Abstract Existence of Cosheafification

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1 Cosheaves Defined

Since pre-cosheaves and cosheaves are under-represented in the literature, we start here.

Definition 1.1 (Pre-Cosheaves and Cosheaves)**.** Suppose $\mathcal{D}$ is a category and $\mathbf{Open}(X)$ is the category of open sets of a topological space. A **pre-cosheaf** valued in $\mathcal{D}$ is a covariant functor $\hat{F} : \mathbf{Open}(X) \to \mathcal{D}$. We usually write the **extension maps** corresponding to an inclusion $U \subset U'$ as $r^{\hat{F}}_{U',U} : \hat{F}(U) \to \hat{F}(U')$, but we will usually omit the superscript $\hat{F}$ if it is clear what pre-cosheaf is being considered.

A pre-cosheaf $\hat{F}$ is a **cosheaf** if for any open cover $\mathcal{U} = \{U_i\}$ of an open set U, we have the following coequalizer diagram

$$\coprod \hat{F}(U_i \cap U_j) \rightrightarrows \coprod \hat{F}(U_i) \to \hat{F}(U).$$

A slightly more conceptual perspective on the definition is the following oft-quoted slogan *"cosheaves send colimits to colimits."* What is meant by this is the following: Given an open cover $\mathcal{U} = \{U_i\}$ of U, we can form its **nerve** $N(\mathcal{U})$ given by taking all finite intersections of elements in the cover. Elements of $N(\mathcal{U})$ are labelled by multi-indices I, specifying which elements of the cover are being intersected. Then we say that $\hat{F}$ is a cosheaf if

$$\hat{F}(U) \cong \varinjlim_{I \in N(\mathcal{U})} \hat{F}(U_I)$$

for any open cover $\mathcal{U}$ of U.

Remark 1.2. Some authors avoid mentioning the nerve and instead specify that $\hat{F}(U) \cong \operatorname{colim}_{U_i \in \mathcal{U}} \hat{F}(U_i)$ under the assumption that $\mathcal{U}$ is "stable under finite intersection," i.e. all finite intersections of elements of $\mathcal{U}$ are included among members of $\mathcal{U}$.

Remark 1.3 (The Empty Set). We will always assume that our pre-cosheaves assigns the initial object of $\mathcal{D}$ to the empty set, when it exists. If we adopt the perspective that the empty set has no covers, then $\hat{F}(\emptyset)$ is the colimit over the empty diagram. The category of cocones is all of $\mathcal{D}$. The universal property of the colimit dictates that the colimit over the empty diagram is the initial object.

Remark 1.4. We will primarily deal with pre-cosheaves and cosheaves valued in abelian categories $\mathcal{A}$, such as $\mathbf{Vect}$ or $\mathbf{Ab}$. In these instances the cosheaf axiom states that the following sequence

$$\bigoplus_{i<j} \hat{F}(U_i \cap U_j) \to \bigoplus_{i} \hat{F}(U_i) \to \hat{F}(U) \to 0$$

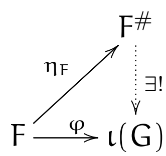
is exact.

2 Cosheafification: Existence of an Adjoint

Classically, when one studies sheaf theory an elegant way of saying that every presheaf has an associated sheaf (it's sheafification), is to say that the inclusion functor

$$\iota : \mathbf{Shv}(X; \mathcal{D}) \hookrightarrow \mathbf{Fun}(\mathbf{Open}(X)^{op}, \mathcal{D}) =: \mathbf{Pre\text{-}Shv}(X; \mathcal{D})$$

has a left adjoint $(-)^{\#}$, i.e. there is a universal natural transformation $\eta : \mathrm{id}_{\mathbf{Pre\text{-}Shv}} \to \iota \circ (-)^{\#}$. Such a subcategory is called **reflective**. This guarantees, for example, that if F is an arbitrary pre-sheaf and G is a sheaf, regarded as a pre-sheaf $G = \iota(G)$, then we have the following universal property:



Pulling back along η_F induces the usual natural isomorphism of **Hom**-sets:

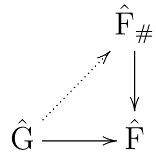
$$\mathbf{Hom}_{\mathbf{Shv}}(F^{\#}, G) \cong \mathbf{Hom}_{\mathbf{Pre\text{-}Shv}}(F, \iota(G))$$

In the case of sheaves, Grothendieck gave us an explicit construction of the functor $(-)^{\#}$, which comes about from applying a certain functor $(-)^{+}$ twice, i.e. $++ = \#$. The requirement on the category $\mathcal{D}$ includes, among other things, that filtered colimits and finite limits commute. If we were to regard a pre-cosheaf $\hat{F} : \mathbf{Open}(X) \to \mathcal{D}$ as a pre-sheaf valued in $\mathcal{D}^{op}$, then the condition that filtered colimits and finite limits commute in $\mathcal{D}^{op}$ would become the condition that cofiltered limits and finite colimits commute in $\mathcal{D}$, which is patently false when $\mathcal{D} = \mathbf{Set}, \mathbf{Vect}$ or $\mathbf{Ab}$.

Consequently, we do not have a clear answer to the question: Does the inclusion functor ι have a *right* adjoint $(-)_{\#}$?

$$\iota : \mathbf{CoShv}(X; \mathcal{D}) \hookrightarrow \mathbf{Fun}(\mathbf{Open}(X), \mathcal{D}) =: \mathbf{Pre\text{-}Coshv}(X; \mathcal{D})$$

so that the dual universal property is satisfied, i.e. if $\hat{F}$ is a pre-cosheaf and $\hat{G}$ is a cosheaf with a morphism $\hat{G} \to \hat{F}$, then there is a unique way of completing the diagram.



In the case where $\mathcal{D} = \mathbf{Set}$, Jon Woolf's paper [3] contains a construction of cosheafification. Unfortunately, this cannot be adapted to categories like $\mathbf{Vect}$ or $\mathbf{Ab}$. For a high-level reason why, Maclane and Moerdijk explain on page 95 of [2] that a sheaf of abelian groups can be identified with an abelian group object in the category of sheaves. Since sheafification preserves finite products, sheafification of pre-sheaves of sets lifts to a functor between abelian group objects. Moreover, since the forgetful functor $\mathsf{for} : \mathbf{Ab} \to \mathbf{Set}$ preserves limits (but not colimits), any sheaf of groups defines a sheaf of sets. Trying to repeat this last line of reasoning for cosheaves of groups fails, i.e. a cosheaf of groups does not, by forgetting, define a cosheaf of sets.

One approach at this point is to verify abstractly wether cosheafification exists, without constructing it. The following theorem of Freyd's gives a criterion for determining when a functor has an adjoint.

Theorem 2.1 (Freyd's Adjoint Functor Theorem)**.** *Let $\mathcal{B}$ be a complete category and $\mathsf{G} : \mathcal{B} \to \mathcal{C}$ a functor, then G has a left adjoint F if and only if G preserves all limits and satisfies the **solution set condition**. This condition states that for each object $x \in \mathcal{C}$ there is a set I and an I-indexed family of arrows $f_i : c \to \mathsf{G}(a_i)$ such that every arrow $f : x \to \mathsf{G}(a)$ can be factored as $x \to \mathsf{G}(a_i) \to \mathsf{G}(a)$, where the first map is $f_i : x \to \mathsf{G}(a_i)$ and the second is G applied to to some $t : a_i \to a$.*

The solution set condition holds nearly all the time, so in practice one only needs to check that G preserves limits, in which case G is a right adjoint (has a left adjoint). Dually, for a functor to be a left adjoint it needs to preserve colimits. Since we wish to be precise, we want to understand the set-theoretic issues in greater detail. To do so, we will use a different theorem that is easier to check.

Theorem 2.2 (Thm 6.28 [1])**.** *Assuming Vopenka's principle (a large cardinal axiom), every full subcategory $\mathcal{B}$ of a locally presentable category $\mathcal{C}$, where $\mathcal{B}$ is closed under colimits, is coreflective, i.e. the inclusion functor $\iota : \mathcal{B} \hookrightarrow \mathcal{C}$ has a right adjoint (a cofree functor).*

We will leave Vopenka's principle as a black box and assume it. Now we aim to prove the following corollary.

Corollary 2.3. *The category of cosheaves of vector spaces is a coreflective subcategory of $\mathbf{Fun}(\mathbf{Open}(X), \mathbf{Vect})$, i.e. cosheafification exists.*

Proof. It is clear that the category of cosheaves is closed under colimits, since we can define the colimit to be the pre-cosheaf, which open-by-open assigns the colimit of vector spaces over that open set. This pre-cosheaf is a cosheaf, because for a fixed cover, each vector space in the diagram is expressed as a colimit and colimits commute with colimits.

It remains to be seen that the category of pre-cosheaves is locally presentable. this means that the category is locally small, has small colimits, has a small set of objects $\mathcal{S}$ that generates $\mathbf{Pre\text{-}Coshv}(X;\mathbf{Vect})$ in the sense that every pre-cosheaf is a colimit of objects in $\mathcal{S}$, and every object is small. The first two statements are easily addressed. $\mathbf{Open}(X)$ is a small category and $\mathbf{Vect}$ is locally small, so the functor category is locally small. Colimits of pre-cosheaves are defined open-by-open. Since $\mathbf{Vect}$ is cocomplete, pre-cosheaves valued in $\mathbf{Vect}$ is also cocomplete. Now we address the existence of a generating set.

Define, for each open set $U \in \mathbf{Open}(X)$, the following pre-cosheaf:

$$\hat{h}_U(V) = \begin{cases} k & \text{if } U \subset V \\ 0 & o.w. \end{cases}$$

We'd like to say that every pre-cosheaf is a colimit of pre-cosheaves of the above form. The corresponding statement for pre-sheaves is proved in pages 41-42 of [2]. We will go ahead and repeat the argument here. Note that if $\hat{G}$ is an arbitrary pre-cosheaf, then

$$\mathbf{Hom}_{\mathbf{Pre\text{-}Coshv}}(\hat{h}_U, \hat{G}) \cong \hat{G}(U).$$

Observe that if $U \subset U'$, then we get a map of pre-cosheaves (a natural transformation) $\hat{h}_{U'} \to \hat{h}_U$. This in turn induces a map

$$\mathbf{Hom}_{\mathbf{Pre\text{-}Coshv}}(\hat{h}_U, \hat{G}) \to \mathbf{Hom}_{\mathbf{Pre\text{-}Coshv}}(\hat{h}_{U'}, \hat{G})$$

which coincides with the internal extension map of $\hat{G}$, that is $r_{U',U} : \hat{G}(U) \to \hat{G}(U')$. In other words, the functor

$$R : \mathbf{Pre\text{-}Coshv}(X;\mathbf{Vect}) \to \mathbf{Pre\text{-}Coshv}(X;\mathbf{Vect})$$

$$G \rightsquigarrow (U \mapsto \mathbf{Hom}_{\mathbf{Pre\text{-}Coshv}}(\hat{h}_U, \hat{G}) \cong \hat{G}(U))$$

is isomorphic to the identity functor. Since adjoints are unique up to isomorphism, then we can conflude that its (left) adjoint must also be isomorphic to the identity functor. However, we will construct explicitly the adjoint, which, combined with the fact that it must be the identity functor, exhibits $\hat{G}$ as the colimit of pre-cosheaves of the form $\hat{h}_U$.

For each pre-cosheaf $\hat{G}$, define the following **category of elements** $J_{\hat{G}}$.[1] The objects of $J_{\hat{G}}$ are pairs

$$(U, x) \qquad \text{where} \qquad U \in \mathbf{Open}(X) \quad x \in \hat{G}(U).$$

A morphism $(U, x) \to (U', x')$ is defined if $U \subset U'$ and $x' = r_{U',U}(x)$. Clearly, there is a projection functor $\pi_{\hat{G}} : J_{\hat{G}} \to \mathbf{Open}(X)$ and by formality, there is a dual functor $\pi^{op}_{\hat{G}} : J^{op}_{\hat{G}} \to \mathbf{Open}(X)^{op}$.

[1] It should be noted that in [2], the category of elements is written $\int \hat{G}$.

Denote by $\mathcal{Y}$ the functor

$$\mathcal{Y} : \mathbf{Open}(X)^{op} \to \mathbf{Pre\text{-}Coshv}(X; \mathbf{Vect}) \qquad U \rightsquigarrow \hat{h}_U.$$

We claim that the left adjoint L to the functor R considered above can be constructed object-wise as follows: for each pre-cosheaf $\hat{G}$ define

$$L(\hat{G}) := \varinjlim \mathcal{Y} \circ \pi_{\hat{G}}^{op}.$$

We claim that $\hat{G}$ is the colimit. This is already given from the fact that L must be isomorphic to the identity functor, but let's at least check how $\hat{G}$ is a co-cone, to make the statement more plausible. For each object (U, x) in $J_{\hat{G}}$ the map to $\hat{G}$ is defined by

$$x \in \hat{G}(U) \cong \mathbf{Hom}_{\mathbf{Pre\text{-}Coshv}}(\hat{h}_U, \hat{G}) \ni \psi_{U,x}$$

where $\psi_{U,x}$ is the natural transformation that sends $1 \in \hat{h}_U(U)$ to $x \in \hat{G}(U)$ and then for any larger open set $U \subset U'$ sends $1 \in \hat{h}_U(U')$ to $r_{U',U}(x)$. Observe that if $U \subset U'$ and $(U', x') \to (U, x)$ is a morphism in $J_{\hat{G}}^{op}$, so $r_{U',U}(x) = x'$, then we have the following commutative diagram:

$$\begin{array}{ccccc}
(U', x') & \rightsquigarrow & \hat{h}_{U'} & & \\
\downarrow & & \downarrow & \searrow^{\psi_{U,x'}} & \\
(U, x) & \rightsquigarrow & \hat{h}_U & \xrightarrow[\psi_{U,x}]{} & \hat{G}
\end{array}$$

At the risk of demonstrating the obvious, the above diagram commutes if the the following diagram commutes for an arbitrary triple of open sets $V \subset V' \subset V''$. We will check it for the interesting "boundary" case $U \subset U' \subset U''$.

$$\begin{array}{ccccc}
\hat{h}_{U'}(U'') = k & \xrightarrow{1} & \hat{h}_U(U'') = k & \xrightarrow{r_{U'',U'}(x)} & \hat{G}(U'') \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle r_{U'',U'}} \\
\hat{h}_{U'}(U') = k & \xrightarrow{1} & \hat{h}_U(U') = k & \xrightarrow{r_{U,U'}(x)=x'} & \hat{G}(U') \\
\uparrow & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle r_{U',U}} \\
\hat{h}_{u'}(U) = 0 & \longrightarrow & \hat{h}_U(U) = k & \xrightarrow{x} & \hat{G}(U)
\end{array}$$

This completes the plausibility check, we use the early observation to conclude that $\hat{G}$ is actually the colimit. The conclusion is that

$$\hat{G} \cong \varinjlim \mathcal{Y} \circ \pi_{\hat{G}}^{op}$$

i.e. $\hat{G}$ is expressible as a small colimit of pre-cosheaves of the form $\hat{h}_U$ where the size of the indexing set is bounded by the product of the cardinality of $\mathbf{Open}(X)$ and the maximum cardinality of $\hat{G}(U)$ over varying U.

Now it remains to check the smallness of objects in $\mathbf{Pre\text{-}Coshv}(X; \mathbf{Vect})$. An object $\hat{G}$ is small if there exists a regular cardinal κ such that $\mathbf{Hom}(\hat{G}, -)$ commutes with directed colimits of diagrams indexed by categories of cardinality at most κ.

Observe that for one of our pre-cosheaves $\hat{h}_U$ is compact since if $(\hat{F}_i)$ is a direct system of pre-cosheaves, then

$$\varinjlim \mathbf{Hom}(\hat{h}_U, \hat{F}_i) \cong \varinjlim \hat{F}_i \cong \mathbf{Hom}(\hat{h}_U, \varinjlim \hat{F}_i).$$

As already shown, for every pre-cosheaf $\hat{G}$ there exists a diagram whose cardinality is the cardinality of $J_{\hat{G}}$, which we will call κ_J. Now, we know how to express $\hat{G}$ as a colimit of $\hat{h}_U$'s. Thus,

$$\begin{aligned}
\mathbf{Hom}(\hat{G}, \varinjlim_i \hat{F}_i) &\cong \mathbf{Hom}(\varinjlim_U \hat{h}_U, \varinjlim_i \hat{F}_i) \\
&\cong \varprojlim_U \mathbf{Hom}(\hat{h}_U, \varprojlim_i \hat{F}_i) \\
&\cong \varprojlim_U \varinjlim_i \mathbf{Hom}(\hat{h}_U, \hat{F}_i) \\
&\cong \varprojlim_i \varinjlim_U \mathbf{Hom}(\hat{h}_U, \hat{F}_i) \\
&\cong \varinjlim_i \mathbf{Hom}(\varinjlim_U \hat{h}_U, \hat{F}_i) \\
&\cong \varinjlim_i \mathbf{Hom}(\hat{G}, \hat{F}_i)
\end{aligned}$$

The third line follows from compactness of $\hat{h}_U$. The fourth line follows from the fact that in $\mathbf{Set}$, κ_J small colimits commute with κ_J -filtered colimits. This completes the proof.

□

References

[1] J. Adamek, J. Adámek, and J. Rosicky. *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[2] S.M. Lane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext (1979). Springer-Verlag, 1992.

[3] Jonathan Woolf. The fundamental category of a stratified space. *Journal of Homotopy and Related Structures*, 4(1):359–387, 2009.