POLYNOMIALITY OF THE BIGRADED SUBDIMENSION OF DIAGONAL HARMONICS

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To my family.

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ABSTRACT

POLYNOMIALITY OF THE BIGRADED SUBDIMENSION OF DIAGONAL HARMONICS

Xinxuan Wang

James Haglund

A sequence of S_n-representation V_n is called representation (multiplicity) stable if after some n, the irreducible decomposition of V_n stabilizes. In particular, Church and Farb (2013) found that if we fix a and b, then the space of diagonal harmonics $DH_n^{a,b}$ exhibits this behavior, and its dimension stabilizes to a polynomial in n eventually. Building on this result, we use the Schedules Formula by Haglund and Loehr (2005) to get an explicit combinatorial polynomial for the dimension of the bigraded spaces $DH_n^{a,b}$. This derivation not only yields the dimension formula but also produces a new stability bound of $a + b$, which is conjectured to be sharp, and determines the exact degree of the dimension polynomial, which is also $a + b$.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

CHAPTER 1

Overview

This chapter provides a broad overview of the thesis. We begin by introducing our central object of study, the Diagonal Harmonics DH_n, defined as a subspace of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. This ring carries a natural action of the symmetric group S_n and admits a bigrading, with its bigraded components denoted by $DH_n^{a,b}$, consisting of polynomials homogeneous of degree a in the x-variables and degree b in the y-variables.

The dimension of DH_n as an S_n-representation was conjectured by Haiman (1994) to be $(n+1)^{n-1}$ and later proven by Haiman (2002). Furthermore, the dimension of each bigraded component $DH_n^{a,b}$ was conjectured by Haglund and Loehr (2005) in terms of number of certain subsets of parking functions of size n, and later proved by Carlsson and Mellit (2018) in a broader context.

This thesis establishes a new result concerning the dimensions $\dim(DH_n^{a,b})$: namely, that for fixed a and b, this dimension eventually becomes a polynomial in n. Moreover, we prove that the stabilization starts no later than $n = a + b$, and conjecture it to be the sharp bound. Chapter 2 introduces the necessary combinatorial background on DH_n, and Chapter 4 presents the explicit polynomial expression for $\dim(DH_n^{a,b})$ and proves the stabilization bound.

We are motivated by a phenomenon discovered by Church and Farb (2013): under certain conditions, sequences of S_n-representations V_n exhibit stabilization in their character and dimension. Notably, the sequence $DH_n^{a,b}$ satisfies their conditions and was among the examples discussed in their foundational work. This result served as the original inspiration for the present thesis. Although the polynomiality of $\dim(DH_n^{a,b})$ can be derived from representation-theoretic considerations, a purely combinatorial proof was previously lacking—this gap motivates our investigation. Chapter 3 provides the necessary background on representation stability.

CHAPTER 2

DIAGONAL HARMONICS

2.1. Catalan Combinatorics

The study of Diagonal Harmonics is closely related to the beautiful Catalan Combinatorics. We would start by introducing the Catalan objects, and then tie it with the study of Diagonal Harmonics.

Definition 2.1.1. A parking function of length n denoted as $p \in PF(n)$ is a sequence of n positive integers, each in the ragne from 1 to n, such that $1 \leq i \leq n$, the sequence contains at least i values that are at most i.

The name "parking function" comes from the following thought experiment: imagine n many cars are trying to park in to n parking spaces, and each of the cars have their preferred parking space. Each car i will go to its preferred parking space, and park there if the space is not already occupied; otherwise they will try the next spot, until they get parked. A parking function describes preferences for which all cars can park.

Example 2.1.2. (2,1,2,1) is a parking function, the parking arrangement would be the first car goes to its preferred spot 2, second car goes to 1, third car goes to 2 but it is occupied so it goes to 3, and the fourth car goes to 1 but it is already occupied, and so are 2 and 3, so it would park at 4. (3,3,1,3) is not a parking function, as the first car would go to its preferred spot 3, the second car would go to 4, and the third car goes to 1, but the fourth car has no where to go since 3 and 4 are both occupied.

Theorem 2.1.3. $|PF(n) = (n+1)^{n-1}|$

To see this, notice that if the parking spaces are circular, then the fourth car could be parked at 1, i.e. if given $n+1$ spaces arranged in a circular fashion, then all kinds of preferences can be parked. There are $(n+1)^n$ many such preferences, each leaving one vacant space; by symmetry there are

$n+1$ choices for preferences that leave space $n+1$ as the vacant space, thus the number $(n+1)^{n-1}$. Parking function first appeared in the study of idealized data storage method popular in theoretical computer science, and it is also crucial in algebraic combinatorics since the dimension of our diagonal harmonics is precisely equal to the number of parking function. We will discover deeper connection between them in Shuffle Theorem. Right now, we want to introduce parking function in another form which is used in the Shuffle Theorem.

Definition 2.1.4. A Dyck path $\Gamma \in Dyck(n)$ is a staircase walk from $(0,0)$ to (n,n) that lies above (but might touch) the diagonal $y = x$.

Theorem 2.1.5. *$|Dyck(n)| = C_n = \frac{1}{n+1}\binom{2n}{n}$ where C_n denotes the Catalan number.*

The Catalan number was first discovered by Mongolian/Chinese mathematician Mingantu around 1730, and was studied more by Euler and Eugène Catalan later. To prove the theorem, we use its recurrence relation

$$C_0 = 1, C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i} \tag{2.1.1}$$

It is clear that $C_0 = 1$. Let (i,i) be the first point of contact with the diagonal, We notice that the number of possible lower portion of Dyck path (under the point $(i.i)$ where it never touches the diagonal is exactly C_{i-1}, and the upper portion has C_{n-i} many possibilities).

We introduce parking function in the form of labelled Dyck path, which we would continue to work with and refer to labelled Dyck path as parking function from now on in this text.

Definition 2.1.6. A labeled Dyck path with length n denoted as $\gamma \in LDyck(n)$ is a Dyck path labelled by the numbers $1,...,n$ such that the labels of consecutive north steps are increasing.

We can see the bijection between labelled Dyck paths and parking functions as following: let $a = (a_1,...,a_n)$ be a parking function, and let b_i count the number of occurrences of i in a. Let $D \in Dyck(n)$ with b_i north steps after the $(i-1)$-st east-step, the fact that a is a parking function ensures D being a Dyck path. Label the b_i north-steps after the $(i-1)$-st east step by the positions of the letter i in a. These combinatorial objects are of special interests to us, because we have the result by Haiman

$$dim(DH_n) = (n+1)^{n-1} \tag{2.1.2}$$

and the Shuffle Theorem by Carlsson and Mellit (2018)

$$Hilb(DH_n) = \sum_{\gamma \in LDyck(n)} q^{dinv(\gamma)} t^{area(\gamma)} \tag{2.1.3}$$

where the Hilbert series, *dinv* and *area* will be defined in the next section.

2.2. Shuffle Theorem

To introduce Hilbert Series and Frobienius Series thus stating the Shuffle Theorem, we need to start with some basic representation theory and symmetric function theory.

Definition 2.2.1. Let G be a finite group. A representation of G is a set of square matrices $\{M(g)|g \in G\}$ such that

$$M(g)M(h) = M(g \cdot h) \ \forall g, h \in G \tag{2.2.1}$$

where $\cdot$ means the group multiplication. We would be mostly focusing on the representations of symmetric groups S_n. Because matrices are linear transformations, one can also think of a representation as a G-module.

Definition 2.2.2. Let V be a vector space and G be a group. Then V is a G-module if there is a multiplication $g\bar{v}$ of elements of V by elements of G such that

1. $g\bar{v} \in V$

2. $g(c\bar{v} + d\bar{w}) = c(g\bar{v}) + d(g\bar{w})$

3. $(gh)\bar{v} = g(h\bar{v})$

4. $1_G\bar{v} = v$

for $g, h \in G, \bar{v}, \bar{w} \in V, c, d \in \mathbb{C}$.

Every G-module is a G-representation. Our most prominent example would be $\mathbb{C}[X_n] = \mathbb{C}[x_1, ..., x_n]$. Given $f(x_1, ..., x_n) \in \mathbb{C}[X_n]$ and $\sigma \in S_n$, then

$$\sigma f = f(x_{\sigma_1}, ..., x_{\sigma_n}) \tag{2.2.2}$$

defines an action of S_n on $\mathbb{C}[X_n]$ thus makes $\mathbb{C}[X_n]$ an S_n-module. Let V be a subspace of $\mathbb{C}[X_n]$, then

$$V = \sum_{i=0}^{\infty} V^{(i)} \tag{2.2.3}$$

where $V^{(i)}$ is the subspace consisting of all elements of V of homogeneous degree i in the x_j. This defines a grading of the space V.

Definition 2.2.3. Hilbert series $Hilb(V; q)$ of V to be the sum

$$Hilb(V; q) = \sum_{i=0}^{\infty} q^i dim(V^{(i)}) \tag{2.2.4}$$

One can also think of the Hilbert series as the generating function of the dimensions.
Beyond the Hilbert series, which encodes information about the dimensions of graded subspaces, the Frobenius series captures the decomposition of these subspaces as S_n-representations. Before defining the Frobenius series, we first introduce a key property of S_n-representations that enables its definition.

Theorem 2.2.4. *(Maschke's Theorem) Let G be a finite group and let V be a nonzero G-module (a G-representation). Then*

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)}$$

where each $W^{(i)}$ is an irreducible G-submodule of V.

Modules that have this property are called completely reducible. Since S_n is a finite group for fixed n, its representations enjoy this property as well. The irreducible S_n-modules are known as Specht modules, denoted by S^{λ}, and are indexed by partitions $\lambda \vdash n$. These modules can be constructed combinatorially; for further details, see Sagan (2001). Now we turn our attention to symmetric functions. Let K be a field and it is usually $\mathbb{C}$, $\sigma \in S_n$, and $f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$. f is a symmetric function if

$$\sigma \cdot f = f(x_{\sigma_1}, \ldots, x_{\sigma_n}) = f$$

for all $\sigma \in S_n$.

Definition 2.2.5. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition. The monomial sytmmetric function corresponding to λ is

$$m_\lambda = m_\lambda(x) = \sum x_{i_1}^{\lambda_1} \sum x_{i_2}^{\lambda_2} \cdots \sum x_{i_l}^{\lambda_l}$$

where the sum is over all all distinct monomials having exponents $\lambda_1, \ldots, \lambda_l$.

Example 2.2.6.

$$m_{2,1} = x_1^2 x_2^2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \ldots$$

The ring of symmetric function is defined to be $\Lambda = \mathbb{C} m_\lambda$, and it is not hard to verify that it is indeed a ring since it is close under product. If we denote Λ^n as the space spanned by all m_λ of degree n, then $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ^n. We are mainly interested in another basis for the symmetric function ring Λ^n.

Definition 2.2.7. A semistandard tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda \vdash n$, is an array with the first row having λ_1 numbers, second row having λ_2 numbers,... the l-th row having λ_l numbers. The numbers are weakly increasing in the rows, and strictly increasing in the columns.

Example 2.2.8. Below is an exmaple of a semistandard tableuax of shape $(3, 2, 1, 1)$

1	3	3
2	4	
3		
5		

Definition 2.2.9. Given a generalized tableau T of shape λ, it has a weight in $\mathbb{C}[x]$

$$x^T = \prod_{(i,j)\in\lambda} x_{T_{i,j}}$$

The weight of in Example 2.2.8 is $x^T = x_1 x_2 x_3^3 x_4 x_5$.

Given a partition λ, the associated Schur function is

$$s_\lambda(x) = \sum_T x^T$$

where the sum is over all semistandard λ-tableaux T.

Example 2.2.10. For $\lambda = (2,1)$, we have the tableaux

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

where $a < c, a \leq b$, so

$$s_{2,1}(x) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + 2x_1x_2x_3 + 2x_1x_2x_4 + \ldots$$

Next we define the Characteristic Map, which bridges the world of representation theory and symmetric function ring.

Definition 2.2.11. A class function on S_n is a function that is constant on the conjugacy classes of S_n. Let R_n be the space of class functions of S_n. The characteristic map is $ch^n : R^n \to \Lambda^n$

$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\mu p_\mu$$

where χ_μ is the value of χ on the class μ.

Define an inner product on Λ^n by

$$< s_\lambda, s_\mu >= \delta_{\lambda\mu}$$

then ch^n preserve these inner products. Another important fact about this map is that $ch^n(S_\lambda) = s_\lambda$ where S_λ denotes the irreducible S_n characters. This makes the characteristic map an isometry since it maps orthonormal basis to another. Now we are ready to define Frobenius series.

Definition 2.2.12. Assume V is a subspace of $\mathbb{C}[X_n]$ fixed by the S_n action. The Frobenius series $F(V;q)$ of V is defined to be the symmetric function

$$\sum_{i=0}^{\infty} q^i \sum_{\lambda \in Par(i)} Mult(S^\lambda, V^{(i)}) \cdot s_\lambda$$

where $Mult(S^\lambda, V^{(i)})$ denotes the multiplicity of the irreducible character S_λ in the character of $V^{(i)}$ under the action.

Note that we can derive the Hilbert Series from a Frobenius Series, namely

$$< Frob(V;q), h_{1^n} >= Hilb(V;q)$$

Now we are ready to introduce the space of Harmonics.

Definition 2.2.13. The space of Harmonics H_n is defined as

$$H_n = \{f(X_n) \in \mathbb{C}[X_n] : \sum_{i=1}^{n} \partial_{x_i}^k f(X_n) = 0 \text{ for all k} > 0\} \tag{2.2.5}$$

H_n is isomporphic to the Ring of Coinvariants as S_n modules, but we will keep refering them as harmonics/coinvariants in this thesis. Given a $f \in \mathbb{C}[X_n]$, S_n acts on f by permuting the indices of the variables, i.e.

$$\sigma \cdot f(x_1, ..., x_n) = f(x_{\sigma_1}, ..., x_{\sigma_n})$$

Now we are ready to define the ring of coinvariants.

Definition 2.2.14. The Coinvariant Ring of S_n is

$$R_n = \frac{\mathbb{C}[X_n]}{I(X_n)^+} \tag{2.2.6}$$

where $I(X_n)^+ = \{f \in \mathbb{C}[X_n] | \sigma \cdot f = f \text{ and } f \text{ is not a constant}\}$.

Haiman (1994) has a detailed proof of the isomorphism and notes that an isomorophism $f : H_n \to R_n$ just sends $h \in H_n$ to $f(h)$, the element of $\mathbb{C}[X_n]$ represented modulo $I(X_n)^+$ by h.

To study the Hilbert series of H_n, we need to use the basis Artin and Milgram (1944) found for R_n:

Theorem 2.2.15. *$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i < i\}$ form a basis for R_n.*

Corollary 2.2.16. $dim(R_n) = \text{n!}$

The Corollary results from directly computing the number of basis elements.

Corollary 2.2.17. By Artin's basis, the Hilbert Series of R_n is

$$Hilb(R_n) = \sum_{\substack{\alpha \models n \\ 0 \leq \alpha_i < i}} q^{|\alpha|} \tag{2.2.7}$$

$$= 1 \cdot (1+q) \cdot (1+q+q^2)...(1+q+...q^{n-1}) \tag{2.2.8}$$

If we introduce a new notation called the $q-$integers where $[k]_q = 1 + q + q^2 + ... + q^{k-1}$, then we can write (5) as $[n]_q! = [n]_q[n-1]_q...[1]_q$. R_n while having interesting S_n-module structure itself, also possesses rich geometric properties.

Theorem 2.2.18. *Borel (1953) $R_n \cong H^*(Fl_n)$ where Fl_n denotes the complete flag variety of n.*

Other than the coinvariant ring R_n, there is actually a family of generalized coinvariant rings that are studied by mathematicians, one of them being the Diagonal Coinvariants DR_n, and we will see how Catalan Combinatorics play an important role in the study of DR_n. Given a polynomial

$f(X_n, Y_n) \in \mathbb{C}[X_n, Y_n]$, S_n acts diagonally by permuting the x and y variables by

$$\sigma \cdot f(x_1, ..., x_n, y_1, ..., y_n) = f(x_{\sigma_1}, ..., x_{\sigma_n}, y_{\sigma_1}, ..., y_{\sigma_n}) \tag{2.2.9}$$

Definition 2.2.19. The Diagonal Coinvariant Ring of S_n is

$$DR_n = \frac{\mathbb{C}[X_n, Y_n]}{I(X_n, Y_n)^+} \tag{2.2.10}$$

where $I(X_n, Y_n)^+$ is the ideal generated by all polynomial $f(X_n, Y_n) \in \mathbb{C}[X_n, Y_n]$ that are invariant under the diagonal action of S_n without constant term.

DR_n also has interesting geometry, as Carlsson and Oblomkov Carlsson and Oblomkov (2018) related DR_n to type A affine Springer fibers using the Lusztig-Smelt paving of these varieties, which led to another proof (the original proof by Haiman) that $dim(DR_n) = (n+1)^{n-1}$, and an explicit monomial basis of DR_n.
Similar to R_n, it is isomorphic to DH_n as S_n-module, which we define below

Definition 2.2.20. The Ring of Diagonal Harmonics is

$$DH_n = \{f(X_n, Y_n) \in \mathbb{C}[X_n, Y_n] : \sum_{i=1}^{n} \partial_{x_i}^r \partial_{y_i}^s f(X_n, Y_n) = 0 \text{ for all } r, s \leq 0, r+s > 0\}$$

The dimension of DR_n becomes much harder to track when we add another set of variables, and the original and first proof of its dimension was by Haiman using geometry of Hilbert Schemes.

Theorem 2.2.21. *Haiman (2002)*

$$dim(DR_n) = (n+1)^{n-1} \tag{2.2.11}$$

which is the same as $|PF(n)|$. The Hilbert and Frobenius series of DR_n had been unsolved for many years too before Calsson and Mellit proved the Shuffle Theorem. First we define what a Hilbert

Series for a bigraded space is.

Definition 2.2.22. Hilbert Series for a bigraded space $V \subseteq \mathbb{C}[X_n, Y_n]$ is defined as

$$Hilb(V_n^{a,b}) = \sum_{a,b \geq 0} q^a t^b \cdot dim(V^{a,b}) \tag{2.2.12}$$

Definition 2.2.23. Frobenius Series for a bigraded space is $W \subseteq \mathbb{C}[X_n, Y_n]$ is defined as

$$Frob(V_n^{a,b}) = \sum_{i,j \geq 0} q^i t^j \sum_{\lambda \vdash n} s_\lambda Mult(S^\lambda, W^{i,j})$$

Similarly we have the relation

$$< Frob(W; q, t), h_{1^n} >= Hilb(V; q, t)$$

Before stating the Shuffle Theorem, we need to introduce two statistics on a parking function (in the labeled Dyck path form).

Definition 2.2.24. The area of a labeled Dyck path $\gamma \in LDyck(n)$ is defined to be the number of complete boxes above the diagonal line and under the path.

Definition 2.2.25. The diagonal inversion, i.e. dinv of a labeled Dyck path $\gamma \in LDyck(n)$ is a pair of cars (s, b), where $s < b$ and either

1. (primary diagonal inversion) s and b are on the same diagonal, and b is on the right of s, or

2. (secondary diagonal inversion) b is in the diagonal above s, and b is on the left of s.

Example 2.2.26. In Figure 2.2.1, we have that $area(\gamma) = 6, dinv(\gamma) = 7$, where the dinv pairs are $(1, 3), (1, 6), (3, 6)$ for primary diagonal inversion, and $(2, 7), (1, 8), (3, 8), (6, 8)$ for secondary diagonal inversion.

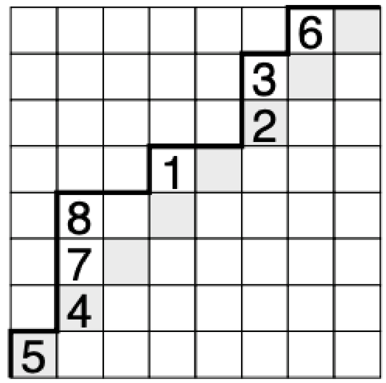


Figure 2.2.1: Labeled Dyck path $\gamma \in LDyck(8)$

Theorem 2.2.27. *(The Shuffle Theorem) Carlsson and Mellit (2018)*

$$Hilb(DR_n) = \sum_{P \in PF(n)} q^{dinv(P)} t^{area(P)} \tag{2.2.13}$$

In fact, the shuffle theorem stated the Frobenius series of DR_n, so the Hilbert Series is a natural corollary, but in this text we would just focus on the Hilbert Series.

After Carlsson and Mellit proved the Shuffle conjecture, Haglund and Loher's derived Schedules Formula also automatically became true. The Schedules Formula offers a somewhat more compact expression, since the sum is over permutations rather than parking functions.

Theorem 2.2.28. *(The Schedules Formula) Haglund and Loehr (2005)*

$$Hilb(DR_n) = \sum_{\sigma \in S_n} t^{maj(\sigma)} \prod_{i=1}^{n} [w_i(\sigma)]_q \tag{2.2.14}$$

Section 2.4 is dedicated for presenting the details of this formula including the defining the two statistics maj and w_i. The Schedules Formula gave rise to another formula for the Hilbert Series:

Theorem 2.2.29. *(W.)*

$$Hilb(DR_n) = \sum_{a,b \geq 0} q^a t^b P(n) \tag{2.2.15}$$

where $P(n)$ is a polynomial in n, and $P(n)$ is given by (permissible is defined in Chapter 4):

$$\sum_{\substack{S\subseteq\{1,2,\ldots,b\}\\ s_1+s_2+\cdots+s_d=b}}\ \sum_{U\subseteq\{1,2,\ldots,s_d\}}\ \sum_{k=0}^{a}\ \sum_{\substack{\tau_l=k+2 \text{ for } l\in U\\ \tau \text{ permissible}}} \Big(\big|D_S\cap W(\tau,U)\big|\Big)\cdot\Big([q^k]\big(\textstyle\prod_{i=1}^{s_d}[\tau_i]_q\big)\Big)\cdot\Big([q^{a-k}]\big([n-s_d]_q!\big)\Big) \tag{2.2.16}$$

Before we dive into more details of the formulas, we should learn about another object of interest in algebraic combinatorics which is the Macdonald Polynomials, and its connection to our subject the Diagonal Harmonics.

2.3. Macdonald Polynomials

Macdonald polynomials, introduced by Ian Macdonald in 1987–1988, revolutionized symmetric function theory by unifying classical polynomial families (Schur, Hall–Littlewood, Jack) through a two-parameter (q,t) framework. Their discovery resolved long-standing problems in algebraic combinatorics and forged unexpected bridges to physics and geometry.

Macdonald Polynomials first emerged from studies of q-analogs of Selberg integrals, prompting Macdonald to construct these polynomials via orthogonalization under a novel scalar product. For more on Macdonald Polynomial's origin story, refer to Haglund (2008). The study of Macdonald polynomials have importance to geometers and physicists, and for us, a primary reason we study it is that it is the 2-parameter generalization of the Schur function, which we introduced in the previous section. Setting the q,t parameter to 0 would give us Schur function. We turn our attention to the modified Macdonald Polynomial $\tilde{H}_\mu(X;q,t)$. Below is the theorem for the existence of Macdonald polynomials that is indexed by partitions, and has 2 extra parameters q and t. In other words, the family of polynomials that satisfy the three triangularity criterions are the Macdonald Polynomials. Before we can introduce the formula, we need to introduce the dominance ordering and the plethystic substitution.

Definition 2.3.1. The dominance order (or majorization order) on partitions is a partial order defined as follows:

Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ and $\mu=(\mu_1,\mu_2,\ldots,\mu_m)$ be two partitions of n (i.e., they both sum to n).

We say that λ **dominates** μ (denoted $\lambda \trianglerighteq \mu$) if:

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i, \quad \text{for all } k \geq 1$$

where we assume that missing parts in a partition are treated as 0 (i.e., extend partitions with trailing zeros if they have different lengths).

Definition 2.3.2. Given a symmetric function f and a formal expression X, the **plethystic substitution** $f[X]$ is defined by substituting the power sum symmetric function p_k as follows:

$$p_k[X] = \sum_{x \in X} x^k.$$

Here, X may represent a sum of variables, an infinite series, or an expression involving other symmetric functions.

Example 2.3.3. Let $X = x_1+x_2+x_3$. Then the plethystic substitution of the power sum symmetric function is:

$$p_2[X] = x_1^2 + x_2^2 + x_3^2.$$

If $X = 1 + q + q^2 + \dots$, then

$$p_k[X] = \frac{1}{1 - q^k}.$$

Similarly, for a Schur function s_λ, we have:

$$s_\lambda[1 + q] = s_\lambda(1, q).$$

Theorem 2.3.4. *The following three conditions uniquely determine a family* $\tilde{H}_\mu(X; q, t)$ *of sym-*

metric functions:

$$\tilde{H}_\mu[X(q-1);q,t] = \sum_{\rho\leq\mu'} c_{\rho,\mu}(q,t)m_\rho(X) \tag{2.3.1}$$

$$\tilde{H}_\mu[X(t-1);q,t] = \sum_{\rho\leq\mu} d_{\rho,\mu}(q,t)m_\rho(X) \tag{2.3.2}$$

$$\tilde{H}_\mu(X;q,t)|_{x_1^n} = 1 \tag{2.3.3}$$

One can regard the previous theorem as the definition of the modified Macdonald Polynomials, but it is implicitly defined, and so one of the major open problems regarding Macdonald Polynomials was finding an explicit combinatorial formula for it. The main task was to find appropriate statistics of labelled partitions so that Macdonald Polynomials can be expressed as a sum of $m_\rho(X)q^{stat1}t^{stat2}$. Fortunately, this was solved by Haiman, Haglund and Loehr and now it is known as the HHL formula.

Theorem 2.3.5. *(HHL Formula)*

$$\tilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}^+} x^\sigma t^{maj(\sigma,\mu)} q^{inv(\sigma,\mu)}$$

where maj and inv are statistics defined on labelled partitions.

More details of the HHL formula can be found at Haglund (2004). Another major open problems surrounding Macdonald Polynomials is its Schur positivity, a fact that beautifully ties Macdonald Polynomials to Diagonal Harmonics. Garsia and Haiman (1993) defined for each partition λ,

$$\Delta_\lambda = det||x_i^{p_j} y_i^{q_j}||_{i,j=1,\ldots,n}$$

An example is shown in Figure 2.3.1 for calculating $\Delta_{3,2}$. Denote the linear span of all the partial derivatives as $L[\partial_x\partial_y\Delta_\lambda]$. They also realized that if the dimension of $L[\partial_x\partial_y\Delta_\lambda]$ is $n!$, then we would

(0,1)	(1,1)	
(0,0)	(1,0)	(2,0)

$$\Delta_{3,2} = \det\left(\begin{bmatrix} x_1^0y_1^0 & x_1^1y_1^0 & x_1^2y_1^0 & x_1^0y_1^1 & x_1^1y_1^1 \\ x_2^0y_2^0 & x_2^1y_2^0 & x_2^2y_2^0 & x_2^0y_2^1 & x_2^1y_2^1 \\ x_3^0y_3^0 & x_3^1y_3^0 & x_3^2y_3^0 & x_3^0y_3^1 & x_3^1y_3^1 \\ x_4^0y_4^0 & x_4^1y_4^0 & x_4^2y_4^0 & x_4^0y_4^1 & x_4^1y_4^1 \\ x_5^0y_5^0 & x_5^1y_5^0 & x_5^2y_5^0 & x_5^0y_5^1 & x_5^1y_5^1 \end{bmatrix}\right)$$

Figure 2.3.1: Computing $\Delta_{3,2}$.

have

$$Frob(L[\partial_x\partial_y\Delta_\lambda]) = \tilde{H}_\mu(X;q,t)$$

which by the definition of Frobenius series, is Schur positive, thus proving the Schur positivity of the Macdonald Polynomials. Therefore the Macdonald Schur Positivity conjecture would be proven if $dim(L[\partial_x\partial_y\Delta_\lambda]) = n!$, which is the very famous $n!$ conjecture, is proven. The $n!$ conjecture was later proven by Haiman, but the proof involves Hilberst Schemes and technqiues from algebraic geometry, so although the positivity is proven, a combinatorial formula for the Schur coefficient $\tilde{H}_\mu(X;q,t)|_{s_\lambda}$ is still open. There are some special cases that have been solved, for example when μ is a hook shape, i.e. $\mu_2 = \mu_3 = \cdots = 1$, and also the case $\mu_i \leq 2$ for $i = 2, 3\ldots$ by Assaf (2018) , the 2 column case where $\mu_i \leq 2$ for all i by Zabrocki (1998), but the general case is still open. There is also a very interesting conjecture called Butler's conjecture, which would have given a "recurrence relation" on the Schur coefficient, in which a special case from a hook shape to an augmented hook shape is proven by Vetter (2024) and Kim et al. (2022), but the general case is still open.

Notice that the definition of $L[\partial_x\partial_y\Delta_\lambda]$, as the linear span of partial derivatives, closely resembles the definition of Diagonal Harmonics, which is the solution space to the system given by the sum of all partial derivatives. In fact, the former is a subspace of the latter.

Theorem 2.3.6. *If $\mu \vdash n$, then $L[\partial_x\partial_y\Delta_\mu]$ is a subspace of DH_n.*

The proof just involves checking the sums and determinants, and can be found at Hicks (2019). Since $L[\partial_x\partial_y\Delta_\mu]$ is a subspace of DH_n, studying one provides insight into the other. This connection allows us to analyze DH_n through the structure of $L[\partial_x\partial_y\Delta_\mu]$ and vice versa, making their study inherently intertwined.

2.4. Schedule's Formula

In this section, we present the proof of Schedule's Formula credited to Haglund and Loehr (2005). Here we recall the Schedules Formula. First we introduce the notion of q-integers, that $[k]_q = 1 + q + ... + q^{k-1}$. Note the at if we let $q = 1$ then we have exactly k.

Theorem 2.4.1. *(The Schedules Formula) Haglund and Loehr (2005)*

$$\sum_{P \in PF(n)} q^{dinv(P)} t^{area(P)} = \sum_{\sigma \in S_n} t^{maj(\sigma)} \prod_{i=1}^{n} [w_i(\sigma)]_q \tag{2.4.1}$$

Here we finally introduce the major index statistics and the w-sequence of permutations.

Definition 2.4.2. For $\sigma \in S_n$, let the 1-st run be defined to be the first increasing subsequence, i-th run be defined to be the i-th increasing subsequence.

- $w_i(\sigma) = |\text{entries } \sigma_j \text{ in the same run as } \sigma_i \text{ and } \sigma_j > \sigma_i| + |\text{entries } \sigma_k \text{ in the next run of } \sigma_i \text{ and } \sigma_k < \sigma_i|$

While calculating w_i, we adjoint 0 at the end of σ. Denote the sequence $w_i(\sigma)$ as $wseq(\sigma)$.

Example 2.4.3.

$$\sigma = 4.25.138.679 \in S_9 \tag{2.4.2}$$

$$wseq(\sigma) = 1.22.212.321 \tag{2.4.3}$$

$$maj(\sigma) = 1 + 3 + 6 = 10 \tag{2.4.4}$$

The dots denote the descents set $S = \{1, 3, 6\}$. Putting everything together, σ gives $t^{10}[2]^4 \cdot [3] \cdot [2] \cdot [1]$. The term "schedules" comes from the idea that the schedule tells you the order (or schedule) of cars being inserted to build up a parking function. We present the "schedules" in the following paragraph.

Insertion Schedule: Let Γ be a parking function. We call $y = x$ as the 0-th diagonal, and $y = x+i$

as the i-th diagonal for $i = 1, 2, 3, \ldots$. Let c be a car not present in Γ yet, and let k be the diagonal such that $k+1$th diagonal is empty, and the k-th diagonal contains no car smaller than c.

- Let $s < c$ be a car in the $k-1$st diagonal of Γ. Move all cars which are in a higher row than s up and to the right once. Place car c directly above car s
- Let $b > c$ be a car in the k-th diagonal of Γ. Move all cars which are in a higher row than b up and to the right once. Place c directly above and to the right of b.
- If $k = 0$, move all cars up and to the right once. Place c to the lower left corner.

Example 2.4.4. In Figure 2.4.1, we present 3 ways of inserting 5 into the parking function using the proposed insertion schedule. The first parking function uses the first algorithm, and $s = 2$. The second uses the second algorithm, and $b = 8$. The third uses the first algorithm, and $s = 4$.

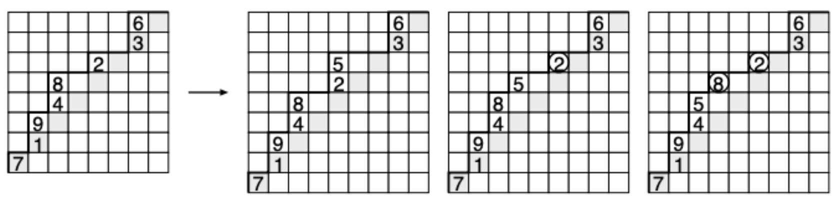


Figure 2.4.1: Inserting a 5 into the 2-diagonal. The cars creating new diagonal inversions with 5 are circled.

Denote the set of all new parking functions as $Insert(\Gamma, c, k)$, and $d_c(\Gamma)$ as the parking function obtained by deleting c, and moving everything that was previously to the right and above c, down and below one step so it is still a parking function. Then we have the following proposition.

Proposition 2.4.5. *Let $\Gamma \in PF(n)$. There is exactly one choice of γ, c and k such that $\Gamma \in Insert(\gamma, c, k)$. In fact k is exactly the highest number of nonempty diagonal of Γ, c is the smallest car in the k-diagonal of Γ, and $\gamma = d_c(\Gamma)$.*

The insertion algorithm is designed such that the proposition holds. We want to stress that there is only one way to build up a parking function. Other than this, this insertion algorithm does what we want with the two parking function statistics area and dinv too.

Lemma 2.4.6. *For any parking function that follows the Insertion Schedule, we have*

$$\sum_{\gamma \in Insert(\Gamma,c,k)} t^{area(\gamma)} q^{dinv(\gamma)} = t^{area(\Gamma)+k} q^{dinv(\Gamma)} [|Insert(\Gamma, c, k)|]$$

We get a proof of the lemma with the help from our example.

Example 2.4.7. For our example in Figure 2.4.1, every one of $\gamma \in Insert(\Gamma, c, k)$ has 2 more areas, because 5 was added on the 2nd diagonal, thus the change in t power.
For the q power, notice that the only new potential dinv are the s and b in the insertion schedule. c would potentially create new dinv with s in the previous diagonal, and b in the same diagonal. In both cases, a dinv is created when c is further to the left. Due to this reasoning, inserting c on the rightmost possible position gives no new dinv, just like in our example, the first γ has the same dinv as our original Γ. As 5 moves to the left to the permitted positions, it creates a new dinv with the car it passed over, whether it was through the first or second insertion schedule, and thus the appearance of $[3] = q^0 + q^1 + q^2$ in our example, and in general $[|Insert(\Gamma, c, k)|]$. Therefore we would have γ have $dinv(\Gamma), dinv(\Gamma) + 1, dinv(\Gamma) + 2, ...dinv(\Gamma) + |Insert(\Gamma, c, k)| - 1$ as claimed.

We need a few more observations before we can get to our final theorem. The end goal is that we want to get a sum in permutations instead of parking functions, while still accounting for dinv and area. We do this by building trees of parking function from a permutation. Let $\tau \in S_n$, we will build a tree of parking function of τ as following. Start with the τ_n which will be a parking function of size 1, and then proceed inductively as such: at the i-th stage, let $c = \tau_{n-i}$, and k be so that τ_{n-i} is in the k-th increasing subsequence (we will refer to it as run in the text) counting from the right. We build a tree of function by adding a child $Insert(\Gamma, c, k)$ to the current parking function Γ. An example of tree of parking functions built from $\tau = 2314$ is shown in Figure 2.4.2. Notice that all of the parking functions generated, has their 0th diagonal being the last run, first diagonal being the second to last run and so on. We denote the set of parking functions generated by a permutation (schedule) τ to be $PF(\tau)$, thus connecting permutation and parking functions.
Now we ask the question, what happens to t and q while we sum over $PF(\tau)$? If we sum up the

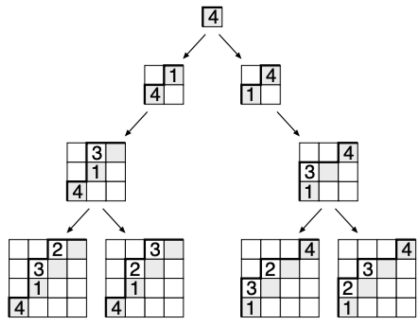


Figure 2.4.2: The tree of parking functions built from $\tau = 2314$.

trees in our examples, we get precisely $t^2[2]_q$. We know exactly what happens when we just insert one τ_i to the current parking function by Lemma 2.4.6. Notice that all of the parking functions in $PF(\tau)$ has area exactly the same as $maj(\tau)$, since every time a descent appears, all of the elements afterwards are moved up one diagonal, which is equivalent to what happens to maj of a permutation when we add a descent in the beginning.

Now let's track the q statistics. Given a permutation $\tau \in S_n$, we start to append 0 at the end of τ, this accounts for the special case $k = 0$ during the insertion schedule. Then the schedule number $w_\tau(c)$ is the number of entries in the same run that is bigger than it plus the number of entries in the next run that are smaller than it, which correspond to the number of b and c's in the Insertion Schedule. Now we finally get

Theorem 2.4.8. *For every permutation τ,*

$$\sum_{\Gamma \in PF(\tau)} t^{area(\Gamma)} q^{dinv(\Gamma)} = t^{maj(\tau)} \prod_c [w_\tau(c)]_q \tag{2.4.5}$$

Theorem 2.4.1 is thus obtained by summing over all the permutations $\sigma \in S_n$, thus getting all of the parking functions in their trees. The proof of the main theorem of the thesis has a similar taste to the proof of the Schedules Fromula.

For now, we have finished telling the combinatorics side of the story. Now we move on to tell the algebraic side of the story.

CHAPTER 3

REPRESENTATION STABILITY

3.1. FI Modules

Church and Farb (2013) discovered a significant property of sequences of representations that ultimately enabled the development of this thesis. They demonstrated that certain sequences of representations exhibit stabilization when considered sufficiently far along the sequence. This chapter introduces the underlying representation theory behind this phenomenon, culminating in the proof of the following theorem:

Theorem 3.1.1. *The coefficient of $q^a t^b$ of $Hilb(DH_n)$, i.e. the dimension of $DH_n^{a,b}$ is eventually a polynomial of n.*

We start our introduction with a motivating example of representation stability.

Definition 3.1.2. The n-th Configuration Space of $\mathbb{C}$ is

$$Conf_n(\mathbb{C}) = \{(z_1, ..., z_n) \in \mathbb{C}^n | \forall i < j, z_i \neq z_j\} \tag{3.1.1}$$

There is a natrual action of S_n on the Configuration space by permuting the indices, and the action also descends to its cohomology $H^i\big(Conf_n(\mathbb{C}); \mathbb{C}\big)$.

Recall that irreducible representations of S_n are indexed by partitions. Denote the irreducible representation indexed by $(\lambda_1, \lambda_2, ..., \lambda_r)$ as $V(\lambda_1, ..., \lambda_r)$. Farb and Church's calculation shows that

$$H^1\big(Conf_2(\mathbb{C})\big) = V(2) \oplus V(1,1) \tag{3.1.2}$$

$$H^1\big(Conf_3(\mathbb{C})\big) = V(3) \oplus V(2,1) \tag{3.1.3}$$

$$H^1\big(Conf_4(\mathbb{C})\big) = V(4) \oplus V(3,1) \oplus V(2,2) \tag{3.1.4}$$

$$H^1\big(Conf_5(\mathbb{C})\big) = V(5) \oplus V(4,1) \oplus V(3,2) \tag{3.1.5}$$

$$H^1\big(Conf_6(\mathbb{C})\big) = V(6) \oplus V(5,1) \oplus V(4,2) \tag{3.1.6}$$

Notice that the irreducible decomposition "stabilizes" starting from 4. To state this more rigorously, let $V(a_1, ..., a_r)$ to be the irreducible S_n-representation corresponding to the partition $\big((n - \sum_{i=1}^{r} a_i), a_1, ..., a_r\big)$, where $a_1 \geq a_2 \geq ... \geq a_n$. Note that this is a partition only when $n - (\sum_{i=1}^{r} a_i) \geq a_1$.

Definition 3.1.3. Church and Farb (2013). A sequence of representations V_n are representation (multiplicity) stable if the decomposition of V_n into S_n-representations as

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V_\lambda \tag{3.1.7}$$

stabilizes, i.e. the coefficients $c_{\lambda,n}$ are eventually independent of n.

Now we introduce the properties needed for representation stability. Let FI be the category of objects being finite sets $\boldsymbol{n} := \{1, \ldots, n\}$, and morphism being injections $\boldsymbol{m} \hookrightarrow \boldsymbol{n}$.

Definition 3.1.4. An FI-module over a commutative ring k is a functor V from FI to the category of k-modules. Denote the k-module $V(n)$ by V_n.

Example 3.1.5.

1. $V_n = H^i(Conf_n(M); \mathbb{Q})$ is an FI-module, where $Conf_n(M) =$ Configuration space of n distinct ordered points on a connected, oriented manifold M.
2. $DH_n^{a,b}$ is an FI-module with the injection map being the canonical inclusion map.
3. $R_J^{(r)}(n)$ is a co-FI-module, where $J = (j_1, \ldots, j_r)$, $R^{(r)}(n) = \oplus R_J^{(r)}(n) =$ r-diagonal coinvariant algebra.

Our diagonal coinvariant algebra is actually a special case of r-diagonal coinvariant algebra where $r = 2$.

Definition 3.1.6. Let K be a field of characteristic 0, and fix $r \geq 1$. For $n \geq 0$, consider the algebra of polynomials

$$K[X^{(r)}(n)] = K[x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}]$$

Then S_n acts on this algebra diagonally by

$$\sigma \cdot x_j^{(i)} := \sigma_{\sigma(j)}^{(i)}$$

It has a natural r-fold multi-grading, where a monomial has multi-grading $J = (j_1, \ldots, j_r)$ if its total degree in the variables $x_1^{(k)}, \ldots, x_n^{(k)}$ is j_k, so $R^{(r)}(n) = \oplus_J R_J^{(r)}(n)$.

Definition 3.1.7. An FI-module is finitely generated if there is a finite set $v_1, \ldots, v_k$ of elements in V_i so that $span(v_1, \ldots, v_k) = V$.

If V is finitely generated, then it enjoys desirable properties that will be useful later in the text, particularly in understanding the dimension of the space.

Definition 3.1.8. For each $i \geq 1$ and $n \geq 0$, let $X_i : S_n \to \mathbb{N}$ be the class function defined by

$$X_i(\sigma) = \text{number of } i\text{-cycles in the cycle decomposition of } \sigma$$

Polynomials in the variables X_i are called character polynomials.

Since the vector space of class functions on S_n is spanned by character polynomials, the character can always be described by a polynomial. Below are two examples of characters written in character polynomials.

Example 3.1.9.

1. $V \simeq \mathbb{Q}^n$, the standard permutation representation of S_n, then $\chi_V(\sigma)$ is the number of fixed points of σ, so $\chi_V = X_1$, the number of 1-cycles in the permutation σ.

2. If $W = \wedge^2 V$, then $\chi_W = \binom{X_1}{2} - X_2$, since $\sigma \in S_n$ fixes the basis elements $x_i \wedge x_j$ for which the cycle decomposition of σ contains the pair $(i)(j)$, and negates those that contains the 2-cycle (i, j).

Notice that in the previous two examples, one character polynomial describes the entire family of characters for all $n \geq 1$, so there is a natural question to ask: is there "the" character polynomial that realizes a sequence of characters χ_n of S_n?

Definition 3.1.10. A sequence χ_n of characters of S_n is eventually polynomial if there exists r, N and a polynomial character polynomial $P(X_1, \ldots, X_r)$ such that

$$\chi_n(\sigma) = P(X_1, \ldots, X_r)(\sigma)$$

for all $n \geq N$ and all $\sigma \in S_n$. The degree of the character polynomial is defined by setting $deg(X_i) = i$.

The central theorem we want to use from Farb and Church's result, is the following considering field of characteristic 0:

Theorem 3.1.11. *Let V be an FI-module over a field of characteristic 0. If V is finitely generated, then the sequence of characters χ_{V_n} is eventually polynomial. In particular, $dimV_n$ is eventually polynomial.*

The dimension being polynomial follows from the fact that

$$dimV_n = \chi_{V_n}(id) = P(n, 0, \ldots, 0) \tag{3.1.8}$$

Farb and Church showed that the characters of r-covariant algebras satisfy the criteria of Theorem 3.1.11, as the co-FI structure they admit also ensures the necessary conditions are met. Thus, we have:

Theorem 3.1.12. *For any fixed $r \geq 1$ and $J = (j_1, \ldots, j_r)$, the characters $\chi_{R_J^{(r)}}$ are eventually polynomial in n of degree at most $|J|$. In particular there exists a polynomial $P_J^{(r)}(n)$ of degree at most $|J|$ such that*

$$dim(R_J^{(r)}(n)) = P_J^{(r)}(n) \text{ for all } n >> 0 \tag{3.1.9}$$

Letting $r = 2$ in the previous theorem brings us to the central result that is the cornerstone of our thesis:

Theorem 3.1.13. *The characters of* $\chi_{DR_n^{a,b}}$ *are eventually polynomial in* n *of degree at most* $a+b$. *In particular there exists a polynomial* $P_{a,b}(n)$ *of degree at most* $a+b$ *such that*

$$dim(DR_n^{a,b}) = P_{a,b}(n) \tag{3.1.10}$$

However, Theorem 3.1.13 establishes only the existence of the polynomial $P_{a,b}(n)$, without providing an explicit formula. The primary goal of this thesis is to make this polynomial explicit, which will be the focus of Chapter 4.

Remark 3.1.14. Notice that Theorem 3.1.12 establishes that many other r-coinvariant algebras also exhibit representation stability. This suggests it would be interesting to investigate their bigraded subdimensions polynomials. However, at present, even the general (total) dimensions of these algebras have not been conjectured. There remain many open problems in this area that are ripe for exploration.

3.2. Stability Range

The *stability range* is defined as the smallest integer N such that for all $n \geq N$, the dimension $\dim V_n$ stabilizes. In Church and Farb (2013), they studied the stability of $DR_n^{a,b}$, and although they did not state an explicit bound, their results on FI-modules imply a stability range of at most $2(a+b)$ for $DR_n^{a,b}$. Moreover, they proved that the dimension becomes a polynomial in n of degree at most $a+b$ once stabilization occurs.

While Church and Farb (2013) gave a proof of the latter fact, there is still gap in existing theory (which we will point out later) to prove that one can find an explicit stability bound that is $2(a+b)$ just using FI-module theory.

Conjecture 3.2.1. $DR_n^{a,b}$ and $DH_n^{a,b}$ starts to stabilizes at the latest from $n = 2(a+b)$

Theorem 3.2.2. *The dimension of* $DH_n^{a,b}$ *is a polynomial in* n *of degree at most* $a+b$ *once they stabilizes.*

We will be quoting the theorem numbers from the Church and Farb (2013) paper in the rest of the section. First, Proposition 3.3.3 states that for $DH_n^{a,b}$, we have stability range

$$N \geq Weight(DH_n^{a,b}) + stab\text{-}deg(DH_n^{a,b}) \tag{3.2.1}$$

where $stab\text{-}deg(DH_n^{a,b})$ denotes the stability degree of $DH_n^{a,b}$. On page 37 "Proof of Theorem 1.11" of the paper Church and Farb (2013), they proved that both the $Weight(DH_n^{a,b})$ and the degree of dimension polynomial to be bounded by $a+b$. Therefore to find a stability range, we just need to find the stability degree.

Due to a gap in the existing theory, we state the following conjecture and proceed under the assumption that it holds for the remainder of this section.

Conjecture 3.2.3. $stab\text{-}deg(DH_n^{a,b}) \leq stab\text{-}deg(\mathbb{C}[X_n, Y_n]^{a,b})$.

Now we investigate $stab\text{-}deg(\mathbb{C}[X_n, Y_n]^{a,b})$. Notice that we have isomorphism $\mathbb{C}[X_n, Y_n]^{a,b} \cong Sym^a(\mathbb{C}^n) \otimes Sym^b(\mathbb{C}^n)$. Since $\mathbb{C}^n$ has weight of 1 and $Sym^a(\mathbb{C}^n) \cong \mathbb{S}^\lambda(\mathbb{C}^n)$ where $\lambda = (a)$, by Proposition 3.4.3, $Sym^a(\mathbb{C}^n) \otimes Sym^b(\mathbb{C}^n)$ has weight $a+b$. By Corollary 4.1.8, we know that for an $FI\#$-module, stability degree is bounded above by the weight. Therefore we only need to prove that $Sym^a(\mathbb{C}^n) \otimes Sym^b(\mathbb{C}^n)$ is an $FI\#$-module to show that $stab\text{-}deg(DH_n^{a,b}) \leq a+b$.

To see that $Sym^a(\mathbb{C}^n) \otimes Sym^b(\mathbb{C}^n)$ is an $FI\#$-module, we know that $\mathbb{C}^n$ is an $FI\#$-module by Example 4.1.2. Since Sym is a Schur functor, by the last paragraph on page 31 of the paper, we know $Sym^a(\mathbb{C}^n)$ is an $FI\#$-module and thus $\mathbb{C}[X_n, Y_n]^{a,b} \cong Sym^a(\mathbb{C}^n) \otimes Sym^b(\mathbb{C}^n)$ is an $FI\#$-module, so we do have $stab\text{-}deg(\mathbb{C}[X_n, Y_n]^{a,b}) \leq a+b$.

Therefore by (3.2.4), we know that $stab\text{-}deg(DH_n^{a,b}) \leq a+b$, and by (3.2.1), we would have a stability range $2(a+b)$ assuming that Conjecture 3.2.3 is true.

Remark 3.2.4. Even if the conjecture is proven, we still wouldn't claim that $2(a+b)$ is a sharp bound, and indeed, we find in Section 4.3 a new bound to be actually $a+b$. In fact, using only tools from FI-module theory typically does not yield sharp bounds. See Hersh and Reiner (2017) for an example where a sharp bound for another S_n-representation—the cohomology of configuration spaces—is obtained using symmetric function theory.

Remark 3.2.5. Once the character polynomial stabilizes, the dimension of the representation immediately stabilizes as well. However, there is no known theorem guaranteeing the converse. Indeed, one can construct toy counterexamples—such as a sequence of sign representations. Nevertheless, in sequences of representations arising "in nature," such pathological behavior appears to be rare. While we do not claim that the stabilization of dimension and character polynomial always occur simultaneously—since we lack a proof—empirical evidence suggests that they usually do.

CHAPTER 4

PROOF OF MAIN THEOREM

4.1. Main formula

Definition 4.1.1. Let $\tau \in \mathbb{N}^n, U \subseteq \{1, 2, ..., s_d\}$, S be the descent set such that $\sum_{i=1}^{d} s_i = b$, define

$$W(\tau) = \{\sigma \in S_n | \, wseq(\sigma) = \tau\}$$

$$D_S = \{\sigma \in S_n | \, Desc(\sigma) = S\}$$

$$W(\tau, U) = \{\sigma \in S_n | \, w_i(\sigma) = \tau_i \text{ for } i \notin U, w_i(\sigma) \geq \tau_i \text{ for } i \in U\}$$

Theorem 4.1.2. *(W.) The dimension of $DR_n^{a,b}$ is given by*

$$\sum_{\substack{S\subseteq\{1,2,\ldots,b\} \text{ s.t.} \\ s_1+s_2+\ldots+s_d=b}} \sum_{U\subseteq\{1,2,\ldots,s_d\}} \sum_{k=0}^{a} \sum_{\substack{\tau_l=k+2 \text{ for } l\in U \\ \tau \text{ permissible}}} \left(\left|D_S \bigcap W(\tau,U)\right|\right) \cdot \left([q^k]\left(\prod_{i=1}^{s_d}[\tau_i]_q\right)\right) \cdot \left([q^{a-k}]\left([n-s_d]_q!\right)\right) \tag{4.1.1}$$

There are two key observations of the w-sequence of $\sigma \in S_n$. Let the descent set $S = \{s_1, s_2, ..., s_d\}$ where $s_1 + ...s_d = b$ be given. First, $w_{s_d+i+1} = n - s_d - i$, i.e. the tail of the w-sequence is

$$\prod_{i=s_d+1}^{n} [w_i] = [n - s_d]_q! \tag{4.1.2}$$

We use the following result in our third parenthesis of (4.1.1).

Theorem 4.1.3. *Knuth (1997)*

$$[q^k]\left([n]_q!\right) = \binom{n+k-1}{k} + \sum_{j=1}^{\infty}(-1)^j\binom{n+k-u_j-1}{k-u_j} + \sum_{j=1}^{\infty}(-1)^j\binom{n+k-u_j-j-1}{k-u_j-j} \tag{4.1.3}$$

where $u_j = \frac{j(3j-1)}{2}$ the pentagonal numbers, and $k \leq n$.

Margolius (2001) noted that $\binom{a}{b} = 0$ when $b < 0$, so we actually have a finite sum, and there are exactly $\lfloor\sqrt{\frac{1}{36} + \frac{2k}{3}}\rfloor$ terms together in both summands. Therefore $[q^k]\Big([n]!\Big)$ is indeed a polynomial in n for $k \leq n$.

Given $k \leq n$ and $\sigma \in S_n$, define a truncation map as follow:

$$\mathrm{tr}(y) = \begin{cases} k+1, & \text{if } y > k+1, \\ y, & \text{otherwise.} \end{cases}$$

The second observation is that

$$[q^k]\Big(\prod_{i=1}^{s_d}[w_i(\sigma)]_q\Big) = [q^k]\Big(\prod_{i=1}^{s_d}[tr\big(w_i(\sigma)\big)]_q\Big) \tag{4.1.4}$$

With this observation in mind, we only need to worry about the counting problem being: Fix a descent set S, given a w-sequence τ with maximum τ_i being $k+1$, how many $\sigma \in S_n$ has a w-sequence that truncate to τ?

This is exactly the first parenthesis in (4.1.1) which is $\Big(\big|D_S \bigcap W(\tau, U)\big|\Big)$. We claim this is a polynomial in n, and we will state and prove this in **Theorem 4.1.10** later in the text. Assuming this theorem, we are ready to give a proof of our main result **Theorem 4.1.2**

Proof of Theorem 4.1.2. First, notice that the first parenthesis $\Big(\big|D_S \bigcap W(\tau, U)\big|\Big)$ is a polynomial in n by **Theorem 4.1.10**, which we assume now and prove later. Second parenthesis $\Big([q^k]\big(\prod_{i=1}^{s_d}[\tau_i]_q\big)\Big)$ is a constant, and third parenthesis $\Big([q^{a-k}]\big([n-s_d]_q!\big)\Big)$ is a polynomial in n by **Theorem 4.1.3**. Now we analyze the 4 summations. First, given a fixed b we want to sum over all possible descent set that gives $s_1 + s_2 + ...s_d = b$. Given a descent set, we want to sum over all the possible $\tau \in \mathbb{N}^n$ such that there exists $\sigma \in S_n$ with $wseq(\sigma) = \tau$. The number of such τ only depends on a and the descent set S, since we noticed in our first observation that the tail of the w-sequence of σ is always $(n - s_d, n - s_d - 1, ..., 2, 1)$. The fact that it also depends on the descent set S comes from the definition of permissible, which we delay until after this proof, as permissible is another restriction on τ that depends on S and ensures $\Big(\big|D_S \bigcap W(\tau, U)\big|\Big) \neq 0$, thus the second and fourth summation. In addition, for the fourth summation we are summing from $\tau_l = k+2$ to avoid double counting: for example, $W\big((1,1,4,5,4,3,2,1),\phi\big)$ is already counted once for $k=3, U=\phi$, so for $U=\{3\}$, we want to calculate $W\big((1,1,5,3,2,1),\{3\}\big)$ to avoid counting $W\big((1,1,4,5,4,3,2,1),\phi\big)$ again. Finally, observe that

$$[q^a]\big(\prod_{i=1}^{n}[w_i(\sigma)]_q\big) = \sum_{k=0}^{a}\bigg([q^k]\big(\prod_{i=1}^{s_d}[w_i(\sigma)]_q\big)\bigg)\cdot\bigg([q^{a-k}]\big(\prod_{i=s_d+1}^{n}[w_i(\sigma)]_q\big)\bigg) \tag{4.1.5}$$

Therefore we have the third summation. Notice none of the summations depend on n, as they only depend on a and b, so we have that (4.1.1) is indeed a polynomial in n that calculates the coefficient of $q^a t^b$ in the Schedule Formula. $\square$

Now we come back to the definition of permissible.

Definition 4.1.4. Let the descent set S and $i \in \{1,2,\ldots,n\}$ be given, and t be so that $s_t \le i < s_{t+1}$. Define the $\max_S w_i$ as

$$\max_S w_i := \begin{cases} (s_{t+1}-i)+(s_{t+2}-s_{t+1}), & \text{if } t < d-1, \\ (s_d - i)+(n-s_d), & \text{if } t = d-1, \\ n - s_d, & \text{if } t = d. \end{cases}$$

Similarly define $\min_S w_i$ as

$$\min_S w_i := \begin{cases} s_{t+1}-i, & \text{if } i \text{ is not a descent position,} \\ 1, & \text{if } i = s_t \text{ for some } t. \end{cases}$$

Note that given σ with descent set S, the maximum and minimum number of $w_i(\sigma)$ are exactly $\max_S w_i$ and $\min_S w_i$, hence the definition.

Definition 4.1.5. Let the descent set S be given, let $\tau \in \mathbb{N}^n$. τ is **permissible** if $\min_S w_i \le \tau_i \le \max_S w_i$, and for elements j in the i-th run $\{\sigma_{s_i}, \sigma_{s_i+1}, ..., \sigma_{s_{i+1}-1}\}$ we have $w_j \le w_{j+1}+1$.

Before we prove our claim about permissibility, we need to identify where n can be, given $\sigma \in S_n$ and $\sigma \in D_S \cap W(\tau, U)$.

Definition 4.1.6. Let the descent set S and U be given, $w \in \mathbb{N}^n$, A maximal spot is σ_k that satisfies one of the following conditions

1. $k = n$ and $w_{s_d}(\sigma) < \max_S w_{s_d}$

2. $k \in \{s_1, ..., s_d\}$ and $k = s_m$, $s_{m-1} \neq s_m - 1$, $w_{s_{m-1}}(\sigma) < \max_S w_{s_{m-1}}(\sigma)$, and $w_{s_m}(\sigma) = \max_S w_{s_m}(\sigma)$.

3. $k \in \{s_1, ..., s_d\}$ and $k = s_m$, $s_{m-1} \neq s_m - 1$, $w_{s_{m-1}}(\sigma) < \max_S w_{s_{m-1}}(\sigma)$, and $k \in U$.

Proposition 4.1.7. *If $D_S \bigcap W(\tau) \neq \phi$, then for $\sigma \in D_s \bigcap W(\tau)$, n can only be at the maximal spots, and there is always at least one maximal spot.*

Proof. First of all, n can only be at the descent positions and the last position since it is bigger than anything else in the permutation. Now separate into 3 cases

1. $k = n$

   If $w_{s_d}(\sigma) = max\ w_{s_d}$, then σ_{s_d} is greater than everything in the next run and thus greater than n, which can't happen.

2. $k \in \{s_1, ..., s_d\}$ and $k = s_m$

   If $s_{m-1} = s_m - 1$, then $s_m = n < s_{m-1}$, which can't happen.

   If $w_{s_{m-1}}(\sigma) = \max_S w_{s_{m-1}}(\sigma)$, then $w_{s_{m-1}}(\sigma) > w_{s_m} = n$, which can't happen.

3. $k \in \{s_1, ..., s_d\}$, $k = s_m$ and $k \in U$

   Note since $k \in U$, we have that $\tau_k \geq v$ for some number v, so naturally $\tau_k = max_S w_k(\sigma)$ is included in this case.

Now we prove the existence of a maximal spot. First, if $w_{s_d} < max\ w_{s_d}$, then n is a maximal spot, so we can have $\sigma_n = n$. If not, then we have $w_{s_d} = max\ w_{s_d}$, and if $w_{s_{d-1}} \neq \max_S w_{s_{d-1}}(\sigma)$, then s_d is a maximal spot. If however $w_{s_{d-1}} \neq \max_S w_{s_{d-1}}(\sigma)$, then find the smallest v such that $w_{s_{v-1}} \neq \max_S w_{s_{v-1}}(\sigma)$. If we get to $v = 2$ and still no v satisfies , then s_1 is a maximal spot. $\square$

Proposition 4.1.8. *Let the descent set S be given, and let $\tau \in \mathbb{N}^n$ be permissible, then $D_S \cap W(\tau) \neq \phi$. The converse of the statement is also valid.*

Proof. Observe that any $\sigma \in D_S \bigcap W(\tau)$ has w_i values as described, so the converse of the statement is true.

Now assume τ is permissible. We can build up a permutation with the desired descent set and the given τ by finding the maximal spot at every stage.

To build this permutation, start by inserting n. $w_i = n$ can only be inserted at a maximal spot, and we know that maximal spots exist by **Proposition 4.1.7**, so we know where n would be inserted. We build up the rest of the permutation in a similar fashion inductively. Now we insert y where $1 \leq y < n$. At this stage we modify our definition of maximal spot to define the maximal spot while trying to insert y. We call "a maximal spot while trying to insert y" as a y-maximal spot. Denote the rightmost available spot in each run $\sigma_{s_i}, ..., \sigma_{s_{i+1}-1}$ as r_i. A y-maximal spot is defined as follow:

1. Let the last available spot be a. If $a > r_{d-1}$ and $w_{r_{d-1}}(\sigma) < a - r_{d-1}$, then a is a y-maximal spot.

2. We know how r_{i-1}, r_i and r_{i+1} compare to each other from their respective w_i values, so we know there exists at least one i where one of the following conditions has to satisfy:

   (a) $r_i > r_{i+1}$ and $r_i > r_{i-1}$

   (b) neither r_{i+1} and r_{i-1} are available spots

   (c) r_{i-1} doesn't exist and $r_{i+1} < r_i$

   (d) r_{i+1} doesn't exist and $r_{i-1} < r_i$

   then r_i is a y-maximal spot.

y is bigger than any number inserted later than y, and so by definition of y-maximal spot, we know that y we have the correct w_i value. At every stage, the condition that for elements in each run $j \in \{\sigma_{s_i}, \sigma_{s_i+1}, ..., \sigma_{s_{i+1-1}}\}$, we have $w_j \leq w_{j+1} + 1$ ensures us that for element in each run, the previous element is smaller than it, thus respecting the descent set, so the resulting permutation would also respect the descent set. The condition that $\min_S w_i \leq \tau_i \leq \max_S w_i$ for all i ensures that at every stage, the resulting w_i values could be actually reached by inserting a number in $\{1, ..., n\}$. The existence of a y-maximal spot while trying to insert y comes from reading the w values of r_i. Continuing this process, we know that at every stage there is at least one y-maximal spot while trying to insert y, so we can construct the permutation with the desired descent set and w_i values. $\square$

Example 4.2.1 Construct a permutation with

$$n = 10, S = \{2, 4, 7\}, \tau = (2, 2, 3, 2, 2, 4, 3, 3, 2, 1)$$

Note this τ values is permissible for the descent set S.

First insert 10. Condition 2 is satisfied in Definition 4.1.6, so 7 is a maximal spot.

$$\bigcirc \bigcirc . \bigcirc \bigcirc . \bigcirc \bigcirc 10. \bigcirc \bigcirc \bigcirc$$

Condition 2.(c) is satisfied in the definition of 9-maximal spot for 2, so 2 is a 9-maximal spot.

$$\bigcirc 9. \bigcirc \bigcirc . \bigcirc \bigcirc 10. \bigcirc \bigcirc \bigcirc$$

Condition 2.(a) is satisfied in the definition of 8-maximal spot for 4, so 4 is a 8-maximal spot.

$$\bigcirc 9. \bigcirc 8. \bigcirc \bigcirc 10. \bigcirc \bigcirc \bigcirc$$

Condition 2.(c) is satisfied in the definition of 7-maximal spot for 1, so 1 is a 7-maximal spot.

7 9. ◯ 8. ◯ ◯10. ◯ ◯◯

Condition 2.(c) is satisfied in the definition of 6-maximal spot for 3, so 3 is a 6-maximal spot.

7 9.6 8. ◯ ◯10. ◯ ◯◯

Condition 2.(c) is satisfied in the definition of 5-maximal spot for 6, so 6 is a 5-maximal spot.

7 9.6 8. ◯ 5 10. ◯ ◯◯

Condition 1 is satisfied in the definition of 4-maximal spot for 10, so 10 is a 4-maximal spot.

7 9.6 8. ◯ 5 10. ◯ ◯4

Condition 1 is satisfied in the definition of 3-maximal spot for 9, so 9 is a 3-maximal spot.

7 9.6 8. ◯ 5 10. ◯ 3 4

Condition 1 is satisfied in the definition of 2-maximal spot for 8, so we have our permutation with the desired descent set and w_i values

7 9.6 8.1 5 10.2 3 4

Corollary 4.1.9. If τ is permissible for the descent set S, then $|D_S \bigcap W(\tau, U)| \neq 0$ for any $U \in 2^{\{1,\ldots,s_d\}}$.

This result follows from the fact that $W(\tau, \phi) \subseteq W(\tau, U)$ for $U \neq \phi$.

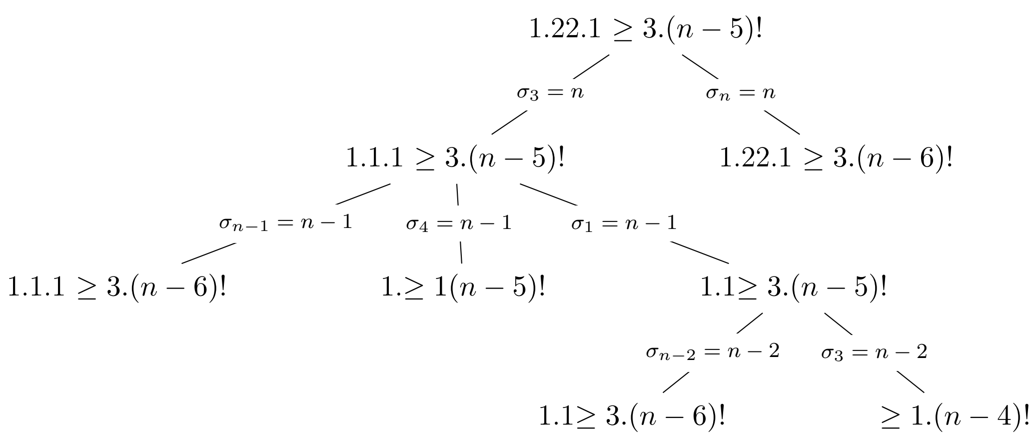
Theorem 4.1.10. *Let S be given and $\tau \in \mathbb{N}^n$ be permissible, then $|D_S \cap W(\tau, U)|$ is a polynomial* *in n.*

4.2. Proof of Theorem 4.1.10

We prove **Theorem 4.1.10** by a recurrence formula. We present an example of such a recurrence before we give the details of the proof.

In the tree below, the children of each node is obtained by removing the biggest entry in the parent, so the first generation is obtained by removing n which we proved to be at the maximal spots. In general, the i-th generation is obtained by removing $n-i+1$ and tracking how S, τ, and U change, which we would define the maps ϕ and ψ in Definition 4.2.1 and 4.2.2 and prove that removing n would affect the corresponding S, τ, U value exactly as we predict in the maps ϕ and ψ.

The base case of the recurrence is when $S = 1$, then we know that $|D_S \cap W(\tau, U)| = n - \tau_1 - 1$ if $U = \{1\}$, and $|D_S \cap W(\tau, U)| = 1$ if $U = \phi$. We also stop at the node obtained by removing $\sigma_{n-i} = n - i$ at the $i+1$th level, as that would be our recurrence step. For the sake of readability in the tree, we denote $D_{1,3,5} \cap W\big((1,2,2,1,3) \sqcup (n-5)!, \{5\}\big)$ as $1.22.1 \geq 3.(n-5)!$, where a dot denotes a descent.



We start our calculation from the bottom. $\geq 1.(n-4)!$ has $S = \{1\} = U$, and for $\sigma \in \geq 1.(n-4)!$ we have $\sigma \in S_{n-3}$, so we get that $\geq 1.(n-4)! = (n-3) - 1 = (n-2) - 2$. If we denote

$F_{n-2} = 1.22.1 \geq 3.(n-5)!$, then we have the recurrence step

$$F_{n-2} = F_{n-3} + (n-2) - 2 \tag{4.2.1}$$

The polynomial that satisfies this relation is

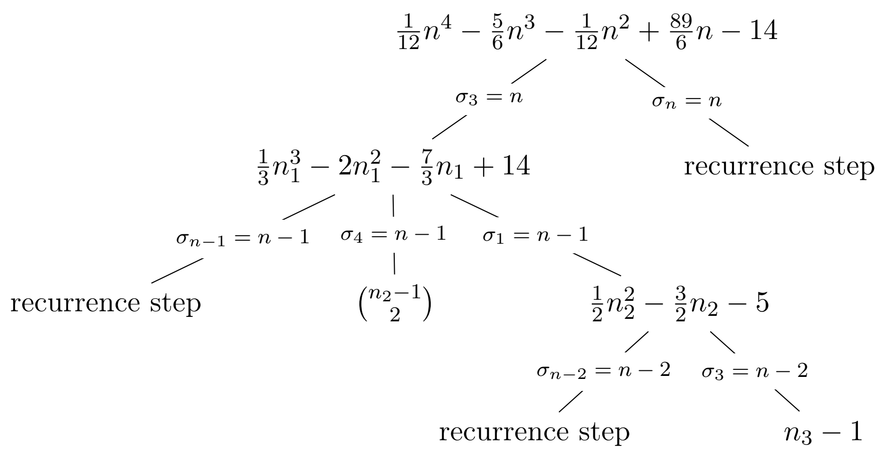
$$\sum_{i=6}^{n-2} i - 2 = \frac{1}{2}(n-2)^2 - \frac{3}{2}(n-2) - 5 \tag{4.2.2}$$

$$= \frac{1}{2}(n-1)^2 - \frac{5}{2}(n-1) - 3 \tag{4.2.3}$$

The summation starts from 6 since the first nonzero term of F_n starts from 6. The resulting polynomial will be computed using Remark 4.2.6, stated later in this thesis.

It's not hard to calculate $1. \geq 1(n-5)!$ as it is just the number of permutations of $n-2$ with descent set being $\{1,2\}$, which would be $\binom{n-2-1}{2} = \frac{1}{2}(n-1)^2 - \frac{5}{2}(n-1) + 3$.

Now we do the recurrence step on $1.1.1 \geq 3.(n-5)!$ in a similar fashion, and we skip the calculation here but provide the final tree of polynomials. For readability, $n_i = n - i$ in the tree.



Having presented the example, we will now establish that the results observed hold true in general. We start by giving details to how S, τ and U change by removing the biggest entry. First, we study how the descent set S change.

Definition 4.2.1. For $m = \sigma_t = s_k$ a maximal spot, define $\phi_m : 2^{\{1,\ldots,b\}} \to 2^{\{1,\ldots,b\}}$ as follow:

1. $m = n$. Then $\phi_m(S) = S$

2. $1 < m < n$.

   (a) If $\tau_{t-1} = 1$, which implies $\sigma_{t-1} < \sigma_{t+1}$, then $s_i \mapsto s_i - 1$ for $i > k$, $s_i \mapsto s_i$ for $i < k$, and then delete $m = s_k$.

   (b) If $\tau_{t-1} \geq 2$, which implies $\sigma_{t-1} > \sigma_{t+1}$, then $s_i \mapsto s_i$ for $1 \leq i < k$, $s_i \mapsto s_i - 1$ for $i \geq k$.

   Note we can tell whether we have $\sigma_{t-1} < \sigma_{t+1}$ or $\sigma_{t-1} < \sigma_{t+1}$ just by inspecting w_{t-1}.

3. $m = 1$, $s_i \mapsto s_i - 1$ and then delete s_1.

Now we define ψ which tracks how τ and U changes.

Definition 4.2.2. Let S be given, $U \subseteq \{1, ..., s_d\}$, and $\tau \in \mathbb{N}^n$ permissible. For $m = \sigma_t$ a maximal spot, define $\psi_m : \mathbb{N}^n \times 2^{\{1,2,...,n\}} \to \mathbb{N}^{n-1} \times 2^{\{1,2,...,n-1\}}$ as follows:

1. $m = n$

   First delete w_n, and for $i \in \{s_d, s_d + 1, ..., n - 1\}$, $w_i \mapsto w_i - 1$. w_i stays the same for all other i. U stays the same.

2. $1 < m < n$. Suppose $m = s_y$.

   (a) $\tau_{t-1} = 1$, i.e. $\sigma_{t-1} < \sigma_{t+1}$, and $i - 1 \notin U$

   First delete w_m. If $y \neq d$, for elements in the same run as m, i.e. $\sigma_{s_{y-1}}, \sigma_{s_{y-1}+1}, ..., \sigma_{s_y - 1}$, we have $w_i \mapsto w_i - 1 + (s_{y+1} - s_y)$. If $y = d$ then $w_i \mapsto w_i + (n - s_y)$

   For elements in the previous run of m, i.e. $\sigma_{s_{y-2}}, \sigma_{s_{y-2}+1}, ..., \sigma_{s_y - 1}$, if $i + \tau_i \geq m - 1$, then $u \mapsto u$ for $u \in U$ and $u < m$ and $u \mapsto u - 1$ for $u > m$ for $i = s_{y-1}, s_{y-1} + 1, ..., s_y - 1$. If $i + \tau_i < m - 1$, then first $u \mapsto u - 1$ for $u > m$ and $U \mapsto U \bigcup \{i\}$ for $i = s_{y-1}, s_{y-1} + 1, ..., s_y - 1$.

   All the other w_i also stays the same.

(b) $\tau_{t-1} \geq 2$, i.e. $\sigma_{t-1} > \sigma_{t+1}$

First delete w_m. For elements in the same run as m, i.e. $\sigma_{s_{y-1}}, \sigma_{s_{y-1}+1}, ..., \sigma_{s_y}$, we have $w_i \mapsto w_i - 1$ if $w_i > 1$, and $w_i \mapsto 1$ if $w_i = 1$.

For $u \in U$ and $u < m$, $u \mapsto u$; for $u > m$ let $u \mapsto u - 1$.

3. $m = 1$.

   Delete w_1.

   If $1 \in U$, then delete 1 and $u \mapsto u - 1$ for all $u \in U$.

If $\tau_{t-1} = 1$ and $t - 1 \in U$, then we need to treat $\psi\big(W(\tau, U)\big)$ as $\psi\big(W(\tau', U)\big) \bigcup \psi\big(W(\tau, U')\big)$ where $\tau'_i = \tau_i$ for $i \neq t-1$ and $\tau_{t-1} = 2$, and $U' = U - \{t-1\}$.

We will do two examples to show how deleting 10 and 9 from a permutation actually affects the descent set and w_i values.

Example 4.2.3. Let $n = 10, U = \{5\}$, $S = \{1,3,5\}, \tau = (1,2,2,1,3,5,4,3,2,1)$. In other words we are looking for maximal spots for permutations of 10, with descent set $S = \{1,3,5\}$, and $w = (1,2,2,1,v,5,4,3,2,1)$ where $v \geq 3$.

First we need to find the maximal spots. $\tau_3 = 2 = \max_S w_3$ so 3 is a maximal spot. When $\tau_5 < 5$, we have that 10 is also a maximal spot. Therefore we have in total of two maximal spots.

1. $m = 3$

   Since $\tau_2 = 2$ we know that $\sigma_2 > \sigma_4$, so following Definition 4.2 1.(b), we have that $\phi_3(S) = \{1,2,4\}$.

   By following Definition 4.2.2(b), we have $\psi_3(W(\tau, U)) = (1,1,1,v,5,4,3,2,1)$ where $v \geq 3$. Formally we write this as

$$\psi_3\Bigg(W\Bigg((1,2,2,1,3,5,4,3,2,1),\{5\}\Bigg)\Bigg) = W\Bigg((1,1,1,3,5,4,3,2,1),\{4\}\Bigg)$$

2. $m = 10$

   By following Definition 4.2.1, we have that $\phi_{10}(S) = \{1,3,5\}$.

By following Definition 4.2.2, we have that $\psi_{10}\big(W(\tau, U)\big) = W\big((1,2,2,1,3,4,3,2,1),\{5\}\big)$.

The descent set and the w_i values indeed change accordingly with the maps ϕ_m and ψ_m when removing 10.

Example 4.2.4. Let $n = 9, S = \{1,3,5\}, U = \{5\}, \tau = (1,1,1,3,5,4,3,2,1)$. In other words we are looking for permutations of 9, with descent set $S = \{1,2,4\}$, and $w = (1,1,1,v,5,4,3,2,1)$ where $v \geq 3$.

First we need to find the maximal spots. 1 is a maximal spot, since $\tau_1 = \max_S w_1 = 1$. 4 is also a maximal spot, and 9 is a maximal spot when $\tau_4 < 5$.

1. $m = 1$

   By following Definition 4.2.1.3 and 4.2.2.3, we get that $\phi_9(S) = \{1,3\}$ and $\psi_9(W(\tau, U)) = (1,1,v,5,4,3,2,1)$ where $v \geq 3$. We can also write this as

$$\psi_9\Bigg(W\big((1,1,1,3,5,4,3,2,1),4\big)\Bigg) = W\big((1,1,3,5,4,3,2,1),3\big) \tag{4.2.4}$$

2. $m = 4$

   In this case $\tau_4 = 5$. Note that since $\tau_2 = 1$ we have that $\sigma_2 < \sigma_4$, so if we follow Definition 4.2.1(a) and 4.2.2(a), we have that $\phi_4(S) = \{1,2\}, \psi_4\big(W(\tau, U)\big) = (1,v,6,5,4,3,2,1)$ where $v \geq 1$. We can also write this as

$$\psi_4\big(W(\tau, U)\big) = W\big((1,1,6,5,4,3,2,1),\{2\}\big) \tag{4.2.5}$$

3. $m = 9$

   In this case $\tau_4 < 5$. By following Definition 4.2.1.2 and 4.2.2.2, we have that $\phi_9(S) = \{1,2,4\}, \psi_9\big(W(\tau, U)\big) = (1,1,1,v,4,3,2,1)$ where $v \geq 3$. We can also write this as

$$\psi_9\big(W(\tau, U)\big) = W\big((1,1,1,3,4,3,2,1),\{4\}\big) \tag{4.2.6}$$

Theorem 4.2.5. *Let S be given, τ be permissible and $U \subseteq \{1, 2, ..., s_d\}$. Then*

$$\left|D_S \bigcap W(\tau, U)\right| = \sum_{m \text{ a maximal spot}} \left|D_{\phi_m(S)} \bigcap \psi_m\big(W(\tau, U)\big)\right| \tag{4.2.7}$$

Proof. Let $\sigma \in D_S \bigcap W(\tau, U)$. First, we will prove that if $\sigma_m = n$ where we proved that m has to be a maximal spot, then removing n we would get exactly a permutation $\sigma' \in D_{\phi_m(S)} \bigcap \psi_m\big(W(\tau; U)\big)$. It's clear to see that $\sigma' \in D_{\phi_m(S)}$ by the definition of ϕ, so we just need to prove that $\sigma' \in \psi_m\big(W(\tau, U)\big)$. We divide into 3 cases as before.

1. If $m = n$, then no elements in the previous run is affected by removing n, but every element in the same run as n is less than n so we have w_i decreases 1 as defined. No other elements would have w_i value changed by removing n.

2. $1 < m < n$. Suppose $m = s_y$

   (a) $\sigma_{m-1} < \sigma_{m+1}$

   For elements σ_i in the same run as n, removing n removes a number that is bigger than σ_i, so we have $w_i - 1$, but in the new permutation since $\sigma_{m+1} > \sigma_{m-1}$, the next run becomes the same run as σ_j, and by the descent set we know every one of the element in the next run is bigger than σ_j, so we need to add the length of the next run, giving us $w_i - 1 + (s_{y+1} - s_y)$. For the case $y = d$ we modified the map by the fact that $\sigma_{n+1} = 0$. For elements σ_i in the previous run of n, if $i + \tau_i \geq m - 1$, then we have $\sigma_i > \sigma_{m-1}$, so by removing n, there might be more elements in the run after n that are smaller than σ_i that are now in the next run of σ_i, and in fact there exist permutations with $w_i(\sigma) = s$ for every $0 \leq s \leq s_{y+1} - s_y$. The reason is that given any $0 \leq s \leq s_{y+1} - s_y$, if there are s elements among σ_{s_y} to $\sigma_{s_{y+1}}$ that are bigger than σ_i, and if we insert m back to its original position, we get a permutation in $D_S \bigcap W(\tau, U)$. Thus the definition of ψ in this case.

   If instead $i + \tau_i < m - 1$, then any thing in the run after m is bigger than σ_i, so although the next run becomes longer, the number of elements in the next run bigger than σ_i

remains the same, thus the definition of ψ in this case.

(b) $\sigma_{m-1} > \sigma_{m+1}$

For elements in the previous run of m, the w_i values do not change because removing n does not merge the next 2 runs. The only values affected is the elements in the same run and it is $w_i \mapsto w_i - 1$.

3. $m = 1$

No elements after n would have their w_i values affected by removing $\sigma_1 = n$, so just delete w_1.

Therefore we have proven that $\sigma' \in \psi_m\big(W(\tau;U)\big)$. Since we have proven that n appear and only appear at the maximal spots, we get the result by summing over all the maximal spots.

$\square$

Now we are ready to give a proof of **Theorem 4.1.10**.

Proof. First we formalize the action of removing the biggest entry and applying the maps ϕ and ψ we defined in Definition 4.2.1 and Definition 4.2.2. Let

$$Z_m : 2^{\{1,\dots,b\}} \times \mathbb{N}^n \times 2^{\{1,\dots,b\}} \to 2^{\{1,\dots,b\}} \times \mathbb{N}^{n-1} \times 2^{\{1,\dots,b\}} \tag{4.2.8}$$

$$D_S \bigcap W(\tau, U) \mapsto D_{\phi_m(S)} \bigcap \psi_m\big(W(\tau, U)\big) \tag{4.2.9}$$

Let $Z^j_{(m_1,\dots,m_j)}(S,\tau,U)$ denote $Z_{m_j} \circ Z_{m_{j-1}} \dots Z_{m_1}(S,\tau,U)$, where m_i is a maximal spot of $Z^{i-1}_{(m_1,\dots,m_{i-1})}(S,\tau,U)$ for all i. For example, in the example we have shown in the beginning of this section, we see that in the first generation, the branch on the left shows $Z\big(\{1,3,5\},(1,2,2,1,3) \sqcup (n-5)!,\{5\}\big) = Z\big(\{1,2,4\},(1,1,1,3) \sqcup (n-5)!,\{4\}\big)$

First, prove that $Z^1_{m_1}(S,\tau,U)$ where $m_1 \neq n$ is a polynomial in $n-1$. By definition of the map ϕ, we know that by at most $s_d - 1$ times of applying ϕ, we can get to the descent set $S = \{1\}$. Let the number of times needed to be applied to S to get to $S = \{1\}$ to be j. We will prove the aforementioned statement by reverse induction on the number of times Z map is applied.

First, prove for the base case $Z^j_{(m_1,...,m_k)}(S,\tau,U)$ where $m_1 \neq n, m_2 \neq n-1, ...m_j \neq n-j+1$, and we know that $S=\{1\}$. If $U=\phi$, then $|Z^j_{(m_1,...,m_j)}(S,\tau,U)|=1$. If $U=\{1\}$, then $|Z^j_{(m_1,...,m_j)}(S,\tau,U)| = n-j-\tau'_1$, where τ' is the resulting τ value in $Z^j_{(m_1,...,m_j)}(S,\tau,U)$. Both cases are polynomials of $n-j$.

Now, assume that $|Z^i_{(m_1,...,m_i)}(S,\tau,U)|$, where $m_1 \neq n, m_2 \neq n-1, ..., m_i \neq n-i+1$, is a polynomial in $n-i$. Prove that $|Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)|$, where $m_1 \neq n, m_2 \neq n-1, ..., m_{i-1} \neq n-i$ is a polynomial in $n-i+1$.

We know that $|Z^i_{(m_1,...,m_i)}(S,\tau,U)| = \sum_{\text{maximal spots } m_i^k} \left| Z_{m_i^k}\big(Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)\big)\right|$. For $m_i^k \neq n-i+1$, i.e. the rightmost position of the permutation after applying Z for $i-1$ times, we know $\left|Z_{m_i^k}\big(Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)\big)\right| = \left|Z^i_{m_1,...,m_{i-1},m_i^k}(S,\tau,U)\right|$ satisfies the induction hypothesis, so each one of them is a polynomial in $n-i$. Denote the sum of them, which is also a polynomial in $n-i$, as $P(n-i)$. Rewrite this as a polynomial in $n-i+1$, and denote it as $Q(n-i+1)$. For $m_i^k = n-i+1$, we know $Z_{n-i+1}\big((Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)\big)$ has the same descent set S, U with $Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)$ and τ up until the last descent α by the definitions 4.2.1.1 and 4.2.2.1 of ϕ and ψ, with the only difference being that the length of τ in $Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)$ being $n-i+1$ and the length of τ in $Z_{n-i+1}\big(Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)\big)$ being $n-i$, so we get

$$\left|Z^{i-1}_{(m_1,...,m_{i-1})}(S,\tau,U)\right| = Q(n-i+1) + \left|Z^i_{(m_1,...,m_{i-1},n-i)}(S,\tau,U)\right| \tag{4.2.10}$$

If we denote the number of permutations in S_{n-i+1} that has descent set $\phi^{i-1}_{(m_1,...,m_{i-1})}(S)$ and is in $\psi^{i-1}_{(m_1,...,m_{i-1})}\big(W(\tau,U)\big)$ as F_{n-i+1}, and the number of permutations in S_{n-i} that has descent set $\phi^{i-1}_{(m_1,...,m_{i-1})}(S)$ and is in $\psi^{i-1}_{(m_1,...,m_{i-1})}\big(W(\tau',U)\big)$ as F_{n-i}, where τ' is of length $n-i$ but $\tau_i = \tau'_i$ for $i=1,...,\alpha$ then (3) can be rewritten as

$$F_{n-i+1} = Q(n-i+1) + F_{n-i} \tag{4.2.11}$$

Then the function F_{n-i+1} that satisfies this recurrence, is a polynomial in $n-i+1$ (see Remark

4.2.6 below for details), namely

$$\left|Z^{i-1}_{(m_1,\ldots,m_{i-1})}(S,\tau,U)\right| = \sum_{l=\alpha+\tau_\alpha}^{n-i+1} Q(l)$$

where α is the last descent place of permutations in $Z^{i-1}_{(m_1,\ldots,m_{i-1})}(S,\tau,U)$, and $Q(\alpha+\tau_\alpha)$ is the first nonzero term of the recursion formula.

Now, by reverse induction, we have proven that $Z^1_{m_1}(S,\tau,U)$ where $m_1 \neq n$ is a polynomial in $n-1$. Using similar reasoning, we can prove that $D_S \bigcap W(\tau,U) = Z^0(S,\tau,U)$ is a polynomial in n. We know that

$$|D_S \bigcap W(\tau,U)| = |Z^0(S,\tau,U)| = \sum_{\text{maximal spots } m_1} |Z_{m_1}(S,\tau,U)|$$

For $m_1 \neq n$, we have that $\sum_{m\neq n} |Z_m(S,\tau,U)|$ is a polynomial in $n-1$ denoted as $P(n-1)$, and after rewriting it into a polynomial of n we get a polynomial $Q(n)$. For $m_1 = n$, we get that $Z_n(S,\tau,U)$ does not change S and U, so we get

$$|Z^0(S,\tau,U)| = Q(n) + Z^1_n(S,\tau,U)$$

By similar reasoning as proving the induction step, we can find that the function that satisfies this recurrence is a polynomial in n, namely

$$|D_S \bigcap W(\tau,U)| = \sum_{l=s_d+\tau_{s_d}}^{n} Q(l)$$

Now we have proved the theorem.

■

Remark 4.2.6. Let $P(x) = \sum_{j=0}^{k} a_j x^j \in \mathbb{Q}[x]$. While calculating $\sum_{i=m}^{n} P(i)$ as performed in the example provided in p37, we can use the formula

$$\sum_{i=1}^{n} i^p = \sum_{i=1}^{p+1} \frac{1}{i} \cdot S(p+1,i) \cdot (n)_i \tag{4.2.12}$$

where $S(n,k)$ denotes the second Stirling number of n and k. The proof of the formula follows from the fact that both the left hand side and the right hand side counts the number of functions $f : [p+1] \to [n]$ where $f(1) = k$ is the maximum of the function.

Note $(n)_i = n \cdot (n-1) \ldots \cdot (n-i+1)$ only has i terms in the product so each $(n)_i$ indeed a polynomial in n; there are only p terms in the sum, so the sum on the right hand side of (4.2.10) is also a polynomial in n. Let $P(x) = \sum_{j=0}^{k} a_j x^j \in \mathbb{Q}[x]$, then we know by Remark 4.2.6 that $\sum_{i=0}^{n} P(i)$ is a polynomial in n, so we also have that $\sum_{i=m}^{n} P(i)$ for some fixed m is a polynomial in n.

Remark 4.2.7. While calculating $D_S \bigcap W(\tau, U)$, if the maximal spot is $n = \sigma_t$, $\tau_{t-1} = 1$ and $t-1 \in U$, then

$$Z_m(S, \tau, U) = Z_m(S, \tau', U) + Z_m(S, \tau, U')$$

where $\tau'_i = \tau_i$ for $i \neq t-1$ and $\tau'_{t-1} = 2$, and $U' = U - \{t-1\}$. This adjustment needs to be made because ϕ and ψ are defined differently for $\tau_{t-1} = 1$ and $\tau_{t-1} \geq 2$.

Remark 4.2.8. While calculating $D_S \bigcap W(\tau, U)$ for $U = \phi$ and $\tau \in \mathbb{N}^n$, the only maximal spot at the first generation is n, and similarly for all the later generation until $n = s_d + \tau_d$. Therefore, $|D_S \bigcap W\big((\tau_1, \ldots, \tau_{s_d}) \sqcup (n - s_d)!, \phi\big)| = |D_S \bigcap W\big((\tau_1, \ldots, \tau_{s_d} \sqcup (\tau_{s_d})!, \phi\big)|$. For example for $S = \{2\}, \tau_1 = 1, \tau_2 = 2$, we have $12.(n-2)! = 12.21$. For any $n \geq 5$, the only satisfying permutation is $(1, 4, 2, 3, 5, \ldots, n-1, n)$. In general, we can also count $|D_S \bigcap W\big((\tau_1, \ldots, \tau_{s_d} \sqcup (\tau_d)!, \phi\big)|$ using our recursion thus get $|D_S \bigcap W\big((\tau_1, \ldots, \tau_{s_d}) \sqcup (n - s_d)!, \phi\big)|$ in the end.

4.3. Stability Range and Degree

In this section, to better investigate the boundary cases, we employ an alternative form of our formula that avoids double-counting certain τ in a different way, while leaving the rest of the formula unchanged. The version below will be used throughout this section:

$$\sum_{\substack{S\subseteq\{1,2,\ldots,b\}\\ s_1+s_2+\ldots+s_d=b}} \sum_{U\subseteq\{1,2,\ldots,s_d\}} \sum_{k=0}^{a} \sum_{\substack{\tau_l=k+1 \text{ for } l\in U\\ \tau_l\leq k \text{ for } l\notin U\\ \tau \text{ permissible}}} \left(\left|D_S \bigcap W(\tau;U)\right|\right)\cdot\left([q^k]\left(\prod_{i=1}^{s_d}[\tau_i]_q\right)\right)\cdot\left([q^{a-k}]\left([n-s_d]_q!\right)\right)$$

Theorem 4.3.1. *The polynomial starts to stabilizes from $a+b$.*

Proof. First we observe that for the 3rd parenthesis in (4.1.1), the Knuth (1997) formula works for all $k=0,1,\ldots,a$ when $n\geq a+b$, so we mainly need to investigate $D_S\bigcap W(\tau,U)$.

Our final polynomial is a sum of polynomials, and it starts to stabilize when we have $D_S\bigcap W(\tau,U)>0$, so considering the boundary case when $S=\{b\}$, $k=a$ so $\tau_b=a+1, b\in U$, we know from our algorithm that we will have a nonzero polynomial in n for $D_S\bigcap W(\tau,U)$ starting from $a+b+1$, so for $n\geq a+b+1$, the polynomial stabilizes. However if we take a closer look at our algorithm, we can actually find that our polynomial starts to stabilizes from $a+b$. We will prove that the formula is still true, i.e. $P(a+b)=0$.

For the boundary case, at the last step, we have one branch being the recurrence branch, and potentially several other branches that add up to a polynomial in n we denote as $p(n)$, and then we use Remark 4.2.6 to calculate the final polynomial in n, which is the polynomial that satisfies the recurrence $F_n=F_{n-1}+p(n)$, and since the first nonzero term is $a+b+1$ in the boundary case, we were calculating

$$\sum_{i=a+b+1}^{n} i^p=\sum_{i=1}^{n} i^p-\sum_{i=1}^{a+b} i^p \tag{4.3.1}$$

therefore when $n=a+b$, we get exactly 0, which is true since $|D_S\bigcap W(\tau,U)|=0$. Therefore we proved that the first parenthesis is indeed correct for all cases for $n=a+b$. For the third parenthesis when we use Knuth (1997) formula, which is true for all k value when $n=a+b$. Therefore stabilization starts from $a+b$ the latest.

Conjecture 4.3.2. $a+b$ is the sharp bound, meaning it is the smallest value of n for which stabilization occurs.

We first consider the boundary case when the first parenthesis fails to be stable. Notice that for $n < a+b$, the algorithm will give the correct formula for $|D_S \bigcap W(\tau, U)|$ for all of the S, τ, U and k, except for the boundary case $k = a+1, S = \{b\}, b \in U$. For example, if $b = 3, S = \{3\}, k = a = 2$, then our formula for $11 \geq 3|(n-3)!$ is true for $n \geq 6$, and is true for $n = 5$ by our last theorem, but becomes negative if $n \leq 4$, when in fact, $|D_S \bigcap W(\tau, U)|$ should be 0. The boundary is also sharp in the sense that, if we relax k so that $k = a = 2$, then the algorithm for $|D_S \bigcap W(\tau, U)|$ will provide the correct polynomial for $n = 4 = a + b - 1$; or if we relax S so that $S = \{1, 2\}$, then $s_d = 2$ so again we would get the correct polynomial for $n = 4$. We claim that the boundary cases would always produce a non-positive number for $n = a+b-1$, and the details of the analysis of the boundary cases for $D_S \bigcap W(\tau, U)$ will be found in our upcoming preprint. Now we investigate the second and third parenthesis of the boundary cases in (4.1.1) in this case. The second parenthesis $[q^a]\left(\prod_{i=1}^{b}[\tau_i]_q\right)$ is nonzero since $\tau_b = a+1$, and the third parenthesis is the constant term in $[n-b]_q!$ which is 1, so the product of the three parenthesis would give a non-positive number, when the correct value should be 0.

Assuming we have successfully proved the claim for the boundary cases of the first parenthesis, we now assert that the boundary cases for the third parenthesis—specifically when $k = 0$, $S = \{b\}$, and $n = a + b - 1$—also yield a value that is strictly smaller than the true value predicted by the polynomial when evaluated at $n = a + b - 1$, using the formula from Knuth (1997). Further details will be provided in our upcoming preprint. Therefore, the product of the 3 parenthesis is smaller than the true value for the boundary cases, and the polynomial in the end would predict lower dimension number, for both cases of boundary cases discussed. In other words, if we denote the polynomial for $dim(DH_n^{a,b})$ as $P(n, a, b)$, then for $n = a + b - 1$, we will have our formula $P(n, a, b) < dim(DH_n^{a,b})$, given the details being filled. $\square$

Conjecture 4.3.3. $DH_n^{a,b}$ is not an $FI\#$-module.

This conjecture is proven once Conjecture 4.3.2 is proven. For an FI-module to be $FI\#$, we need stabilization to start immediately from 0. Since by Conjecture 4.3.2 the sharp bound is $a+b$, this would be a proof that $DH_n^{a,b}$ is not $FI\#$ combinatorially.

Theorem 4.3.4. *The degree of the polynomial of coefficient of q^a for the descent set S with $maxS = d$ of $Hilb(DH_n)$ is $d+a$.*

Proof. If we fix the descent set S and increase a by 1, then the polynomial increases by degree 1. This is by the observation that

$$\sum_{j=0}^{a} \prod_{i=1}^{d} [w_i(\sigma)]\Big|_{q^j} \cdot [n-d]!\Big|_{q^{a-j}}$$

increases its degree by 1 when a increases by 1. By increasing a by 1, $[w_i(\sigma)]\Big|_{q^j}$ does not change, so we only need to track how the degree of $[n-d]!\Big|_{q^{a-j}}$ changes. In the formula for $[n]!|_{q^k}$, the first term is always $\binom{n+k-2}{k}$, and so the first term of $[n-d]!\Big|_{q^{a-j}}$ is always $\binom{(n-d)+(a-j)-2}{a-j}$. This term has the highest degree and it is of degree $a-j$, so increasing a by 1 would increase the degree of the whole formula by 1.

If $a = 0$, then the inductive formula reduces to

$$\sum_{\substack{S \subseteq \{1,2,\ldots,b\} \\ \text{s.t. } s_1+s_2+\ldots+s_d=b}} P_S(n)$$

where $P_S(n)$ denotes the polynomial in n which represents the number permutations of n with the descent set S. We can do the same recursion on the descent set S by using the map ϕ defined in Definition 4.2.1, where all of the descents are maximal spots, and we treat all ϕ_m in the condition that $\tau_{m-1} = 1$ and $m-1 \in U$. In this way, every time we apply ϕ, there is one branch that has the maximum of $\phi(S)$ as $max\ \phi^{i+1}(S) = max\ \phi^i(S) - 1$, so it takes exactly $d-1$ times of applying ϕ to get to $S = [1]$, which is the base case $n-1$. When we do the recurrence starting with the base case $S = [1]$, applying ϕ for $d-1$ times would produce a degree d polynomial in the end. Therefore by increasing a, we would get a polynomial of degree $d+a$. $\square$

Corollary 4.3.5. The degree of the coefficient of $q^a t^b$ of $Hilb(DH_n)$ is $a+b$.

This is because that among all the descent sets with $maj(S) = b$, the biggest d can be is exactly b when $S = \{b\}$.

Remark 4.3.6. We actually improved both the stability range and the degree of the dimension polynomial. In Section 3.2, we used the theory of FI-modules to conjecture a stability bound for $\dim(DR_n^{a,b})$ to be $2(a+b)$, but we proved in the thesis that stabilization actually starts from $a+b$, and conjectured it to be the sharp bound. Furthermore, while FI-module theory gives an upper bound of $a+b$ for the degree of $\dim(DH_n^{a,b})$, we showed that the degree is *exactly* $a+b$.

We provide a table of polynomials of $dim(DR_n^{a,b}) = [q^a t^b]\left(\sum_{\sigma\in S_n} t^{\mathrm{maj}(\sigma)} \prod_{i=1}^{n} [w_i(\sigma)]_q\right)$ given $n \geq a+b$ in Table 4.3.1.

/	$\mathbf{q}^0$	$\mathbf{q}^1$	$\mathbf{q}^2$	$\mathbf{q}^3$
t^0	1	$n-1$	$\frac{1}{2}n^2-\frac{1}{2}n-1$	$\frac{1}{6}n^3-\frac{7}{6}n$
t^1	$n-1$	n^2-2n	$\frac{n^3}{2}-n^2-\frac{3n}{2}+1$	$\frac{1}{6}n^4-\frac{1}{6}n^3-\frac{5}{3}n^2+\frac{2}{3}n+1$
t^2	$\frac{n^2}{2}-\frac{n}{2}-1$	$\frac{n^3}{2}-n^2-\frac{3n}{2}+1$	$\frac{n^4}{4}-\frac{n^3}{2}-\frac{7n^2}{4}+n+1$	$\frac{1}{12}n^5-\frac{1}{12}n^4-\frac{5}{4}n^3+\frac{1}{12}n^2+\frac{13}{6}n+1$
t^3	$\frac{n^3}{6}-\frac{7n}{6}$	$\frac{n^4}{6}-\frac{n^3}{6}-\frac{5n^2}{3}+\frac{2n}{3}+1$	$\frac{n^5}{12}-\frac{n^4}{12}-\frac{5n^3}{4}+\frac{n^2}{12}+\frac{13n}{6}+1$	$\frac{n^6}{36}-\frac{23n^4}{36}-\frac{n^3}{2}+\frac{19n^2}{9}+3n-1$

Table 4.3.1: Polynomials for $dim(DR_n^{a,b})$

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