\sigma]^\vee) \rightarrow P(S_\sigma^\vee) = [\sigma]^\vee$$

and thus we can define a natural transformation from  $P^2$  to  $\text{id}_{D(S)}$  when restricted to elementary injectives. However, by Theorem 2.1 we know that every injective looks like such a sum, so this works for injective sheaves concentrated in a single degree. However, it is clear that  $P$  sends a complex of injectives, before taking the totalization of the double complex to the projective resolutions of a complex of skyscraper cosheaves. Applying  $P$  to the quasi-isomorphism relating the double complex of projective cosheaves to the complex of skyscrapers, extends the natural transformation to the whole derived category. However, since  $P$  preserves quasi-isomorphisms, this natural transformation is in fact an equivalence. This shows  $P_{D(S)}^2 \cong \text{id}_{D(S)}$ . Repeating the argument starting from co-sheaves shows that

$$P : D^b(S(X)) \leftrightarrow D^b(C(X)) : P$$

is an adjoint equivalence of categories. □

## 4 Verdier Duality

### 4.1 Linear Duality

There is an endofunctor on the category of finite dimensional vector spaces **vect** given by sending a vector space to its dual  $V \rightsquigarrow V^*$ . This functor has the effect of taking a cellular sheaf  $(F, \rho)$  to a cellular co-sheaf  $(F^*, \rho^*)$ , since the co-restriction maps get dualized into restriction maps. It is contravariant since a sheaf morphism  $F \rightarrow G$  gets sent to a co-sheaf morphism in the opposite direction  $F^* \leftarrow G^*$  as one can easily check. We can promote this functor to the derived category.

**Definition 4.1** (Linear Duals). Define  $V : D^b(S_f(X))^{\text{op}} \rightarrow D^b(C_f(X))$  as follows

- $V(F^\bullet) = (F^*)^{-\bullet}$ , i.e. take a sheaf in slot  $i$ , dualize its internal co-restriction maps  $\rho_{\sigma,\tau}^{F^i}$  to restriction maps  $r_{\tau,\sigma}^{F^{i*}}$  to obtain a co-sheaf and then put it in slot  $-i$ .
- $V$  sends differentials between sheaves  $d^i$  to their adjoints in negative degree  $\partial^{-i-1} := (d^i)^*$

$$\star(\dots \longrightarrow F^i \xrightarrow{d^i} F^{i+1} \longrightarrow \dots) = \dots \longrightarrow [(F^{i+1})^*]^{-i-1} \xrightarrow{\partial^{-i-1}} [(F^i)^*]^{-i} \longrightarrow \dots$$

We will reserve the right to abuse notation and let  $V$  map from co-sheaves to sheaves in the obvious manner, i.e.  $V : D^b(C_f(X))^{\text{op}} \rightarrow D^b(S_f(X))$  or formally equivalent  $V : D^b(C_f(X)) \rightarrow D^b(S_f(X))^{\text{op}}$ .

**Lemma 4.2.**  $V : D^b(C_f(X)) \rightarrow D^b(S_f(X))^{\text{op}}$  is an equivalence of categories.

*Proof.* It is clear that if  $\alpha : I^\bullet \rightarrow J^\bullet$  is a map in the category of complexes of sheaves homotopic to zero  $\alpha \simeq 0$ , i.e. there exists a map  $h : I^\bullet \rightarrow J^{\bullet-1}$ , written  $h : I \rightarrow J[-1]$  such that  $\alpha^n - 0^n = d_J^{n-1}h^n + h^{n+1}d_I^n$ . Writing out how  $V$  acts carefully we see that  $V(\alpha) : V(J) \rightarrow V(I)$  and  $V(h) : V(J[-1]) = V(J)[+1] \rightarrow V(I)$  defines a homotopy between  $V(\alpha)$  and  $V(0) = 0$  by setting  $(h^*)^\bullet = V(h)^{\bullet-1}$ .

$V$  thus sends  $K^b(\text{Inj} - S_f)^{\text{op}}$  to  $K^b(\text{Proj} - C_f)$  and composed twice  $VV : K^b(\text{Inj} - S_f) \rightarrow K^b(\text{Inj} - S_f)$  is naturally isomorphic to the identity functor, so it is an equivalence. We can repeat the arguments for co-sheaves and use formality to put the  $\text{op}$  where we want.  $\square$

## 4.2 Verdier Duality

**Definition 4.3** (Verdier Dual). The Verdier dual functor  $D : D(S_f(X)) \rightarrow D(S_f(X))^{\text{op}}$  is defined as  $D := \mathcal{H}om(-, \omega_X^\bullet)$ . Recall that  $\mathcal{H}om(F, G)$  is a sheaf whose value on a cell  $\sigma$  is given by  $\mathbf{Hom}(F|_{\text{st}(\sigma)}, G|_{\text{st}(\sigma)})$ , i.e. natural transformations between the restrictions to the star of  $\sigma$ .

The complex of injective sheaves  $\omega_X^\bullet$  is called the dualizing complex of  $X$ . It has in negative degree  $\omega_X^{-i}$  the sum over the one-dimensional elementary injectives concentrated on  $i$ -cells  $[\gamma^i]$ . The maps between use the orientations on cells to guarantee it is a complex.

$$\dots \longrightarrow \bigoplus_{|\tau|=i+1} [\tau] \xrightarrow{\oplus[\gamma:\tau]} \bigoplus_{|\gamma|=i} [\gamma] \xrightarrow{\oplus[\sigma:\gamma]} \bigoplus_{|\sigma|=i-1} [\sigma] \longrightarrow \dots$$

The Verdier dual of  $F$  is the complex of sheaves  $D^\bullet F := \mathcal{H}om(F, \omega_X^\bullet)$ . Written out explicitly it is

$$\dots \longrightarrow \bigoplus_{|\tau|=i+1} [\tau]^{F(\tau)*} \xrightarrow{\oplus[\gamma:\tau]_\rho^*} \bigoplus_{|\gamma|=i} [\gamma]^{F(\gamma)*} \xrightarrow{\oplus[\sigma:\gamma]_\rho} \bigoplus_{|\sigma|=i-1} [\sigma]^{F(\sigma)*} \longrightarrow \dots$$

## 4.3 Verdier Duality Recovered

**Proposition 4.4.** The functor  $P : D^b(S_f(X)) \rightarrow D^b(C_f(X))$  composed with linear duality  $V : D^b(C_f(X)) \rightarrow D^b(S_f(X))^{\text{op}}$  gives Verdier duality, i.e.  $D \cong VP$ .

*Proof.* Just check by hand. □

*Remark 4.5.* We could have used well-known facts about Verdier duality to prove a weaker version of our main theorem by restricting to finitely generated stalks.

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## References

- [1] Aluffi, Paolo “Algebra. Chapter 0”. Graduate Studies in Mathematics Volume 104. American Mathematical Society 2009.
- [2] Curry, Justin; Ghrist, Robert; Robinson, Michael “Euler Calculus with Applications to Signals and Sensing” to appear in AMS Proceedings of Symposia in Applied Mathematics, arXiv:1202.0275.
- [3] Kashiwara, Masaki; Schapira, Pierre “Sheaves on Manifolds”
- [4] Kashiwara, Masaki “The Riemann-Hilbert Problem for Holonomic Systems” Publ. RIMS, Kyoto Univ. 20 (1984) pp. 319-365.
- [5] Ladkani, Sefi “On Derived Equivalences of Categories of Sheaves over Finite Posets”
- [6] Nadler, David “Fukaya Categories as Categorical Morse Homology” arXiv:1109.4848v1
- [7] Shepard, Allen “A Cellular Description of the Derived Category of a Stratified Space” Brown University PhD Thesis, May 1985.