COMBINATORIAL EXPANSIONS OF MACDONALD AND LLT POLYNOMIALS

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ABSTRACT

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In 1987, Ian Macdonald introduced a special family of symmetric polynomials $H_\mu(X;q,t)$. These polynomials, now known as Macdonald polynomials, can be written as $H_\mu(X;q,t) = \sum_{\lambda\vdash n} K_{\lambda,\mu}(q,t)s_\lambda(X)$, a sum over Schur functions $s_\lambda(X)$, a basis for the ring of symmetric functions. Macondald conjectured that $K_{\lambda,\mu}(q,t) \in \mathbb{N}(q,t)$, i.e., have positive coefficients. Shortly after, a more natural form of these polynomials was introduced, $\tilde{H}_\mu(X;q,t)$. Written in the Schur basis, $\tilde{H}_\mu(X;q,t) = \sum_{\lambda\vdash n} \tilde{K}_{\lambda,\mu}(q,t)s_\lambda(X)$ where $\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)}K_{\lambda,\mu}(q,1/t)$. In 2001, Mark Haiman showed $\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}(q,t)$ using algebraic geometry. Since then, it has been a major open problem to find a combinatorial interpretation for $\tilde{K}_{\lambda,\mu}(q,t)$. We prove a new formula for $\tilde{K}_{\lambda,\mu}(q,t)$ when $\mu = (n-k-1,2,1^{k-1})$ in terms of statistic on Standard Young Tableau. Using this formula, we then prove a special case of a conjecture due to Lynne Butler in 1994 on the change of Schur coefficients from a hook shape to an augmented hook shape.

In 1997, Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon introduced a new family of symmetric polynomials, now known as LLT polynomials. In 2005, Jim Haglund, Mark Haiman, and Nick Loehr showed how to write Macdonald polynomials as a sum of LLT polynomials. Thus, a combinatorial formula for Macdonald polynomials can be derived from a combinatorial formula for LLT polynomials. In 2020, Alex Abreu and Antonio Nigro showed that if G is an indifference graph, then $LLT_G(q) = \sum_{\sigma\leq\mathbf{m}}(q-1)^{n-\ell(\lambda(\sigma))}q^{wt_G(\sigma)}e_{\lambda(\sigma)}$. Using this expansion of the LLT polynomials into the e-basis, we prove a combinatorial formula for the coefficients of s_λ when $\lambda = (n-k,1^k)$ or $\lambda = (n-k-1,2,1^{k-1})$.

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CHAPTER 1

INTRODUCTION

Upon introducing Macdonald Polynomials in 1987, an enormous amount of new combinatorics has ensued. It took over a decade to show the Schur coefficients of Macdonald polynomials are positive, but the proof was not combinatorial. Because of the positivity of the Schur coefficients, combinatorialists expect a set of mathematical objects and algebraic statistics to be associated with these statistics. In recent years, a plethora of different objects have been used in Macdonald combinatorics, but one has become the most common: fillings of Young tableau. We define a statistic on Standard Young Tableau depending on an augmented hook shape $\mu = (n-k-1, 2, 1^{k-1})$ and use this statistic to prove a combinatorial formula for $\tilde{H}_\mu(X; q, t)$ when $\mu = (n-k-1, 2, 1^{k-1})$. Using a known formula for $\tilde{H}_\mu(X; q, t)$ when $\mu = (n-k, 1^k)$, we prove a special case of a conjecture from 1994 due to Butler.

We then turn our study to LLT polynomials of a family of graphs. Haglund, Haimain, and Loehr showed how to expand Macdonald polynomials into a sum of LLT polynomials in 2005. Since then one approach to proving a combinatorial formula for Macdonald polynomials is to study LLT polynomials. It is know that when G is a graph associated to a unit-interval order, then $LLT(G)$ is symmetric and Schur-positive. In 2020, Abreu and Nigro proved a formula for the e-expansion of $LLT_G(q)$. Using this formula, we can describe the Schur coefficients in a natural way using the Pieri rules. We introduce a multivariate version of this formula for the LLT polynomials and prove the coefficients of s_λ are related to inversions in Standard Young Tableau when $\lambda = (n-k, 1^k)$ or $\lambda = (n-k-1, 2, 1^{k-1})$.

1.1. Outline

This thesis is organized into three parts. The first part is Chapter 2 which provides extensive background on symmetric functions and the developments of the theory of Macdonald Polynomials. The second part is contained in Chapters 3 and 4. In chapter 3, we give details on various known Schur expansions of Macdonald polynomials. In chapter 4, we prove our new formula for the

Schur expansion of the augmented hook shape Macdonald polynomial. The third part in contained in chapters 5 and 6. In chapter 5, we provide extensive background on LLT polynomials and recent formulas for their e-expansion. In chapter 6, we prove the new multivariate formula for the coefficients of s_λ when λ is a hook or an augmented hook shape.

CHAPTER 2

SYMMETRIC FUNCTION WORLD

We begin by covering the introductory notions in symmetric function theory. We also explore the classical results in symmetric function theory along with other important results for this thesis. Most of this material can be found in Haglund (2007), Macdonald (2015), Sagan (2013), Stanley (2011), and Stanley (1997), among others.

2.1. Basic Symmetric Functions

Here, we will define important notation for symmetric functions.

Definition 2.1.1 (Symmetric Function)**.** Let K be a field (often $\mathbb{Q}$), $\sigma \in S_n$ be a permutation, and $f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$. We say f is a *symmetric function* if

$$\sigma f = f(x_{\sigma_1}, \ldots, x_{\sigma_n}) = f$$

for all $\sigma \in S_n$.

Often, we will want f to be a function of countably many variables. In this case $f = f(x_1, x_2, \ldots)$ and we can view f as a formal power series of each x_i. In this case, f is symmetric if we permute the variables in any way and still obtain the original f. We will also use the notation X_n and X to denote the set of variables $\{x_1, \ldots, x_n\}$ and $\{x_1, x_2, \ldots\}$ respectively.

Example 2.1.2. As an example, if $n = 3$, then the following are symmetric functions over $\mathbb{Q}$:

1. $x_1 + x_2 + x_3$
2. $x_1^k + x_2^k + x_3^k$ for any $k \in \mathbb{Q}$
3. $(x_1 + x_2 + x_3)^k$ for any $k \in \mathbb{Q}$.

We may also have countably many variables.

Example 2.1.3. The following are examples of symmetric functions over $\mathbb{Q}$:

1. $\sum_{i=1}^{\infty} x_i$
2. $\sum_{i=1}^{\infty} x_i^k$ for any $k \in \mathbb{Q}$
3. $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i x_j$.

2.1.1. Bases For the Ring of Symmetric Functions

We let Λ be the ring of symmetric functions in X and Λ^n the sub-vectorspace of Λ consisting of the homogeneous degree n functions. Next, we will consider various bases of Λ^n which will be important throughout. We begin by considering partitions.

Definition 2.1.4 (Partitions)**.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Then, we say λ is a *partition* of n, denoted as $\lambda \vdash n$. We use the notation $Par(n)$ to denote the set of partitions of n

Example 2.1.5. The following is a complete list of partitions for $n = 4$, $Par(4)$:

1. (4)
2. (3,1)
3. (2,2)
4. (2,1,1)
5. (1,1,1,1,1).

Often it is useful to compare two partitions λ and μ of the same n. Is there a way to decide when one partition is bigger than the other in some way? Here is one widely used partial order on partitions.

Definition 2.1.6 (Dominance Order on Partitions)**.** Let $\lambda, \mu \vdash n$. We say λ *dominates* μ, denoted

$\lambda \geq \mu$ if the following is true:

$$\lambda \geq \mu \iff \sum_{i=1}^{min(i,\ell(\lambda))} \lambda_i \geq \sum_{i=1}^{min(i,\ell(\mu))} \mu_i.$$

We now consider a basis for Λ.

Definition 2.1.7 (Monomial Symmetric Functions)**.** Let $\lambda \vdash n$, so $\lambda \in Par(n)$. We define the *monomial symmetric function* $m_\lambda = m_\lambda(X) = m_\lambda(X_n)$ to be the sum of all monomials in the x_i (finite for X_n and infinite for X) where the multiset of exponents of a monomial is equal to the multiset formed by the parts of λ.

We will consider 3 examples of m_λ.

Example 2.1.8. Here are examples with no variables given, a finite set of variables, and an infinite set of variables.

1. $m_{2,1} = \sum_{i<j} x_i^2 x_j + x_i x_j^2$
2. $m_{3,2,1}(X_3) = x_1^3 x_2^2 x_3 + x_1^3 x_2 x_3^2 + x_1^2 x_2^3 x_3 + x_1^2 x_2 x_3^3 + x_1 x_2^3 x_3^2 + x_1 x_2^2 x_3^3$
3. $m_4(X) = \sum_{i=1}^{\infty} x_i^4$.

Now that we have defined m_λ, we will consider 3 special families of symmetric functions that are bases for Λ^n.

Definition 2.1.9 (Elementary Symmetric Functions)**.** For any n, let 1^n be the partition of n into $1's$. Then, define $e_n = m_{1^n}$ to be the *elementary symmetric functions.*

Definition 2.1.10 (Power-Sum Symmetric Functions)**.** For any n, let (n) be the partition of n into exactly one part. Then, define $p_n = m_n$ to be the *power-sum symmetric functions.*

Definition 2.1.11 (Complete Homogeneous Symmetric Functions)**.** For any n, let

$$h_n = \sum_{\lambda \in Par(n)} m_\lambda.$$

We call h_n the *complete homogeneous symmetric functions.*

Theorem 2.1.12 (Bases of Λ^n)**.** *With the above 3 families of symmetric functions in mind, let* $\lambda \vdash n$ *and consider:*

1. $e_\lambda = \prod_i e_{\lambda_i}$
2. $p_\lambda = \prod_i e_{\lambda_i}$
3. $h_\lambda = \prod_i e_{\lambda_i}$.

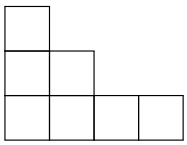
Then, $\{e_\lambda, \lambda \vdash n\}$*,* $\{p_\lambda, \lambda \vdash n\}$*, and* $\{h_\lambda, \lambda \vdash n\}$ *are each individually bases for* Λ^n*.*

2.1.2. Schur Functions

With the above symmetric functions in mind, we will now develop an extremely important basis for Λ^n. First, we must introduce Young Diagrams and their fillings.

Definition 2.1.13 (Young Diagram)**.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, then we call Y_λ a *Young Diagram* of shape λ where Y_λ is created by placing λ_1 boxes in the bottom row, λ_2 boxes in the second row, up through λ_k boxes in the kth row.

As an example, let $\lambda = (4, 2, 1)$, then Y_λ is:



We now consider fillings of Y_λ. For convenience, we will let $Y = Y_\lambda$

Definition 2.1.14 (Young Tableaux)**.** Let Y be a Young Diagram of shape λ, where $|\lambda| = n$. We say that a filling of Y with positive integers is a *Standard Young Tableaux* if rows and columns

are strictly increasing. We call the filling a *Semistandard Young Tableaux* if the rows are weakly increasing and the columns are strictly increasing.

Continuing with our example of $\lambda = (4, 2, 1)$ and Y_λ, the following are Semistandard and Standard fillings of Y_λ, respectively.

Example 2.1.15 (Semistandard Young Tableau of Shape (4,2,1))**.**

4			
2	3		
1	2	2	3

Example 2.1.16 (Standard Young Tableau of Shape (4,2,1))**.**

5			
3	6		
1	2	4	7

Now that we understand two different types of fillings of Young Diagrams, we will now consider the total number of fillings given a specific alphabet.

Definition 2.1.17 (Kostka Numbers)**.** Let $\mu = \{1^{\mu_1}, 2^{\mu_2}, \ldots, n^{\mu_n}\}$. Let $SSYT(\lambda)$ be the set of semistandard Young tableaux of shape λ. Then, we define:

$$K_{\lambda,\mu} = \#\{T \in SSYT(\lambda) | \text{ content of } T \text{ is } \mu\}$$

and we call $K_{\lambda,\mu}$ the *Kostka Numbers.*

In other words, $K_{\lambda,\mu}$ count the total number of semistandard Young tableaux of shape λ with content μ. We note that counting the standard Young Tableaux of shape λ (SYT(λ)) is the special case where $\mu = (1^n)$. In particular, $|SYT(\lambda)| = K_{\lambda,(1^n)}$. In 1953, Frame, Robinson, and Thrall proved a combinatorial formula for $|SYT(\lambda)|$. To understand the formula, we need one more definition,

Definition 2.1.18 (Hook-length)**.** Let $u \in Y_\lambda$, i.e., u is a cell in the Young Diagram Y_λ. Let $h(u)$ be the total number of cells in the same row and weakly to the right of u and the number of cells in the same column and strictly above u. We call $h(u)$ the *hook-length* of u.

We can now state the famous Hook-Length Formula first proven by Frame, Robinson, and Thrall. For a detailed history, see Sagan (2013):

Theorem 2.1.19 (Hook-Length Formula)**.** *Let $\lambda \vdash n$. Then,*

$$|SYT(\lambda)| = K_{\lambda,(1^n)} = \frac{n!}{\prod_{u \in Y_\lambda} h(u)}.$$

Though first proven in 1953, Greene et al. (1979) proved the formula using probabilistic methods that is much more interesting and directly uses the hook content of cells. The Kostka Numbers often appear when discussing symmetric functions. We will now see one such instance of great importance.

Proposition 2.1.20. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ be a partition, then an equivalent definition of the monomial symmetric functions, $m_\lambda(X)$, is*

$$m_\lambda(X) = \sum_{\alpha \sim \lambda} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $\alpha \sim \lambda$ means α rearranges to λ with possibly extra 0s.

With this equivalent definition of the monomial symmetric functions in mind, we have our first definition of the Schur functions.

Definition 2.1.21 (Schur Functions)**.** For $\lambda \vdash n$, we can define the *Schur functions*, $s_\lambda(X)$, as

$$s_\lambda(X) = \sum_{\alpha, T} \prod_i x_i^{\alpha_i}$$

where α is a weak composition of n and T is a tableaux of shape λ and content α.

There are several other equivalent definitions of Schur functions. One stems from the Vandermonde determinant.

Definition 2.1.22. Let $\lambda \vdash n$, then define the following determinant function:

$$a_{(\lambda_1+n-1,\lambda_2+n-2,\ldots,\lambda_n)}(x_1,x_2,\ldots,x_n) = det\begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{bmatrix}.$$

With the above determinant function, we can now list the equivalent definitions of Schur functions. Each equivalent definition has use in its own right. Here are three of the important ones.

Proposition 2.1.23 (Equivalent Definitions of Schur Functions)**.** *The following are equivalent definitions of Schur Functions*

1. *Monomial Expansion: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ be a partition, then*

$$s_\lambda(X) = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu(X).$$

2. *Jacobi's Bialternant Formula: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \vdash n$, then,*

$$s_\lambda(X) = \frac{a_{(\lambda_1+n-1,\lambda_2+n-2,\ldots,\lambda_n)}(x_1,x_2,\ldots,x_n)}{a_{(n-1,n-2,\ldots,0)}(x_1,x_2,\ldots,x_n)}.$$

3. *Jacobi-Trudi Identity: Let $\ell(\lambda)$ be the number of parts of λ. Then,*

$$s_\lambda(X) = det(h_{\lambda_i-i+j})_{i,j}^{\ell(\lambda)}.$$

2.2. RSK Algorithm

We now proceed to a discussion of the famous Robinson-Schensted-Knuth (RSK) Algorithm. For a detailed account of the RSK Algorithm, see Stanley (1997) or Sagan (2013). We use this algorithm to create a bijection between two line arrays of words and pairs of SSYT of the same shape. We start with a word $w = w_1...w_n$ where $w_i \in [n]$. We then construct a sequence of pairs of SSYT through the RSK algorithm. We denote this sequence as $RSK_i(w) = (P_w^i, Q_w^i)$ where P_w^i is known as the insertion tableau obtained from inserting w_i into P_w^{i-1} and Q_w^i is the tableau formed by recording the location of the new box from P_w^{i-1} to P_w^i. Here is a complete description of the RSK algorithm as found in Sagan (2013). Let $\pi \in S_n$ and suppose that π is the following in 2-line notation:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \pi_1 & \pi_2 & \cdots & \pi_{n-1} & \pi_n \end{pmatrix}$$

We then construct the sequence of tableaux pairs $RSK_i(\pi) = (P_\pi^i, Q_\pi^i) = (P^i, Q^i)$, starting with $(P^0, Q^0) = (\emptyset, \emptyset)$, and ending with $(P^n, Q^n) = (P, Q)$, the output of the RSK algorithm. The RSK algorithm iterativley inserts π_k into π_{k-1} so that P^k is a partial tableaux- a tableaux where the rows and columns are increasing. We then place k into Q^{k-1} so that the shape of Q^k is the same as the shape of P^k. Here is the precise insertion algorithm:

1. Let R be the bottom row of P^{k-1} and $x = \pi_k$.

2. While x is less than some element in row R, let y be the smallest element in R that is greater than x, and replace y by x in row R. Now, define x to be y and let R be the next row up in P^{k-1}.

3. After the previous step, x is now greater than every element in row R, so place x at the end of this row. This gives us P^k. Construct Q^k for Q^{k-1} by placing k in the same position that the last x was placed.

We note that this can be generalized so π is not a permutation and the recording value is not the

identity, but we begin with an example of this situation. We begin with an example where w is a permutation in S_n, and the recording word value is $12\dots n$.

Example 2.2.1. As an example, let $w = 425163$, then we compute RSK on the two line array:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 1 & 6 & 3 \end{pmatrix}$$

We start by inserting 4 into the empty tableau, obtaining

$$\begin{aligned}(P_w^0, Q_w^0) &= \left(\quad , \quad \right) \xleftarrow{4} \\ &= \left(\begin{array}{|c|}\hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \end{array} \right) = (P_w^1, Q_w^1)\end{aligned}$$

where 1 is inserted into the newly inserted position in P_w. Next, we insert 2, obtaining:

$$\begin{aligned}(P_w^1, Q_w^1) &= \left(\begin{array}{|c|}\hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline \end{array} \right) \xleftarrow{2} \\ &= \left(\begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \right) = (P_w^2, Q_w^2)\end{aligned}$$

Next, we insert 5, obtaining:

$$\begin{aligned}(P_w^2, Q_w^2) &= \left(\begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \right) \xleftarrow{5} \\ &= \left(\begin{array}{|c|c|}\hline 4 & \\ \hline 2 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \right) = (P_w^3, Q_w^3)\end{aligned}$$

Next, we insert 1, obtaining:

$$(P_w^3, Q_w^3) = \left(\begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \, , \begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \cline{1-2} \end{array} \right) \xleftarrow{1}$$

$$= \left(\begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 5 \\ \cline{1-2} \end{array} \, , \begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \cline{1-2} \end{array} \right) = (P_w^4, Q_w^4)$$

Next, we insert 6, obtaining:

$$(P_w^4, Q_w^4) = \left(\begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 5 \\ \cline{1-2} \end{array} \, , \begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \cline{1-2} \end{array} \right) \xleftarrow{6}$$

$$= \left(\begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & 6 \\ \cline{1-3} \end{array} \, , \begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 5 \\ \cline{1-3} \end{array} \right) = (P_w^5, Q_w^5)$$

Finally, we insert 3, obtaining:

$$(P_w^5, Q_w^5) = \left(\begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 5 & 6 \\ \cline{1-3} \end{array} \, , \begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 5 \\ \cline{1-3} \end{array} \right) \xleftarrow{3}$$

$$= \left(\begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & 5 \\ \hline 1 & 3 & 6 \\ \cline{1-3} \end{array} \, , \begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & 6 \\ \hline 1 & 3 & 5 \\ \cline{1-3} \end{array} \right) = (P_w, Q_w)\,.$$

We now discuss important properties of the RSK algorithm. The first, is the main theorem due to

Robinson (1938) and Schensted (1961).

Theorem 2.2.2 (Robinson (1938) and Schensted (1961))**.** *The RSK map is a bijection between permutations in S_n and pairs of Standard Young Tableaux of the same shape $\lambda \vdash n$. In particular, if $f^\lambda = |SYT(\lambda)|$, then*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

The proof of this theorem uses the invertibility of the RSK map. One of the main motivations for Schensted work on the RSK algorithm was finding the longest increasing or decreasing subsequence of a permutation π. He proved the following theorem in Schensted (1961):

Theorem 2.2.3 (Schensted (1961))**.** *Let $\pi \in S_n$ and let $P(\pi)$ be the P-tableaux from the RSK algorithm applied to π. Then, the longest increasing subsequence in π is the length of the first row of $P(\pi)$ and the longest decreasing subsequence is the length of the first column of $P(\pi)$.*

Another interesting aspect of the RSK algorithm is due to Knuth. Much of the material can be found in Schensted (1970). We begin with some definition.

Definition 2.2.4 (P-equivalent and Knuth Relations)**.** We give 3 different definitions due to Knuth's work.

1. Let $\pi, \sigma \in S_n$. We say that π and σ are *P-equivalent* if $P(\pi) = P(\sigma)$.

2. Let $x < y < z$ and $\pi, \sigma \in S_n$. We say π and σ differ by a *Knuth relation* if either of the following is true:

   (a) $\pi = \pi_1 \dots xyz \dots \pi_n$ and $\sigma = \pi_1 \dots yzx \dots \pi_n$

   (b) $\pi = \pi_1 \dots xzy \dots \pi_n$ and $\sigma = \pi_1 \dots zxy \dots \pi_n$.

   (c) We say $\pi, \sigma \in S_n$ are *Knuth Equivalent* if there is a sequence of Knuth relations transforming π to σ.

With these definitions in mind, we have an important theorem found in Schensted (1970).

Theorem 2.2.5 (Schensted (1970))**.** *Let $\pi, \sigma \in S_n$, then π and σ are Knuth equivalent if and only if they are P-equivalent.*

The next interesting RSK result concerns the inverse of permutations. This work is due to Schützenzerberger. The proof involves shadow diagrams. For complete details, see Sagan (2013).

Theorem 2.2.6. *Let $\pi \in S_n$, then*

$$P(\pi^{-1}) = Q(\pi) \text{ and } Q(\pi^{-1}) = P(\pi).$$

In particular, note that if $\pi = \pi^{-1}$, then $P(\pi) = Q(\pi)$. This also shows that the number of involutions in S_n is equal to the total number of Standard Young Tableaux for any n.

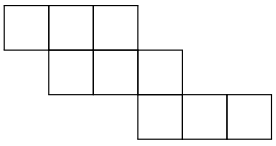
There are many other important properties of the RSK algorithm. We will see others later. For now, we move consider a similar algorithm due to Schützenzerberger, Jeu de Taquin.

2.2.1. Jeu de Taquin

We consider a new algorithm performed on skew tableaux.

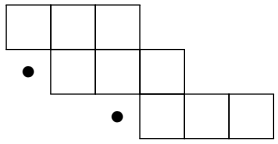
Definition 2.2.7 (Skew-Diagram)**.** Let $\mu \vdash n$, and let $\nu \vdash k$ where $k <= n$, $\ell(\nu) <= \ell(\mu)$ and $\nu_i <= \mu_i$ for $1 \leq i \leq \ell(\nu)$. We call $Y_{\mu/\nu}$, the *skew diagram of shape μ/ν*, where we start with the diagram Y_μ and remove the boxes corresponding to Y_ν.

Here is an example of a skew diagram where $\mu = (6, 4, 3)$ and $\nu = (3, 1)$.

$\mu/\nu =$ 

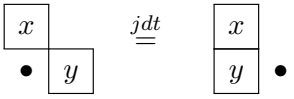
We will now describe how to perform Jeu de Taquin on skew tableaux using our example for

$\mu = (6, 4, 3)$ and $\nu = (3, 1)$. Consider the diagram with added bullets denoting the *corner squares*:

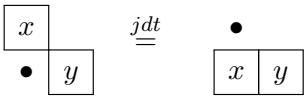
$\mu/\nu =$ 

If we pick a corner square, we can perform the following jdt moves.

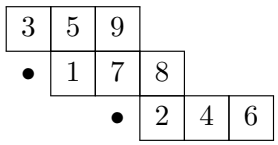
1. If $x > y$, then



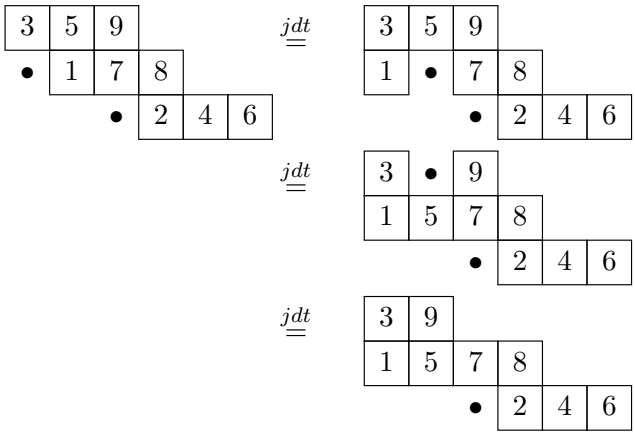
2. If $x <= y$, then



Using our example, suppose we have the following filling:

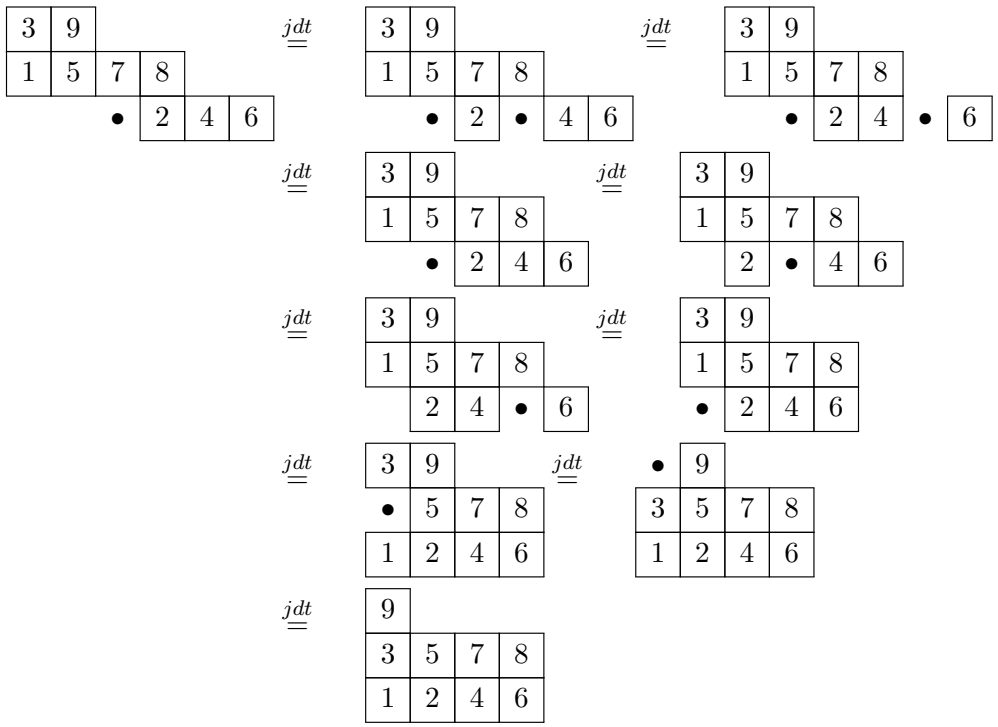
$\mu/\nu =$ 

and we move the top left corner square using jdt moves. We then obtain the tableaux:



We can now use this last tableau and perform jdt moves until no corner squares remain (i.e., we

obtain a standard tableaux):



We now have a standard Young Tableaux. We now discuss some important properties of Jeu de Taquin. If P and Q are skew tableaux and there is some sequence of jdt moves transforming P to Q, we say that P and Q are jdt-equivalent. The following theorem can be found in Sagan (2013), originally due to Schützenzerberger.

Theorem 2.2.8. *If P and Q are standard skew tableau, then P and Q are jdt-equivalent if and only if they are Knuth equivalent.*

Note that there are also dual version of Knuth relations, RSK and jdt that result in dual equivalence relations. See Sagan (2013) for further details.

Another important characteristic about Jeu de Taquin and RSK corresponds to the reading word of a tableau.

Definition 2.2.9. Let T be a tableau. We define the *reading word of T*, rw(T), to be the word formed by the rows of numbers in tableau from top to bottom.

Example 2.2.10. If we have the tableau

$$T = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 2 & 5 & \\ \hline 1 & 3 & 6 \\ \hline \end{array} \quad ,$$

then $rw(T) = 425136$.

Let $jdt_a(T)$ denote a Jeu de Taquin move of a in T. Then the following holds.

Lemma 2.2.11 (Stanley (1997))**.** *A Jeu de Taquim slide converts the reading word of a tableau into a Knuth-equivalent word. In particular,*

$$rw(jdt_a(T)) \overset{K}{\sim} rw(T).$$

Further, we obtain an important theorem on equivalence classes of tableau.

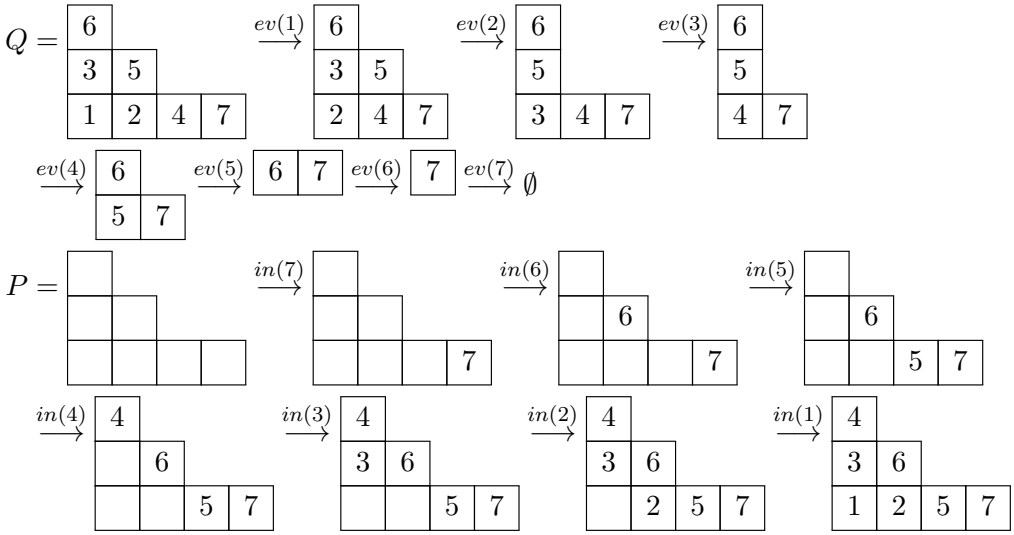
Theorem 2.2.12 (Stanley (1997))**.** *Each Jeu de Taquin equivalence class contains exactly one straight shape tableau.*

We now consider another important property of jdt and RSK.

Definition 2.2.13 (Evacuation)**.** Let Q be a partial skew tableau, and let m be the smallest number in Q. Erase m from its cell, replace it with a bullet, and perform jdt moves on this bullet square. This process is called *evacuation*.

Now, start with a standard tableau Q and a corresponding diagram Y of the same shape. Starting with $i = 1$, evacuate Q until empty. For each evacuation, there is now a cell that is in Y that is no longer in the evacuated Q tableau. Place $n - i + 1$ in this cell. The resulting tableau is the *evacuation of Q*, denoted evQ. Here is an example:

Example 2.2.14 (Evacuation Tableau)**.**



Thus,

$$ev(Q) = \begin{array}{|c|c|c|c|}\hline 4 & \multicolumn{3}{c}{} \\ \hline 3 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 5 & 7 \\ \hline \end{array}$$

We can now state an interesting theorem about the Q-tableau from RSK and evacuation tableau. See Sagan (2013) for details.

Theorem 2.2.15. *Let $\pi \in S_n$, and let π^r be the permutation in S_n where $\pi^r_i = \pi_{n-i+1}$, i.e., the reversed permutation. Then,*

$$Q(\pi^r) = ev(Q(\pi))^t.$$

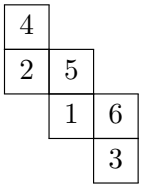
With the previous theorem in mind, we have an important, useful proposition. Recall that there is exactly one unique standard, straight tableaux in each jdt equivalence class. For a given $\beta \in S_n$, we can construct the insertion tableaux through a sequence of jdt moves.

Proposition 2.2.16. *Let $\beta \in S_n$, where $\beta = \beta_1\beta_2\cdots\beta_n$ and form a skew diagram, T_β, by placing β_{i+1} below β_i if $\beta_i > \beta_{i+1}$, and to the right otherwise. Let $jdt(\beta)$ be the tableau obtained by performing jdt on T_β until we have a standard tableau. Then, $jdt(\beta) = P_\beta$ and $jdt(\beta^{-1}) = Q_\beta$ from*

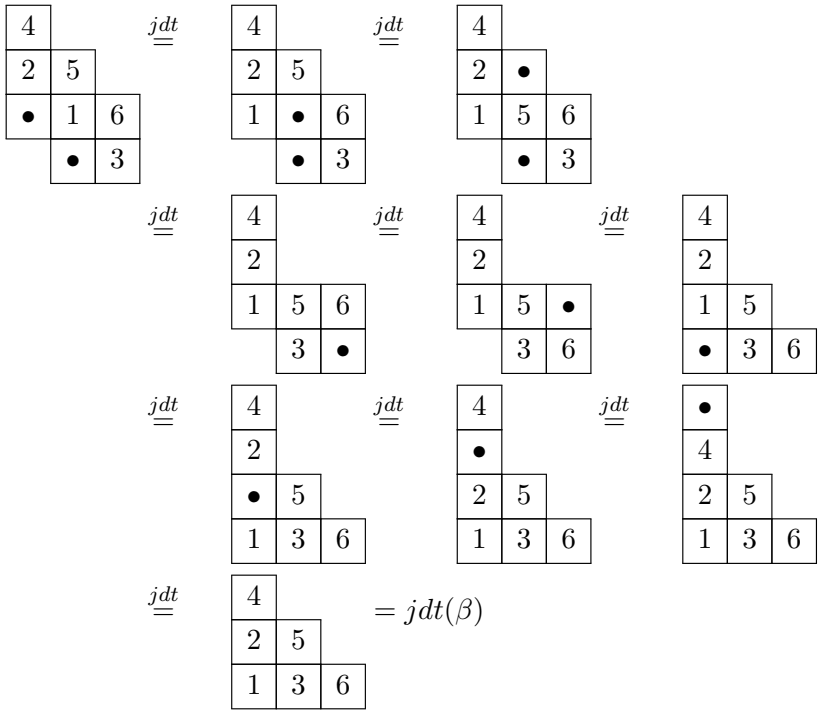
$RSK(\beta)$.

We will show an example obtaining P_β. Note that $jdt(\beta^{-1}) = Q_\beta$ follows from $jdt(\beta) = P_\beta$ and the previously mentioned properties of RSK.

Example 2.2.17. As an example, let $\beta = 425163$, then the skew tableau associated to β would be:



Note that for a given skew tableau, there may be multiple corner squares, as is the case in this example. It is known that the result of performing jdt is independent of which corner square we start with. The following is the jdt algorithm performed on this skew tableau:



Now, note that the final tableau, $jdt(\beta)$ is precisely equal to P_β from $RSK(\beta)$. If we were to compute $jdt(\beta^{-1})$, we would obtain Q_β.

With the above in mind, we see an important property of evacuation.

Proposition 2.2.18 (Schützenzerberger Involution Stanley (1997))**.** *The map $Q \to ev(Q)$ is an involution.*

With this background on basic symmetric functions and two important combinatorial algorithms, we will now turn to the main symmetric functions under consideration, Macdonald Polynomials.

2.3. Development of Macdonald Polynomials

In this section, we will introduce many important properties of Macdonald Polynomials and the goal of this thesis. Much of the background can be found in the classic text Macdonald (2015) with new developments in Haglund (2007).

We begin by defining two important functions for a given partition λ. We use these notations throughout.

Definition 2.3.1. Let $\lambda \vdash n$

1. $n(\lambda) = \sum_i (i-1)\lambda_i$
2. $z_\lambda = \prod_i i^{n_i} n_i!$

where n_i is the number of parts of λ that are equal to i.

We can now state the three fundamental identities concerning the infinite product:

$$\prod_{i,j} (1 - x_i y_i)^{-1}$$

where $x_i's$ and $y_i's$ are independent variables.

Theorem 2.3.2 (Macdonald (2015))**.** *Let $\lambda \vdash n$ be a partition, p_λ, m_λ, h_λ, and s_λ the symmetric*

functions defined previously. Then,

$$\begin{aligned}\prod_{i,j}(1-x_iy_i)^{-1} &= \sum_{\lambda} z_\lambda^{-1} p_\lambda(x)p_\lambda(y) \\ &= \sum_{\lambda} h_\lambda(x)m_\lambda(y) \\ &= \sum_{\lambda} s_\lambda(x)s_\lambda(y)\end{aligned}$$

summed over all partitions λ*.*

With the above theorem in mind, we define a scalar product $\langle \cdot\ ,\ \cdot \rangle$ on Λ such that

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$$

for all $\lambda, \mu \vdash n$. This is known as the *Hall Scalar Product*. Note that $\delta_{\lambda,\mu}$ is 1 when $\lambda = \mu$, and 0 otherwise. We use an alternative notation that sometime appears in the literature. If P is a statement, then we say $\chi(P) = 1$ if P is a true statement, and $\chi(P) = 0$ otherwise. Hence, in this case, $\delta_{\lambda,\mu} = \chi(\lambda = \mu)$.

We note two important properties of this scalar product. First,

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$$

giving an orthogonal basis for Λ. Second,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$$

giving an orthonormal basis for Λ, showing the importance of the Schur functions. Now, for any $f \in \Lambda$, and any basis b_λ of Λ, let $f|_{b_\lambda}$ denote the coefficient of b_λ when f is expressed in that basis. As an example,

$$f|_{m_\lambda} = \langle f, h_\lambda \rangle$$

by the above properties of the Hall Scalar product. Further, because the Schur basis is orthonormal with respect to the inner product,

$$f|_{s_\lambda} = \langle f, s_\lambda \rangle$$

2.3.1. Plethysm

The last notation we need before turning to Macdonald polynomials is for a plethystic substitution. An abstract description of plethysm can be found in Macdonald (2015), but we will follow Haglund (2007) for a more understandable definition.

Definition 2.3.3 (Plethystic Substitution)**.** Let $E(t_1, t_2, \dots)$ be a formal power series of rational functions in the variables $t_1, t_2, \dots$. Define the *plethystic substitution* of E into p_k by

$$p_k[E] = E(t_1^k, t_2^k, \dots).$$

In other words, the kth-plethystic substitution of a formal power series is computed by replacing each variable with its kth power. We note for ease of notation, if we have the notation X inside of plethystic brackets, we really mean

$$X = p_1(X) = x_1 + x_2 + x_3 + \cdots .$$

We now remark on taking the negative inside of plethystic brackets. Let $X = (x_1, x_2, \dots)$. Suppose we want to consider the negative of each variable, namely $Z = (-x_1, -x_2, \dots)$. From the definitition of plethysm,

$$p_k(Z) = \sum_i (-1)^k x_i^k \neq p_k[-X].$$

Because of this, we use a special notation for using negatives with plethysm. We let

$$p_k[\epsilon X] = \sum_i (-1)^k x_i^k.$$

We only defined plethysm in terms of the power-symmetric function. However, for any $f \in \Lambda$, we can write f in the p_λ basis, and then compute the plethysym. Thus, it makes sense to consider plethsym on any symmetric function. Next, consider the important involution ω defined as follows:

$$\omega(p_k) = (-1)^{k-1} p_k$$

and extend it to p_λ by

$$\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda.$$

Again, because p_λ is a basis for Λ, we can compute $\omega(f)$ for any symmetric function by first writing f in the p_λ basis. Here are two interesting examples of the involution:

1. $\omega(e_\lambda) = h_\lambda$
2. $\omega(s_\lambda) = s_{\lambda'}$.

With the definitions of plethysm, ϵ, and ω in mind, for any $f \in \Lambda$

$$\omega(f(X)) = f[-\epsilon X].$$

We will now consider symmetric functions over $\mathbb{Q}(q,t)$ and introduce Macdonald polynomials.

2.3.2. Macdonald Polynomials

Here, we consider the symmetric functions developed by Macdonald in chapter 6 of Macdonald (2015). Macdonald denotes these functions as $P_\lambda(x;q,t)$, a class of rational functions depending on two parameters q and t. This family of functions satisfies a modified Hall Scalar product that now depends on the parameters q and t.

Definition 2.3.4. Let $\lambda \vdash n$. Then, let:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Note that when $q = t$, we obtain the usual Hall Scalar product.

We now extend our definition of symmetric functions Λ. Let $F = \mathbb{Q}(q,t)$, the field of rational functions in q and t. Then, define $\Lambda_F = \Lambda \otimes F$ to be the $F - algebra$ of symmetric functions with coefficients in F. In other words, we extend our ring of symmetric functions Λ so that the coefficients are no longer restricted to $\mathbb{Q}$, but are in the field extension of $\mathbb{Q}$ by q and t. Macdonald proves the following:

Theorem 2.3.5 (Macdonald (2015) Chapter 6, Theorem 4.7)**.** *For each partition λ, there is a unique symmetric function $P_\lambda = P_\lambda(x;q,t) \in \Lambda_F$ such that:*

$$P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda,\mu} m_\mu$$

where $u_{\lambda,\mu} \in F$ and $u_{\lambda,\lambda} = 1$ and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0$$

if $\lambda \neq \mu$.

This family of symmetric functions contains a few important subfamilies. First, when $q = t$, $P_\lambda(x;t,t) = s_\lambda(x)$, the usual Schur functions. When $q = 0$, $P_\lambda(x,;0,t) = P_\lambda(x;t)$, which are known as Hall-Littlewood polynomials. Finally, let $q = t^\alpha$ and divide P_λ by $(1-t)^{|\lambda|}$. If we let $t \to 1^-$, then the P_λ are equal to another family of symmetric functions, the Jack polynomials. The P_λ are now known as *Macdonald Polynomials*. Now that we have defined this special family of polynomials and seen that they do in fact exist, we will discuss some of their properties and modifications that have occurred over the years. Throughout, our goal is to reformulate Macdonald polynomials in combinatorial terms. Often, we use various properties of tableaux to derive special formulas. We will define various tableaux statistics.

Definition 2.3.6. Let Y_λ be the Young diagram of shape λ. Let $c = (i,j)$ be a cell in Y_λ corresponding to the square in the ith row and jth column. We define the following 4 quantities:

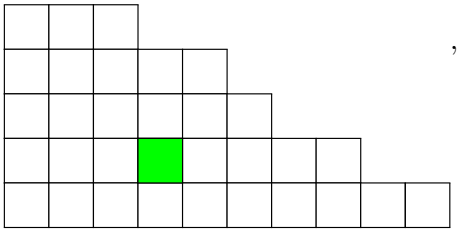
1. Let $arm(c) = a$ be the number of cells strictly to the right of c and in Y_λ.

2. Let $coarm(c) = a'$ be the number of cells strictly to the left of c and in Y_λ.

3. Let $leg(c) = l$ be the number of cells strictly above c and in Y_λ.

4. Let $coleg(c) = l'$ be the number of cells strictly below c and in Y_λ.

For instance, if we have the Young diagram

$Y_\lambda =$  ,

then the green square corresponds to the cell $c = (2, 4)$ and has $arm = 4$, $coarm = 3$, $leg = 2$, and $coleg = 1$.

With the definitions above, we can now consider a modification of the Macdonald polynomials, known as the Macdonald Integral Form Polynomials where:

$$J_\mu(X; q, t) = \prod_{c \in \mu} (1 - q^{a(c)} t^{l(c)+1}) P_\mu(X; q, t).$$

With these modified symmetric functions in mind, we consider the work of Mark Haiman in 2000, proving a famous open problem posed by Macdonald. See Haiman (2000) for full details.

First, Macdonald showed that the plethystic form of the Schur functions are a basis for Λ_F. These modified Schur functions are defined as $s_\lambda[X(1-t)]$, see Macdonald (2015) Chapter 3, 4.5 and Chapter 6, 8.9 for more details. Because of this, we can express the J_μ in the $s_\lambda[X(1-t)]$ basis, obtaining:

$$J_\mu(X; q, t) = \sum_\lambda K_{\lambda,\mu}(q, t) s_\lambda[X(1-t)]$$

and our goal is to understand the $K_{\lambda,\mu}(q,t)$. When $q=0$, $K_{\lambda,\mu}(0,t)=K_{\lambda,\mu}(t)$. The $K_{\lambda,\mu}(t)$ were introduced through the transition matrix from $s_\lambda(x)$ to the Hall-Littlewood functions $P_\mu(X;t)$. A famous conjecture in Foulkes (1974), says that there is some positive integer statistic $c(T)$ associated to semistandard Young tableaux such that:

$$K_{\lambda,\mu}(t)=\sum_{T\in SSYT(\lambda,\mu)} t^{c(T)},$$

where $SSYT(\lambda,\mu)$ is the set of semistandard Young tableaux of shape λ and weight μ. A statistic known as *charge* was discovered by Lascoux and Schützenzerberger, proving the conjecture of Foulkes. With this is mind, $K_{\lambda,\mu}(q,t)$ are now known as the q,t-Kostka-Foulkes polynomials. Macdonald proved many special cases regarding $K_{\lambda,\mu}(q,t)$ in Macdonald (2015) including:

Proposition 2.3.7. *The following are true regarding $K_{\lambda,\mu}(q,t)$:*

1. $K_{\lambda,\mu}(0,0)=\delta_{\lambda,\mu}$
2. $K_{\lambda,\mu}(0,1)=K_{\lambda,\mu}$
3. $K_{\lambda,\mu}(q,t)=K_{\lambda',\mu'}(t,q)$
4. $K_{\lambda,\mu}(1,1)=\frac{n!}{h(\lambda)}$.

However, the famous conjecture due to Macdonald is that $K_{\lambda,\mu}(q,t)\in\mathbb{N}[q,t]$, i.e., the coefficients are non-negative integers. Garsia and Haiman modified the q,t-Kostka-Foulkes polynomials in the following way:

$$\tilde{K}_{\lambda,\mu}(q,t)=t^{n(\mu)}K_{\lambda,\mu}(q,1/t).$$

We will refer to the Macdonald polynomials with the modified q,t-Kostka-Foulkes polynomials as the modified Macdonald polynomials, denoted $\tilde{H}_\mu(x;q,t)$. In Haiman (2000), Haiman showed that $\tilde{K}_{\lambda,\mu}(q,t)\in\mathbb{N}[q,t]$, proving the conjecture of Macdonald. However, Haiman's proof involved deep algebraic geometry. It is our goal to discover some combinatorial formula for the $\tilde{K}_{\lambda,\mu}(q,t)$. Macdonald's open conjecture is the following:

Conjecture 2.3.8. *Let $\lambda, \mu \vdash n$, then there exist algebraic statistics, $qstat(T, \mu)$ and $tstat(T, \mu)$, such that*

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{T \in SYT(T)} q^{qstat(T,\mu)} t^{tstat(T,\mu)}.$$

This conjecture has been solved in a few cases, but is largely open. We will solve one of the open cases later. We now construct the famous combinatorial formula for Macdonald polynomials first conjectured by Haglund in Haglund (2004) and proven by Haglund, Haiman, and Loehr in Haglund et al. (2005b).

2.3.3. HHL Formula for Macdonald Polynomials

We now look at the famous combinatorial formula for Macdonald polynomials, now known as the HHL Formula. Let $\mu \vdash n$, we define a filling of μ to be a function $\sigma : Y_\mu \to \mathbb{Z}_+$, i.e., we fill the diagram of shape μ with positive integers, and no other restrictions. For a given filling σ, we define the x-weight of the filling to be

$$x^\sigma = \prod_{c \in Y_\mu} x_{\sigma(c)},$$

i.e., we take the product of x_i for each i in the filling of Y_μ. From Haglund (2004), we have two algebraic statistics on fillings, $inv_\mu(\sigma)$ and $maj_\mu(\sigma)$. We say a *descent* in a filling is a cell $u = (i, j) \in Y_\mu$ such that $\sigma(u) > \sigma(v)$ where $v = (i + 1, j)$. In other words, a cell is a descent if it is larger than the number in the cell directly below it. We defined $Des(\sigma)$ to be the set of cells that are descents in a given filling. Next, we define an *attack* relation between cells. Let $u, v \in Y_\mu$, then we say u attacks v if either of the following hold:

1. u and v are in the same row with u to the left of v

2. u and v are in consecutive rows,u in the higher row, and v is in the lower row and to the left.

Then, we say that u and v form an inversion pair if u attack v and $\sigma(u) > \sigma(v)$. We denote the set of inversions for a given filling as $Inv(\sigma)$. With these definitions in mind, we can now define the

algebraic statistics formulated in Haglund (2004):

$$maj_\mu(\sigma) = \sum_{u \in Des(\sigma)} leg(u) + 1$$

and

$$inv_\mu(\sigma) = |Inv(\sigma)| - \sum_{u \in Des(\sigma)} arm(u).$$

Sometimes, we will simply write $inv(\sigma)$ or $maj(\sigma)$ when μ is clear. Now, consider the following function:

$$C_\mu(x; q, t) = \sum_{\sigma:\mu \to \mathbb{Z}_+} q^{inv_\mu(\sigma)} t^{maj_\mu(\sigma)} x^\sigma.$$

The following is the famous theorem in Haglund et al. (2005b), giving rise to the HHL formula:

Theorem 2.3.9 (HHL Formula Haglund et al. (2005b) Theorem 2.2)**.** *For any μ,*

$$C_\mu(x; q, t) = \tilde{H}_\mu(x; q, t).$$

Now that we have seen the famous HHL formula, we now present an important theorem regarding the modified Macdonald polynomials, which is vital to the proof of the HHL Formula and also found in Haglund et al. (2005b).

Theorem 2.3.10 (Macdonald Polynomial Triangularity)**.** *The following 3 conditions define a unique family of symmetric functions:*

1. $\tilde{H}_\mu[X(q-1)]; q, t] = \sum_{\rho \leq \mu'} c_{\rho,\mu}(q, t) m_\rho(X)$
2. $\tilde{H}_\mu[X(t-1)]; q, t] = \sum_{\rho \leq \mu} d_{\rho,\mu}(q, t) m_\rho(X)$
3. $\tilde{H}_\mu(x; q, t)|_{x_1^n} = 1.$

Later, we will use these 3 properties to show a conjectured combinatorial formula for the modified

Macdonald Polynomials for the augmented hook shape is in fact equal to the Macdonald Polynomial.

CHAPTER 3

SCHUR EXPANSION OF MACDONALD POLYNOMIALS

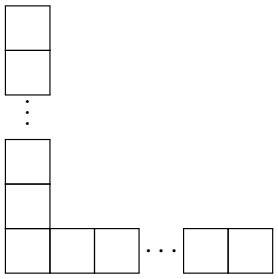
With the background information from Chapter 2, we now consider various combinatorial descriptions of the Schur expansion of Macdonald polynomials. In particular if we write the modified Macdonald polynomials in the Schur basis,

$$\tilde{H}_\mu(x;q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda(x),$$

our goal is to find a combinatorial description of $K_{\lambda,\mu}(q,t)$. Doing this for arbitrary μ is an extremely difficult challenge. Thus, we will consider various families of μ that have been solved and describe some of the combinatorics behind the proofs. Then, we will consider a conjecture due to Lynne Butler in the early 1990s that gives us a possible approach to tackle the general problem, see Butler (1994) for full details.

3.1. Combinatorial Formulas for Hook Shape Macdonald Polynomials

The first shape we consider is known as the hook shape. By a hook, we mean $\mu = (k, 1^{n-k})$, so that the first row has k boxes, and the first column has $n-k+1$ boxes. Here is a basic example of a hook shape diagram:



Now define the function B_μ for any shape μ as:

$$B_\mu = \sum_{c \in Y_\mu} q^{a'(c)} t^{l'(c)}.$$

By slightly modifying a result from Macdonald in Macdonald (2015), we have the following result

Theorem 3.1.1. *For any μ and $\lambda = (k, 1^{n-k})$ a hook shape,*

$$\tilde{K}_{\lambda,\mu}(q,t) = e_{n-k}[B_\mu - 1].$$

Thus, we can express the Schur coefficient associated to a hook shape for any Macdonald polynomial indexed by μ using this plethystic evaluation and the coarm and coleg of cells in a diagram. However, it would be nice to have more concrete combinatorial expansions. We will see more of these, starting with μ a hook shape. Before describing this formula, we need to consider statistics on skew tableau.

Definition 3.1.2. Let T be a skew tableau, and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n into k parts. Let $T^{(i)}$ be the parts of the tableau T that contain the numbers $\alpha_{i-1} + 1$ through α_i where $\alpha_0 = 0$, where we standardize the numbers, i.e., the α_i numbers are rewritten as 1 through α_i maintaining their order. Then, we call the set of $T^{(i)}$ the α-sectionalization of T. Further, if we have some statistics $stat(T)$, then we can define $stat(T, \alpha)$ to be the α-sectionalization of the statistic, evaluated as:

$$stat(T, \alpha) = \sum_{i=1}^{k} stat(T^{(i)}).$$

Recall the previous definition of maj_μ dealing with descents and legs in a diagram. We will now modify the maj statistic to match Stembridge.

Definition 3.1.3. Let $T \in SYT(\lambda)$, then define the *descent set*, Des(T), to be the set of i such that $i + 1$ is in a row above i in T. With the descent set in mind, consider two different statistics on T used by Stembridge:

1. $maj(T) = \sum_{i \in Des(T)} i$
2. $comaj(T) = \sum_{i \notin Des(T)} |\lambda| - i$

With the above in mind, the following is a result in Stembridge (1994), a purely combinatorial

formula for the hook shape Macdonald polynomial using statistic on tableau.

Theorem 3.1.4 (Stembridge (1994))**.** *Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n into k parts. Let $rev(\alpha) = (\alpha_k, \ldots, \alpha_1)$ be the reverse composition of α. When $\mu = (n-k, 1^k)$,*

$$K_{\lambda,\mu} = \sum_{T \in SYT(\lambda)} q^{maj(T,\mu)} t^{comaj(T, rev(\mu'))}.$$

We will now show an example confirming the two formulas due to Macdonald and Stembridge, are in fact equivalent.

Example 3.1.5. As an example, consider $\mu = (4,1)$ and $\lambda = (3,1,1)$, then we can compute $B_{(3,1,1)}$ as the sum of the coefficients in the following diagram:

t			
1	q	q^2	q^3

so $B_\mu = q^3 + q^2 + q + t + 1$. When computing $\tilde{K}_{\lambda,\mu}(q,t)$ from Macdonald's plethystic formular, we obtain:

$$\begin{aligned}
\tilde{K}_{\lambda,\mu}(q,t) &= e_2[B_\mu - 1] \\
&= e_2[q^3 + q^2 + q + t] \\
&= q^5 + q^4 + q^3 + q^3 t + q^2 t + qt.
\end{aligned}$$

We now compare this two Stembridge's formula. Here, we need the following 6 standard tableau:

5			5			5			4			4			3		
4			3			2			3			2			2		
1	2	3	1	2	4	1	3	4	1	2	5	1	3	5	1	4	5

Because $\mu = (4,1)$, when we compute the μ-sectionalization, we are really just ignoring the entry 5. Hence, $maj(T,\mu)$ for the above 6 tableau is $3,2,1,5,4,3$ respectively. Next, $\mu' = (2,1,1,1)$, and $rev(\mu') = (1,1,1,2)$. When computing $comaj(T, rev(\mu'))$, we need only consider $T^{(4)}$ from the

$rev(\mu')$-sectionalization. Thus, comaj for the 6 tableau is $0, 0, 0, 1, 1, 1$. Thus, $K_{\lambda,\mu} = q^3 + q^2 + q + q^5t + q^4t + q^3t$, and since

$$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q, 1/t)$$

and $n(\mu) = 1$,

$$\begin{aligned}\tilde{K}_{\lambda,\mu}(q,t) &= t(q^3 + q^2 + q + q^5 1/t + q^4 1/t + q^3 1/t) \\ &= q^3t + q^2t + qt + q^5 + q^4 + q^3\end{aligned}$$

which matches the above plethystic formula as desired.

3.2. Combinatorial Formulas for Two Column Macdonald Polynomials

3.2.1. Fishel Two-Column Formula

Shortly after Macdonald's conjecture, many people solved special cases of the q,t-Kostka Polynomials. In 1995, Fishel worked on the case where $\mu_1 \leq 2$, i.e., the two-column case. Here, we describe her results. Her work begins with a formula from Stembridge on the two column case. In particular, in Stembridge (1994), we have

Theorem 3.2.1 (Stembridge (1994))**.** *Let* $\mu = (2^r, 1^{n-2r})$*, then*

$$K_{\lambda,\mu}(q,t) = \sum_{s=0}^{r} q^{r-s}(t^{n-r}q;t^{-1})_s \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda,(2^r,1^{n-2r})}(t).$$

From this formula involving Macdonald's charge polynomial, in Fishel (1995), Fishel finds statistics c_r and cut_r such that

$$K_{\lambda,\mu}(q,t) = \sum_{(\alpha(0),L)\in\mathcal{M}_0^0} q^{cut_r(\alpha(0),L)} t^{c_r(\alpha(0),L)}$$

where $\mathcal{M}_0^0$ is a set of rigged configurations of the tableaux of shape λ, developed by Kirillov and Reshetikhin in the 1980s. See Kerov et al. (1988) and Kirillov and Reshetikhin (1988) for full details on rigged configurations. Though nice to have a combinatorial formula, it is quite difficult to state and requires new combinatorial objects to prove it. So people continued to search for a more concise

combinatorial formula.

3.2.2. Lapointe and Morse Two-Row Formula

First appearing on the arXiv in 1998 and then published in a book in 2003, Lapointe and Morse found another formula that can be translated into a formula for the two-column Macdonald polynomial in Lapointe and Morse (2003). Their work is deeply algebraic with a series of complex operators on tableaux. They produce the following theorem:

Theorem 3.2.2 (Lapointe and Morse (2003) Theorem 27)**.** *Let* $\mu = (2m + \ell + a, \ell)$*, then*

$$H_\mu(x; q, t) = \sum_{|T|=|\mu|} stat(T) s_{shape(T)}.$$

Here, Lapointe and Morse found a statistic, stat(T), that is a q, t statistic on tableaux computed after finding something known as the domino vector of a tableaux. As stated, this is a result for a two-row Macdonald polynomial. Through the transposition property of q, t-Kostka polynomials, we can translate this into a result for the two-column Macdonald polynomials. Again, though it is great that we have another result towards a combinatorial expression for the Schur expansion of Macdonald polynomials, the complexity as stated leaves much to be desired.

3.2.3. Zabrocki Two-Column Formula

In 1998, Zabrocki worked on algebraic operators on Macdonald polynomials that led to a formula for the two-column Macdonald polynomials in Zabrocki (1998). In particular, Zabrocki wanted to find a vertex operator on Macdonald polynomials in the same spirit as the vertex operators on other families of symmetric functions. In particular, on the homogeneous symmetric functions, Schur functions, and Hall-Littlewood symmetric functions, there are vertex operators, h_m, S_m, H_m^t, respectively, such that for $m \geq \mu_1$,

1. $h_m h_\mu[X] = h_{(m,\mu)}[X]$
2. $S_m s_\mu[X] = s_{(m,\mu)}[X]$

3. $H_m^t H_\mu[X;t] = H_{(m,\mu)}[X;t]$.

Zabrocki's goal was to find a vertex operator that extends to the Macdonald polynomials. In other words, he wanted to find an operator H_m^{qt} such that

$$H_m^{qt} H_\mu[X;q,t] = H_{(m,\mu)}[X;q,t]$$

For general partitions, this condition is not enough to show uniqueness. However, for special cases, it is. In Zabrocki (1998), he proved the following theorem.

Theorem 3.2.3 (Zabrocki (1998) Theorem 2.6)**.** *The operator*

$$H_2^{qt} = H_2^t + q\omega H_2^{1/t} \omega R^t$$

gives the desired formula

$$H_2^{qt} H_{(2^a,1^b)}[X;q,t] = H_{(2^{a+1},1^b}[X;q,t].$$

Using this result, Zabrocki then proves the following on the Schur expansion of two-column Macdonald polynomials.

Theorem 3.2.4 (Zabrocki (1998) Corollary 3.7)**.** *Let $\mu = (2^a, 1^b)$, then there are statistics $a_\mu(T)$ and $b_\mu(T)$ such that:*

$$H_{(2^a,1^b)} = \sum_{T \in SYT(2a+b)} q^{b_\mu(T)} t^{a_\mu(T)} s_{shape(T)}(x)$$

where $a_\mu(T)$ and $b_\mu(T)$ are defined from a series of tableaux operators $H_2^{s_i}$ where s_i is either $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ *or* $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$.

Understanding the full details requires lots of examples and is beyond the scope here. However, each operator $H_2^{s_i}$ and the statistics $a_\mu(T)$ and $b_\mu(T)$ boils down to finding the relative locations on sets of numbers depending on μ and determining if they are of type $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ or $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$. Thus, the combinatorics involved is quite interesting, though difficult. However, we continue on our journey

to finding simpler combinatorial formulas for the q,t-Kostka polynomials for two columns. Next, is the best known result and by far the simplest.

3.2.4. Haglund, Haiman, and Loehr Two-Column Formula

We now consider the two-column formula derived in Haglund et al. (2005a). We consider this the best two-column formula to date given its simple characterization in terms of filling of Young diagrams. Further, the algebraic statistics are easy to state and compute.

Definition 3.2.5 (Yamanouchi Words)**.** Let w be a word in $\mathbb{Z}_+^n$. We say w satisfies the Yamanouchi condition if for any k, the final part of w beginning at k,$w_k w_{k+1} \dots w_n$ has partition content. In other words, the content of $w_k \dots w_n$ is $\{1^{\lambda_{k_1}}, \dots, j^{\lambda_{k_j}}\}$ where $\lambda_{k_1} \geq \lambda_{k_2} \geq \cdots \geq \lambda_{k_j}$. Let $Yam(\lambda)$ be the set of Yamanouchi words with content $\{1^{\lambda_1}, \dots, j^{\lambda_j}\}$.

Example 3.2.6. As an example, let us compute $Yam(3,2)$. Now, the full set of words with content $\{1^3, 2^2\}$ are

$$\{11122, 11212, 11221, 12112, 12121, 12211, 21112, 21121, 21211, 22111\}.$$

From these words, the following satisfy the Yamanouchi condition:

$$\{12121, 12211, 21121, 21211, 22111\}$$

We have the following theorem.

Theorem 3.2.7 (Haglund et al. (2005a) Proposition 9.2)**.** *Let $\mu_1 \leq 2$, then*

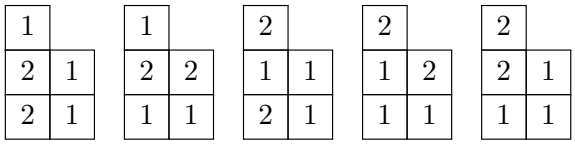
$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{\substack{\sigma:\mu\to\mathbb{Z}_+ \\ w(\sigma)\in Yam(\lambda)}} q^{inv(\sigma)} t^{maj(\sigma)}$$

where $w(\sigma)$ is the reading word of the filling σ.

The proof of this theorem comes from a crystal structure on the set of fillings. In the construction

of the crystal structure, the set of fillings, Σ_μ, are partitioned into components where $q^{inv(\sigma)}t^{maj(\sigma)}$ is constant on each component. The authors hoped to extend the crystal structure and algebraic statistics to other families, so far, such an extensions has been elusive, even for the case $\mu_1 = 3$. We will continue with our previous example and compute $\tilde{K}_{(3,2),(2,2,1)}(q,t)$.

Example 3.2.8. Note that for $\lambda = (3,2)$, $Yam(\lambda) = \{12121, 12211, 21121, 21211, 22111\}$. Hence, there are five terms in $\tilde{K}_{(3,2),(2,2,1)}(q,t)$ corresponding to the following five fillings of μ.



These five tableau have q,t weights $q^2t^0, q^0t^3, q^1t^1, q^1t^2, q^0t^2$, respectively. Thus,

$$\tilde{K}_{(3,2),(2,2,1)}(q,t) = q^2 + t^3 + qt + qt^2 + t^2.$$

This formula is simple to compute compared to all other known formulas. Unfortunately, if we try to extend this to larger shapes, it fails. Even in the simplest case $\mu = (3,3)$, the formula fails. The hope is to modify the definition of Yamanouchi and find a formula when $\mu_1 = 3$. Though this has been unsuccessful thus far, we do have a few other larger families that have been solved.

3.3. Other Combinatorial Formulas for Families of Macdonald Polynomials

In the last 10 years, several new combinatorial formulas have been proven. There are two very different formulas concerning "doubly-augmented hook" Macdonald polynomials. These have $\mu_2 \leq 2$, i.e. they are extended two-column Macdonald polynomials. The main work on these families are found in Loehr (2017) and Assaf (2018). Additionally, the best known result can be found in Blasiak (2016). Here, he solves the "three-column" case, i.e., $\mu_1 \leq 3$. We now give an overview of these three different combinatorial formulas.

3.3.1. Loehr's Combinatorial Formula Using RSK Variants

Here, we consider the modified Macdonald polynomial with $\mu_1 \leq 3$ and $\mu_2 \leq 2$. The main new results in Loehr (2017) concerns new modified RSK algorithms. With these modified RSK algorithms, Loehr is able to translate Haglund's combinatorial formula and algebraic statistics into a Schur expansion. Recall the following algebraic statistics

$$maj_\mu(\sigma) = \sum_{u \in Des(\sigma)} leg(u) + 1$$

and

$$inv_\mu(\sigma) = |Inv(\sigma)| - \sum_{u \in Des(\sigma)} arm(u)$$

where σ is a filling of the diagram μ and $Inv(\sigma)$ is the set of inversions pairs that are attacking in μ. We now describe and equivalent characterization of $inv_\mu(\sigma)$. Consider any set of 3 cells in the Ferrer's diagram of the form:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \cdots \boxed{c},$$

where we set $b = \infty$ if a and c are in the bottom row. After standardization, we call the triple (a, b, c) an inversion triple if and only if $a < b < c$, $b < c < a$, or $c < a < b$. Then, $inv_\mu(\sigma)$ is equal to the total number of inversion triples in the diagram. Now, consider any $\pi \in S_n$, we will associate π to a tableau T by filling the diagram left to right, top to bottom with π. In other words, we sometimes associate a permutation π to a tableaux such that the reading word of the tableau, $rw(T)$ is equal to π. With this in mind, we have the following theorem.

Theorem 3.3.1 (Loehr (2017) Theorem 5)**.** *For all $n \geq 1$ and $\mu \vdash n$ with $\mu_1 \leq 3$ and $\mu_2 \leq 2$,*

$$\tilde{H}_\mu(X; q, t) = \sum_{\lambda \in Par(n)} \left(\sum_{T \in SYT(\lambda)} q^{inv_\mu(rw(T)^{-1})} t^{maj_\mu(rw(T)^{-1})} \right) s_\lambda(X).$$

Loehr notes that if we consider the simplest extension and let $\mu = (4)$, the theorem fails to hold. Thus, a different approach must be used for other cases. In order to prove this case, Loehr develops

a series of new rules for the RSK algorithm. His new RSK algorithm is built on "p-row insertion" which modifies the usual row insertion of RSK with a few special new rules depending on a parameter p. Just like the usual row insertion of RSK, p-row insertion is invertible. With the modification in mind, Loehr proves several properties of the modified RSK algorith, RSK_p. Here is a summary of the important properties found in Loehr (2017).

Theorem 3.3.2 (Properties of RSK^p in Loehr (2017))**.** *Let $RSK_p(\sigma) = (P_p(\sigma), Q_p(\sigma))$, then*

1. ***Theorem 14*** *RSK_p is a bijection from S_n to $\cup_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda)$.*
2. ***Theorem 18*** *For all $w \in S_n$, $Des(w) = Des(Q_p(w))$.*
3. ***Theorem 19*** *If $w \in S_n$ is the reading word of a partial standard tableau, then $P_p(w) = P(w) = w$ and $Q_p(w) = Q(w)$.*
4. ***Theorem 22*** *By extending Knuth relations to a p-Knuth relation, $\sim_p$, for all partial permutation w, $w \sim_p P_p(w)$*
5. ***Theorem 24*** *For all w, $Q_p(w) = Q(w)$.*

Many of these properties are identical to the usual RSK properties. Now, when $\mu_1 \leq 3$ and $\mu_2 \leq 2$, let $m_1(\mu)$ equal the total number parts of μ equal to 1 and $p(\mu) = m_1(\mu) + 1$. Now, define

$$RSK^{\mu}(w) = (P^{\mu}(w), Q^{\mu}(w)) = (Q_p(w^{-1}), P_p(w^{-1})).$$

This definition may seem strange, but Loehr is able to show this definition of RSK^{μ} satisfy the following extremely useful theorem.

Theorem 3.3.3 (Theorem 7 Loehr (2017))**.** *Let $\mu \vdash n$, and suppose we can construct an RSK-like algorithm $RSK^{\mu} = (P^{\mu}, Q^{\mu})$ that satisfies the following 3 conditions:*

1. *RSK^{μ} is a bijection from S_n to $\cup_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda)$.*
2. *For all $w \in S_n$, $IDes(w) = Des(P^{\mu}(w))$.*

3. For all $w \in S_n$, the values of $inv_\mu(w)$ and $maj_\mu(w)$ depend only on $Q^\mu(w)$ and not on $P^\mu(w)$.

Then, for all λ

$$\tilde{K}_{\lambda,\mu} = \sum_{T \in SYT(\lambda)} q^{\tilde{a}_\mu(T)} t^{\tilde{b}_\mu(T)}$$

where

$$\tilde{a}_\mu(T) = inv_\mu(w)$$

and

$$\tilde{b}_\mu(T) = maj_\mu(w)$$

for all $w \in S_n$ such that $Q^\mu(w) = T$. Or, equivalently,

$$\tilde{a}_\mu(Q) = inv_\mu((RSK^\mu)^{-1}(P, Q))$$

and

$$\tilde{b}_\mu(Q) = maj_\mu((RSK^\mu)^{-1}(P, Q)).$$

Here, Loehr shows his definition of RSK^μ satisfies the above theorem, which then proves Theorem 5 in Loehr (2017). Theorem 7 is of great interest because it provides a general framework for solving the Schur expansion problem. If we can develop an RSK variant that satisfies those properties, then we can find a Schur expansion for Macdonald polynomials. We will use this framework in the next chapter to prove a new positive combinatorial Schur expansion.

3.3.2. Assaf's Combinatorial Formula Using Dual Equivalence Graphs

Here, we give a combinatorial formula for Macdonald polynomials with restriction $\mu_2 \leq 2$. This work appears in Assaf (2018) and utilizes the theory of dual equivalence graphs. We consider a dual equivalence relation on the set of permutations.

Definition 3.3.4. Let $w \in S_n$ and $1 < i < n$. We define the ith *elementary dual equivalence* relation d_i the map such that if i is between $i-1$ and $i+1$ in w, then $d_i(w) = w$. Otherwise, d_i interchanges i with whichever of $i-1$ and $i+1$ is further away from i in w.

Now, suppose we consider all permutations in S_n with a fixed major index. Then, the d_i partition this set into equivalence classes, where two permutations are in the same equivalence class if there is a sequence of the d_i transforming one into the other. As an example, 2314 and 1423 are dual-equivalent as $d_2(2314) = 1324$ and $d_3(1324) = 1423$. It turns out that there are exactly two equivalence classes for S_4 with major index 2. They are $\{2314, 1324, 1423\}$ and $\{2413, 3412\}$.

Definition 3.3.5. Let $w \in S_n$ and $IDes(w) = \{i_1, i_2, \ldots, i_k\}$. We define the *de-standardization* of w to be the word obtained by changing 1 through i_1 to 1, $i_1 + 1$ through i_2 to 2, etc. We call the *weight* of the de-standardization to be the composition whose ith part is the number of $i's$ of the word.

As an example, consider the permutation $381265974 \in S_9$ with $iDes = \{2, 4, 5, 7\}$. The de-standardization is 251143542 with weight $(2, 2, 1, 2, 2)$. Now let $dst_k(w)$ be the de-standardization of the subword $w_k \cdots w_n$. Suppose that for all k, $dst_k(w)$ has at least as many $i-1's$ as $i's$. In other words, every suffix of $dst(w)$ has partition weight. Then, we call w a *super-standard* word. As an example, let $w = 518296734$, then $dst(w) = 213132211$ with weight $(4, 3, 2)$, and for all k, $dst_k(w)$ has partition weight and is thus a super-standard word. To demonstrate, $dst_4(w) = 132211$ which has weight $(3, 2, 1)$. With this in mind, the following is shown in Assaf (2018).

Theorem 3.3.6 (Theorem 7 Assaf (2018))**.** *Every permutation is dual-equivalent to a unique super-standard permutation. Further, the quasi-symmetric generating function of a dual equivalence class of a super-standard permutation is equal to the Schur function indexed by its weight.*

With this theorem in mind and the previous knowledge about maj being preserved in dual-equivalence classes, Assaf gives the following combinatorial formula for the single column Macdonald polynomial:

$$\tilde{H}_{(1^n)}(X; q, t) = \sum_{\lambda \vdash n} \left(\sum_{u \in SS(\lambda)} t^{maj(u)} \right) s_\lambda(X)$$

where $SS(\lambda)$ is the set of super-words with weight λ. Now, with the previous results about dual-

equivalence in mind, Assaf considers a modification which will turn into a formula for other families of Macdonald polynomials.

Definition 3.3.7. Let $w \in S_n$ and $1 < i < n$. We define the ith *elementary twisted dual equivalence* relation $\tilde{d}_i$ the map such that if i is between $i-1$ and $i+1$ in w, then $\tilde{d}_i(w) = w$. Otherwise, $\tilde{d}_i$ cyclically rotates $i-1, i, i+1$ so that i lies on the other side of $i-1$ and $i+1$.

The significance of these equivalence relations comes from preserving the number of inversions. Notice that when the relative order of $i-1, i, i+1$ satisfies the second part of the definition of twisted dual equivalence, the cyclical rotation preserves the number of inversions. Now, for a permutation w and w_i and w_j in w, we say w_i and w_j are potential μ-descents or μ-inversions if they form possible descents or inversions in the filling of μ by w. Finally, we have the following set of involutions:

Definition 3.3.8. Let $1 < i < n$, then define

$$D_i^{\mu} = \begin{cases} \tilde{d}_i(w) & \text{if both } i-1 \text{ and } i+1 \text{ are potential } \mu\text{-descents or potential } \mu\text{-inversions with } i \\ d_i(w) & \text{otherwise.} \end{cases} \tag{3.1}$$

As shown in Proposition 12 in Assaf (2018), maj_{μ} and inv_{μ} are fixed if we apply D_i^{μ}. Though not appearing in the statements of any of the combinatorial formulas for Assaf, the D_i^{μ} involutions are vital to the proofs. Assaf uses these involutions to build modified Foata maps. The Foata map is a bijection on S_n where the major index of a permutation is equal to the inversion number of its image. The Foata map is indexed by a number x at each step. We will let γ_x be the Foata map relative to x. Assaf defines a family of maps $\phi_k(w)$ defined as

$$\phi_k(w) = w_1 \cdots w_k \gamma_{w_k}(w_{k+1} \cdots w_n).$$

With this definition in mind and the previous comments on preserving inversions and major index, Assaf shows the following combinatorial formula.

Theorem 3.3.9 (Corollary 19 Assaf (2018)). *Let* $\mu = (n-k, 1^k)$ *and* $\psi_\mu = \phi_{k+1}\phi_{k+2}\cdots\phi_{n-1}$, *then*

$$\tilde{H}_\mu(X;q,t) = \sum_{\lambda\vdash n}\left(\sum_{u\in SS(\lambda)} q^{inv_\mu(\psi_\mu(u))}t^{maj_\mu(\psi_\mu(u))}\right)s_\lambda(X)$$

Now, we have seen similar combinatorial formulas for hook shape partitions. Assaf's goal is to extend this to partitions where $\mu = (n-2b-a, 2^b, 1^a)$. Her process is very similar in nature developing the formula for the hook shape. She first defines a map called β_x that swaps certain adjacent indices in a permutation. She then uses this map to create a family of bijections called $\sigma_{(k,m)}$ that applied b_x to $w_{k+1}\cdots w_{k+m}$. This family of maps is then used to construct a recursive family of maps $\phi_{a,b}$ that are very much in the spirit of the Foata map. The full details of these various maps are beyond the scope, but the overall goal is to have a bijection similar to the bijection for the hook shape that preserves inv and maj. The main result is now stated.

Theorem 3.3.10 (Corollary 23 Assaf (2018)). *Let* $\mu = (n-2b-a, 2^b, 1^a)$ *and* $\psi_\mu = \phi_{(a+2,b-1)}\phi_{(a+4,b-2)}\cdots\phi_{(a+2b,0)}\phi$ *then*

$$\tilde{H}_\mu(X;q,t) = \sum_{\lambda\vdash n}\left(\sum_{u\in SS(\lambda)} q^{inv_\mu(\psi_\mu(u))}t^{maj_\mu(\psi_\mu(u))}\right)s_\lambda(X).$$

This gives the first combinatorial formula for the family of Macondald polynomials indexed by $\mu = ((n-2b-a, 2^b, 1^a)$. Later, we give a new combinatorial formula for the case where $\mu = (n-2-a, 2, 1^a)$ and it is interesting to note that these formulas are quite different. We now briefly discuss the best known result for Schur expansions of Macdonald polynomials.

3.3.3. Blasiak's Three-Column Formula

We now consider a conjecture due to Haglund in Haglund (2004) and proven by Blasiak in Blasiak (2016). This formula depends on the expansion of Macdonald polynomials into a positive sum of LLT polynomials. We will briefly define LLT polynomials in this section with much greater detail

about LLT polynomials in Chapter 5. As defined in the proof of the HHL formula, consider

$$F_D(X;q) = \sum_{\substack{\sigma:\mu\to\mathbb{Z}_+ \\ Des(\sigma,\mu)=D}} q^{inv_\mu(\sigma)} x^\sigma$$

where D is a fixed descent set. We will see later that F_D is an LLT product of ribbons. Now, define

$$L_D = \sum_{s\in D} leg(s) + 1$$
$$A_D = \sum_{s\in D} arm(s).$$

With these definitions in mind, it is shown in Haglund et al. (2005b) that

$$\tilde{H}_\mu(X;q,t) = \sum_D t^{L_D} q^{-A_D} F_D(X;q)$$

where D is any potential set of descents in a filling of μ. Now, in Haglund (2004), this formula is slightly modified so that we have:

$$\tilde{H}_\mu(X;q,\vec{t}) = \sum_D F_D(X;q) \prod_{s\in D} t_s,$$

which can be viewed as a multi-t-variate version of the modified Macdonald polynomials. If we set $t_s = t^{leg(s)+1}$, we obtain the original formula. We now describe a weight associated to special fillings that leads to Haglund's conjecture. Let F_1 be the filling of μ whose reading word is the identity, and let F_2 be the filling of μ whose reading word is the reverse of the identity. Now, for $1 \le a < b \le n$, let $\mu(a) = A$ and $\mu(B) = B$ be the squares that a and b occupy in μ respectively. Consider the following weight contribution of a and b

$$wt(\mu,a,b) = \begin{cases} q & \text{if (A,B) is an inversion in } F_2 \\ q^{-arm(A)} t_a & \text{if A is a descent in } F_2 \text{ with } B = South(A) \\ 1 & \text{otherwise.} \end{cases}$$

The following is conjectured in Haglund (2004):

Conjecture 3.3.11 (Conjecture 3 Haglund (2004))**.** *If* $\mu_1 \leq 3$*, then*

$$\tilde{H}_\mu(X;q,\vec{t}) = \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} \prod_{(a,b) \in Inv(T)} wt(\mu,a,b) \right) s_\lambda.$$

In Blasiak (2016), this conjecture is proven. This gives a positive combinatorial formula for the Schur coefficients of the LLT product indexed by 3 partitions and gives the positive combinatorial formula for the modified Macdonald polynomials with at most 3 columns. The full details of the proof are beyond the scope here. It is important to see this formula because it is still the best known result for LLT products to this day and gives a combinatorial formula for the largest family of Macdonald polynomials. We now turn to an old conjecture from the 1990s on the Schur coefficients of Macdonald polynomials.

3.3.4. Butler's Conjecture

Our overarching goal is to find a new method of computing the Schur coeficients of Macdonald polynomials. In Butler (1994), we find an interesting pattern in the coefficients for a fixed λ, and variable μ. To illustrate this phenomenon, we consider two examples for $n = 5$.

μ	$\tilde{K}_{221,\mu}(q,t)$
5	$q^8+q^7+q^6+q^5+q^4$
4,1	$q^5+q^4t+q^4+q^3t+q^2t$
3,2	$q^4+q^3t+q^2t^2+q^2t+qt^2$
3,1,1	$q^3t+q^2t^2+qt^3+q^2t+qt^2$
2,2,1	$q^2t^2+qt^3+t^4+q^2t+qt^2$
2,1,1,1	$qt^4+t^5+qt^3+t^4+qt^2$
1,1,1,1,1	$t^8+t^7+t^6+t^5+t^4$

Table 3.1: Table of values of $\tilde{K}_{221,\mu}(q,t)$

Example 3.3.12. First, let us consider the coefficients in 3.1. Let us compare for $\mu = 5$ and $\mu = 4,1$: the coefficients are $q^8+q^7+q^6+q^5+q^4$ and $q^5+q^4t+q^4+q^3t+q^2t$, respectively. Here we see q^5 and q^4 occur in both, leaving $q^8+q^7+q^6$ and $q^4t+q^3t+q^2t$, respectively. Notice that $(q^8+q^7+q^6)q^{-4}t = q^4t+q^3t+q^2t$. This is just one example, but does this pattern persist. Let's look

μ	$\tilde{K}_{311,\mu}(q,t)$
5	$q^7+q^6+2q^5+q^4+q^3$
4,1	$q^5+q^4+q^3t+q^3+q^2t+qt$
3,2	$q^3t+q^3+2q^2t+qt^2+qt$
3,1,1	$q^2t^2+q^3+q^2t+qt^2+t^3+qt$
2,2,1	$qt^3+q^2t+2qt^2+t^3+qt$
2,1,1,1	$t^5+qt^3+t^4+qt^2+t^3+qt$
1,1,1,1,1	$t^7+t^6+2t^5+t^4+t^3$

Table 3.2: Table of values of $\tilde{K}_{311,\mu}(q,t)$

at the next two, $\mu=(4,1)$ and $\mu=(3,2)$. Following the same steps, we have $q^5+q^4t+q^4+q^3t+q^2t$ and $q^4+q^3t+q^2t^2+q^2t+qt^2$, respectively. Here we see q^4, q^3t, and q^2t occur in both, which is a bit more complex than the first case. However, if we ignore q^4 and q^2t, we have $(q^5+q^4t+q^3t)q^{-2}t=q^3t+q^2t^2+qt^2$. We can actually continue in this way and find that for two partitions μ and ν, as long as we only have to move one square to convert μ into ν, exactly two coefficients are fixed, and the remaining are exactly q^at^b away from the others. In our case, if $\mu=(4,1)$ and $\nu=(3,2)$, we have $a=-2$ and $t=1$. One can check that if we let $\mu=(4,1)$ and $\nu=(3,1,1)$, then $a=-3$ and $t=2$. Let's now look at a slightly more complicated example with the coefficients in 3.2. Let $\mu=(4,1)$ and $\nu=(3,2)$, so we are comparing $q^5+q^4+q^3t+q^3+q^2t+qt$ with $q^3t+q^3+2q^2t+qt^2+qt$. We now have 6 coefficients to consider. Notice that q^3t, q^3, q^2t, and qt all appear in both. However, if we ignore q^3, q^2t, and qt, then $(q^5+q^4+q^3t)q^{-2}t=q^3t+q^2t+qt^2$. Here, exactly 3 of the coefficients are fixed, and the other 3 change. In our case, if $\mu=(4,1)$ and $\nu=(3,2)$, we have $a=-2$ and $b=1$. Comparing to the case above, we got the same values of a and b. Now this is beginning to look interesting.

Recall the dominance order on partitions. We can use this dominance order to form a Hasse diagram for partitions. In a diagram of a partially ordered set, we say that x cover y if $x>y$ in the partial order, and there does not exist a z such that $x>z>y$. We can apply this to the ordering for partition and the associated Hasse diagram. Suppose $\mu>\nu$ and μ covers ν. When this is the case, we have two possible scenarios. Suppose $\mu-\nu=(i,j)$, then either:

1. $\nu-\mu=(i-a,j+1)$

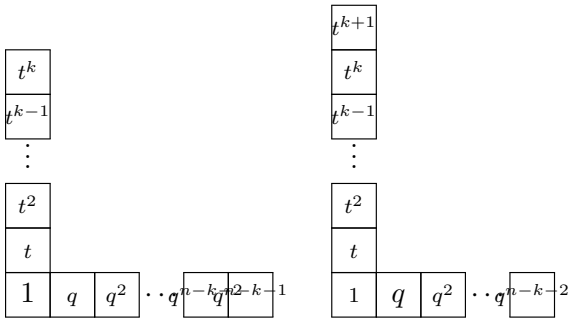
2. $\nu - \mu = (i-1, j+a)$.

In our example, let $\mu = 5$ and $\nu = (4,1)$, then $\mu - \nu = (5,1)$ and $\nu - \mu = (1,2)$. Thus, we are in the first case with $a = 4$. For a given covering as in this case, associate the term $q^{-a}t$. If the covering falls into the second case, associate $q^{-1}t^a$. With this in mind, we have the following conjecture due to Butler.

Conjecture 3.3.13 (Conjecture 2.7.1 Butler (1994))**.** *Let $\lambda, \mu, \nu \vdash n$ and $\mu > \nu$ such that μ covers ν such that $q^a t^b$ is the monomial associated to the covering. Then exactly f_2^λ coefficients change from $\tilde{K}_{\lambda,\mu}(q,t)$ to $\tilde{K}_{\lambda,\nu}(q,t)$, and they change exactly by $q^a t^b$, where f_2^λ is the number of Standard Young Tableau of shape λ with 2 in the first column.*

We can actually extend this conjecture even further. Suppose we have a chain of partitions $\mu_1 > \mu_2 > \cdots > \mu_k$ where μ_i covers μ_{i+1}. Then we can calculate the change of the coefficients from $\tilde{K}_{\lambda,\mu_1}(q,t)$ to $\tilde{K}_{\lambda,\mu_k}(q,t)$ as the product of $q^{a_i}t^{b_i}$ for each covering $\mu_i > \mu_{i+1}$. Consider the following example between hook shape partitions. In fact, the exact weight corresponding to the conjecture is very much related to B_μ.

Example 3.3.14. Let $\mu = (n-k, 1^k)$ and $\nu = (n-k-1, 1^{k+1})$. We can calculate the total weight in the conjecture as follows. Recall how we fill partitions to generate B_μ. The (q,t) filling from B_μ for these partitions is as follows:



Recall that B_μ is the sum of all the weights in these fillings. Note also that $B_\mu = B_\nu + q^{n-k-1} - t^{k+1}$. Butler's conjecture in this case says that the coefficients in B_μ will change by $q^{-(n-k-1)}t^{k+1}$.

With the previous example in mind, to reformulate the conjecture, let μ_ν be the square in μ but not in ν and let ν_μ be the square in ν but not in μ. Associate the term $q^{a_\mu}t^{b_\mu}$ and $q^{a_\nu}t^{b_\nu}$ as in B_μ and B_ν, respectively. Then, Butler's conjecture says the terms that change in $\tilde{K}_{\lambda,\mu}(q,t)$ to $\tilde{K}_{\lambda,\nu}(q,t)$ will change by $q^{a_\nu - a_\mu}t^{b_\nu - b_\mu}$.

We now prove Butler's conjecture for the hook shapes using two different formulas that we have seen.

Example 3.3.15. Recall when $\lambda = (n-k, 1^k)$, we have

$$\tilde{K}_{(n-k,1^k),\mu}(q,t) = e_k[B_\mu - 1].$$

Now, suppose we have μ and ν one square is moved from μ to ν. In particular, say that $B_\mu = B + 1 + q^{a_\mu}t^{b_\mu}$ and $B_\nu = B + 1 + q^{a_\nu}t^{b_\nu}$. Here we include the weight of 1 from the bottom left corner outside of the B term for ease when considering the known formula for the q,t-Kostka. Thus, B has exactly $n-2$ terms. With Butler's conjecture in mind, we can easily compute the number of Standard Young tableaux of shape λ with 2 above 1. In particular, the first column must have $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$, and the rest of the $k-1$ entries in the first column can be any subset of the other $n-2$ remaining numbers. In particular, we have $\binom{n-2}{k-1}$ such tableaux. Now, we need to compare the two formulas:

$$\tilde{K}_{(n-k,1^k),\mu}(q,t) = e_k[B_\mu - 1] = e_k[B + 1 + q^{a_\mu}t^{b_\mu} - 1] = e_k[B + q^{a_\mu}t^{b_\mu}]$$

and

$$\tilde{K}_{(n-k,1^k),\nu}(q,t) = e_k[B_\nu - 1] = e_k[B + 1 + q^{a_\nu}t^{b_\nu} - 1] = e_k[B + q^{a_\nu}t^{b_\nu}].$$

Thus, in both cases, we are computing $e_k[-]$ of a sum of $n-1$ terms, $n-2$ of which are the same. When doing this, our result is the sum of all products of k distinct elements in the sum. Now, if the k terms are all from terms in B, we get the same result for μ and ν. Thus, our only difference is when we have $k-1$ terms from the B terms, and the kth term is $q^{a_\mu}t^{b_\mu}$ for μ and $q^{a_\nu}t^{b_\nu}$ for ν. This means we have a total of $\binom{n-2}{k-1}$ terms that are multiplied by $q^{a_\mu}t^{b_\mu}$ for μ and the same number

of terms multiplied by $q^{a_\nu}t^{b_\nu}$. These $\binom{n-2}{k-1}$ terms are the same because we are choosing size $k-1$ subsets from the same B. Thus, the difference in these terms is from μ to ν is exactly $q^{a_\nu - a_\mu}t^{b_\nu - b_\mu}$ as desired. Further, the number of terms matches the conjectured number of terms from Butler.

Unfortunately, the previous example does not give a nice combinatorial description of $\tilde{K}_{(n-k,1^k),\mu}$. However, the following example does.

Example 3.3.16. Let $\mu = (n-k, 1^k)$, and recall the inv_μ and maj_μ statistics from the HHL formula in Haglund (2007). Because we are considering fillings of a hook shape, these statistics are easier to compute. In particular, if we fill μ with the word w, $maj_\mu(w) = maj(w_1 \cdots w_{k+1})$, the usual maj on words. In other words, maj for a hook is simply the maj of the column filling. Further, when we compute $inv_\mu(w)$, we are simply computing the usual inversion number, $inv(w_k \cdots w_n)$, the row of the filling. Recall that we had the Foata map, ϕ, such that $inv(w) = maj(\phi(w))$. With this map, w_1 is actually fixed. In this case, because maj_μ and inv_μ are both computed on w and the only common term is w_k, if we apply the Foata map to $w_k \cdots w_n$, we will not affect maj_μ because w_k is fixed. We now modify how we fill a hook shape diagram. Let $w_1 \cdots w_{n-k}$ fill the first row right to left, and let $w_{n-k+1} \cdots w_n$ fill the first column, starting from the second row and going up (note w_{n-k} is at the bottom of the first column). Because we are considering all possible words, we will still obtain the same weights, but it is going to be easier after we note the following important property about the RSK map. If $w_i > w_{i+1}$, then $i+1$ will be above i in the insertion Q tableaux of RSK(w). Also note that $maj(w_n \cdots w_{n-k+1}w_k$ is the same as $comaj(w_k \dots w_n)$. We obtain the following which is a reformulation of the Stembridge formula for hook shapes. First, let

$$maj_\mu(T) := \sum_{1 \le i \le n-k-1} i\chi_i(T) \tag{3.2}$$

and

$$comaj_\mu(T) := \sum_{n-k \le i \le n-1} (n-i)\chi_i(T). \tag{3.3}$$

where $\chi_i(T)$ is 1 if $i+1$ is above i in T, then

$$\tilde{K}_{\lambda,(n-k,1^k)}(q,t) = \sum_{T \in SYT(\lambda)} q^{maj_\mu(T)} t^{comaj_\mu(T)}. \tag{3.4}$$

Why were we interested in this in terms of Butler's Conjecture? It is now easy to prove. Suppose $\mu = (n-k, 1^k)$ and $\nu = (n-k-1, 1^{k+1})$. The above formula says that if $n-k$ is above $n-k-1$ in T, then $\chi_i(T)$ is true and we get a contribution of $n-k-1$ to maj_μ, but when we switch to ν, we no longer have this contribution, and instead, we now get $n-(n-k-1) = k+1$ contributing to $comaj_\nu$. Thus, for the same Standard Young tableau, the q,t- weight changes by $q^{-(n-k-1)}t^{k+1}$, which is the exact amount conjectured by Butler. Further, the number of Standard Young Tableaux of shape λ with 2 above 1 is the same as those with $n-k$ above $n-k-1$, hence the number of terms that change is also the same as the conjecture. Thus, Butler's conjecture is true for hook shapes.

Now that we have covered all of the necessary background material on RSK, symmetric functions, Schur expansions of Macdonald Polynomials, and Butler's conjecture, it is now time to consider the main result in this direction.

CHAPTER 4

COMBINATORIAL SCHUR EXPANSION OF AUGMENTED HOOK MACDONALD POLYNOMIALS

We now consider our main result concerning Macdonald polynomials and Butler's conjecture. In particular, we give a combinatorial formula for the Schur expansion of Macdonald polynomials indexed by an "augmented hook." This formula is a summation over Standard Young tableaux with statistics that strongly resemble maj and comaj. The proof of the formula is similar to the proof of the HHL formula as found in Haglund et al. (2005b) or in Appendix A of Haglund (2007). We show that the formula satisfies the same 3 uniqueness conditions that Macdonald Polynomials satisfy via 2 sign-reversing involutions. After proving this formula, we will then show how the formula immediately implies Butler's conjecture is true when our two partitions are a hook shape and an augmented hook shape.

4.1. Statement of Theorem

Let $\mu = (k-1, 2, 1^{n-k-1})$ be an augmented hook. Let λ be a partition of n and $SYT(\lambda)$ be the set of standard Young tableaux of shape λ. We define two separate algebraic statistics on Standard Young Tableaux. For T a Standard Young Tableaux, let $\chi_i(T)$ to evaluate to 1 if $i+1$ is above i in T, and 0 otherwise. Let $\chi_\mu(T^k)$ be defined according to jeu-de-taquin (jdt) of $k-2$, $k-1$, k, and $k+1$ in T and the chart in Table 4.1, then define

$$amaj_\mu(T) := \sum_{1 \le i \le k-1} i\chi_i(T) - (k-2)\chi_\mu(T^k), \tag{4.1}$$

and

$$acomaj_\mu(T) := \sum_{k \le i \le n-1} (n-i)\chi_i(T) + \chi_\mu(T^k). \tag{4.2}$$

Tableau, T	$\chi_\mu(T)$
1 2 3 4	0
2 1 3 4	0
3 1 2 4	1
4 1 2 3	0
3 2 1 4	1
4 2 1 3	0
4 3 1 2	1
3 4 1 2	0
2 4 1 3	1
4 3 2 1	1

Table 4.1: Table of $\chi_\mu(T)$

Note that we use the terms *amaj* and *acomaj* because both statistics are nearly the same as *maj* and *comaj*, but slightly modified for the augmented hook case. Let us compute the values of *amaj* and *acomaj* in an example.

Example 4.1.1. Let $\mu = (5, 2, 1^2)$ and consider the following SYT of shape $(4, 3, 2)$

$$T = \begin{array}{|c|c|c|c|}\hline 5 & 9 & \multicolumn{2}{c}{} \\ \hline 3 & 4 & 7 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 6 & 8 \\ \hline \end{array}$$

them, we can compute $amaj$ and $acomaj$ as follows:

$$amaj_\mu(T) := \sum_{1 \leq i \leq 5} i\chi_i(T) - (k-2)\chi_\mu(T^6),$$

and

$$acomaj_\mu(T) := \sum_{6 \leq i \leq 8} (9-i)\chi_i(T) + \chi_\mu(T^6).$$

In T, $\chi_i(T) = 1$ for $i \in \{2, 4, 6, 8\}$. Now, we need to compute $\chi_\mu(T^6)$. We compute this by first evacuating 1, 2, and 3 from T. When we do this, we have the following tableau:

$$ev_3(T) = \begin{array}{|c|c|c|}\hline 9 & \multicolumn{2}{c}{} \\ \hline 5 & 7 & \multicolumn{1}{c}{} \\ \hline 4 & 6 & 8 \\ \hline \end{array}$$

and to find T^k, we remove all numbers greater than $k+1$ and standardize $k-2$, $k-1$, k, and $k+1$ to 1, 2, 3, and 4 respectively. Thus we have:

$$T^k = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$$

and using Table 4.1, we have $\chi_\mu(T^k) = 1$. Thus, our statistics on T are:

$$amaj_\mu(T) := 2 + 4 - (6-2) * 1 = 2,$$

and

$$acomaj_\mu(T) := (9-6) + (9-8) + 1 = 5.$$

It turns out this tableau will contribute a weight of q^2t^5 to $s_{4,3,2}$ in $\tilde{H}_{(5,2,1^2)}(X; q, t)$.

Now, consider the following symmetric function:

$$V_\mu(X;q,t) = \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} q^{amaj_\mu(T)} t^{acomaj_\mu(T)} \right) s_\lambda. \tag{4.3}$$

This leads to our main result:

Theorem 4.1.2. *For $\mu = (k-1, 2, 1^{n-k-1})$ an augmented hook and $V_\mu(X;q,t)$ as defined above,*

$$V_\mu(X;q,t) = \tilde{H}_\mu(X;q,t).$$

4.2. Background Towards the Proof of 4.1.2

We will use tools tools developed in section 2 in order to prove the result. The most important tools are the RSK algorithm and Jeu-de-Taquin. We begin by showing how we actually evaluated $\chi_\mu(T)$ in Tablea 4.1.

4.2.1. Evaluating $\chi_\mu(T)$

We now describe how to evaluate $\chi_\mu(T)$ for a given tableau. Locate $k-2$, $k-1$, k, and $k+1$ in T. Now, evacuate numbers 1 through $k-3$ from T. This means to start with 1, convert it to a corner square, relabel 2 through n with 1 through $n-1$, and perform jdt on this skew tableau. Repeat this process $k-4$ more times. In the resulting tableau, $k-2$, $k-1$, k, and $k+1$ will be relabeled as 1, 2, 3, and 4 respectively. These four numbers will form a subtableau that is standard, call this T^k. We then define $\chi_\mu(T)$ on this subtableau of 4 numbers. We have all of the values for χ_μ in the previously shown 4.1.

Now, for a standard Young tableau of size n, we can compute T^k as above, which will be a standard Young tableau of size 4. However, it will be natural to evaluate χ_μ on permutations. We will now extend the definition of $\chi_\mu(T)$ to $\chi_\mu(\beta)$ where β is a permutation in S_4.

β	β^{-1}	$\chi(\beta)$
1234	1234	0
1243	1243	0
1324	1324	1
1342	1423	0
1423	1342	1
1432	1432	1
2134	2134	0
2143	2143	1
2314	3124	1
2341	4123	0
2413	3142	0
2431	4132	1
3124	2314	0
3142	2413	1
3214	3214	1
3241	4213	0
3412	3412	0
3421	4312	1
4123	2341	0
4132	2431	0
4213	3241	1
4231	4231	0
4312	3421	1
4321	4321	1

Table 4.2: Table of $\chi_\mu(\beta)$

In Table 4.2, we listed β^{-1} for a specific reason: $\chi(\beta) = \chi(jdt(\beta^{-1}))$. This will be important as we convert our Schur expansion from a sum over tableau to a sum over permutations.

4.2.2. Relation Between β and β^{-1} and RSK

We now discuss the relationship between a permutation and its inverse. Let $\beta = \beta_1\beta_2\cdots\beta_n$ and $\beta^{-1} = \beta_1^{-1}\beta_2^{-1}\cdots\beta_n^{-1}$. Suppose $\beta_i = j > \beta_{i+1} = l$, then $i+1$ is to the left of i in β^{-1}. In particular, $\beta_j^{-1} = i$ and $\beta_l^{-1} = i+1$, and $l < j$. We can extend this to β_{k-2}, β_{k-1}, β_k, and β_{k+1}. We can map these values to a permutation in S_4, say σ, and then if we compute σ^{-1}, this will be the relative order of $k-2$, $k-1$, k, and $k+1$ in β^{-1}. Now, recall the RSK algorithm and the fact that $Q_\beta = jdt(\beta^{-1})$. Now, if we want to compute χ_μ on Q_β, we have $\chi_\mu(Q_\beta) = \chi_\mu(Q_\beta^k)$, where we

evacuate 1 through $k-3$ from Q_β and look at the subtableau formed by the newly labelled 1, 2, 3, and 4. By the properties of jdt and the fact that $k-2$, $k-1$, k, and $k+1$ are consecutive numbers, this subtableau is equivalent to $jdt(\sigma^{-1})$. Thus, $\chi_\mu(Q_\beta) = \chi_\mu(\sigma)$. Further, note that $i+1$ is above i in Q_β if $\beta_i > \beta_{i+1}$. Thus, we can compute $amaj_\mu(Q_\beta)$ and $comaj_\mu(Q_\beta)$ completely in terms of β.

4.2.3. Gessel's Fundamental Quasisymmetric Functions

Our goal now is to follow the framework of the proof for of HHL formula in Haglund et al. (2005b). In order to do this we need to consider a few more symmetric function properties that we can find in Haglund (2007). First, we define Gessel's fundamental quasisymmetric function.

Definition 4.2.1. Consider two alphabets $A_+ = \{1, 2, \ldots, n\}$ and $A_- = \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ and $A_\pm = A_+ \cup A_-$, and a total order on these alphabets $\mathcal{TO}$. Let σ be a word in the alphabet $A_\pm$. We define the standardization of σ as $std(\sigma) = \tilde{\sigma}$ where equal positive letters are increasing left to right, and equal negative letters are decreasing left to right, and any unequal letters respect the total order. Let $D \subseteq [n-1]$. Then we define *Gessel's fundamental quasisymmetric function*, $F_{n,D}(X)$, as:

$$F_{n,D}(X) = \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i = a_{i+1} \implies i \notin D}} x_{a_1} x_{a_2} \cdots x_{a_n}$$

where $a_i \in A_+$. We can extend this definition to the *super fundamental quasisymmetric function*:

$$\tilde{F}_{n,D}(X,Y) = \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i = a_{i+1} \in A_+ \implies i \notin D \\ a_i = a_{i+1} \in A_- \implies i \in D}} x_{a_1} x_{a_2} \cdots x_{a_n}$$

where if $\bar{a} \in A_-$, then $x_{\bar{a}} = y_a$.

We use this to give an equivalent definition of s_λ. Let $SYT(\lambda)$ be the set of standard Young tableau of shape λ. Let $T \in SYT(\lambda)$, then we define $Des(T)$ to be the set of i such that $i+1$ appears in a row above i in T. Then, the following holds:

$$s_\lambda(X) = \sum_{T \in SYT(\lambda)} F_{n,Des(T)}(X)$$

With the above preliminaries in mind, we will derive an equivalent formula for V_μ as a sum over $\beta \in S_n$.

4.3. Proof of 4.1.2

We now prove the main theorem by showing the formula $V_\mu(X;q,t)$ satisfies the 3 Macdonald polynomial conditions. First, we recall this theorem from Haglund et al. (2005b).

Theorem 4.3.1. *The following 3 conditions uniquely determine a family $\tilde{H}_\mu(X;q,t)$ of symmetric functions:*

- *Condition 1 (C1):*
$$\tilde{H}_\mu[X(q-1);q,t] = \sum_{\rho \leq \mu'} c_{\rho,\mu}(q,t) m_\rho(X)$$

- *Condition 2 (C2)*
$$\tilde{H}_\mu[X(t-1);q,t] = \sum_{\rho \leq \mu} d_{\rho,\mu}(q,t) m_\rho(X)$$

- *Condition 3 (C3)*
$$\tilde{H}_\mu(X;q,t)|_{x_1^n} = 1.$$

We will show that V_μ satisfies these 3 conditions, and is thus equal to $\tilde{H}_\mu$. First, note that by definition, C3 is satisfied since x_1^n only appears when $\lambda = (n)$ since only one word has content 1^n, and this means the single row tableau is filled with all 1s. Further, for this tableau, $amaj_\mu$ and $acomaj_\mu$ evaluate to 0. Hence, the coefficient of x_1^n is 1. Now, the hard part is to prove C1 and C2. We will now derive an equivalent expression for V_μ. From our description of Gessel's fundamental quasisymmetric function and expanding Schur functions in the previous section,

$$\begin{aligned} V_\mu(X;q,t) &= \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} q^{amaj_\mu(T)} t^{acomaj_\mu(T)} \right) s_\lambda \\ &= \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} q^{amaj_\mu(T)} t^{acomaj_\mu(T)} \right) \left(\sum_{T' \in SYT(\lambda)} F_{n,Des(T')}(X) \right) \end{aligned}$$

and we now note that if this product is expanded, we get a sum over pairs of standard Young tableau of the same shape where the q and t powers depend on one, and the Gessel fundamental quasisymmetric function depends on the other. I.e., we obtain:

$$\begin{aligned} V_\mu(X;q,t) &= \sum_{\lambda\vdash n}\left(\sum_{T\in SYT(\lambda)} q^{amaj_\mu(T)}t^{acomaj_\mu(T)}\right)\left(\sum_{T'\in SYT(\lambda)} F_{n,Des(T')}(X)\right) \\ &= \sum_{\lambda\vdash n}\left(\sum_{(T',T)\in SYT(\lambda)\times SYT(\lambda)} q^{amaj_\mu(T)}t^{acomaj_\mu(T)}F_{n,des(T')}(X)\right) \end{aligned}$$

Suppose $\beta \to (T', T)$ under RSK, then, this can be rewritten as

$$\begin{aligned} V_\mu(X;q,t) &= \sum_{(T',T)\in SYT(\lambda)\times SYT(\lambda)} q^{amaj_\mu(T)}t^{acomaj_\mu(T)}F_{n,des(T')}(X) \\ &= \sum_{\beta\in S_n} q^{amaj_\mu(Q_\beta)}t^{acomaj_\mu(Q_\beta)}F_{n,des(P_\beta)}(X) \end{aligned}$$

Computing $amaj_\mu(Q_\beta)$ and $acom_\mu(Q_\beta)$ can be done with only knowing β. In particular, we already showed how to compute $\chi_\mu(\beta)$ which is consistent with Q_β, and computing $\chi_i(Q_\beta)$ is equivalent to checking if $\beta_i > \beta_{i+1}$. Hence, we can write $amaj_\mu(Q_\beta)$ and $acomaj_\mu(Q_\beta)$ as $amaj_\mu(\beta)$ and $acomaj_\mu(\beta)$, respectively. Further, $des(P_\beta)$ is the same as the set of i such that $i+1$ is to the left of i in β. Equivalently, this is equal to $des(\beta^{-1})$, which is the set of i such that $\beta_i^{-1} > \beta_{i+1}^{-1}$. With this in mind, we have:

$$V_\mu(X;q,t) = \sum_{\beta\in S_n} q^{amaj_\mu(\beta)}t^{acomaj_\mu(\beta)}F_{n,des(\beta^{-1})}(X)$$

We will now convert this formula in a similar fashion to the Appendix in Haglund (2007). Using equations (A.4) and (A.5) in Haglund (2007), the above formula can be rewritten as:

$$V_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}^+} q^{amaj_\mu(std(\sigma))}t^{acomaj_\mu(std(\sigma))}x^\sigma,$$

where σ is a filling of μ from left to right starting with the first row (the reason for viewing it like

this will be clear later), and std(σ) is the standardization of σ with respect to the total order. Now, using $(A.5)$ and (6.25) (see (A.15) for comparison) in Haglund (2007), along with our formula for V_μ in terms of permutations and Gessel fundamentals, we obtain:

$$\omega^Y V_\mu[X+Y;q,t] = \sum_{\beta \in S_n} q^{amaj_\mu(\beta)} t^{acomaj_\mu(\beta)} \tilde{F}_{n,des(\beta^{-1})}(X,Y).$$

Now, replace x_i with αx_i and y_i by $-x_i$, then the previous equation implies (see A.18 in Haglund (2007) book for reference):

$$V_\mu[X\alpha - X;q,t] = \sum_{\tilde{\sigma}:\mu \to \mathbb{A}_\pm} q^{amaj_\mu(std(\tilde{\sigma}))} t^{acomaj_\mu(std(\tilde{\sigma}))} \alpha^{pos(\tilde{\sigma})} (-1)^{neg(\tilde{\sigma})} x^{|\tilde{\sigma}|},$$

where $|\tilde{\sigma}|$ is obtained by replacing each negative letter $\bar{j}$ by j, pos and neg denote the number of positive and negative letters in $\tilde{\sigma}$ respectively, and $std(\tilde{\sigma})$ is with respect to the total order on $A_\pm$. With this in mind, we can now prove $C1$ and $C2$ by utilizing two different sign-reversing involutions and two different total orders.

4.3.1. Proof of C1

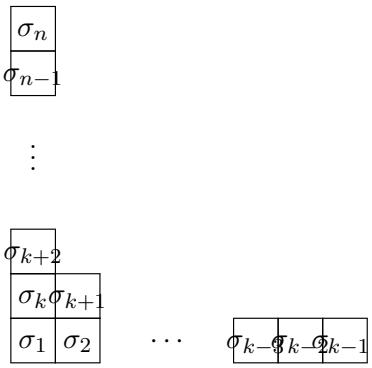
In order to prove $C1$, we will construct a sign-reversing involution on fillings of μ. We will use the total order:

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

When $\alpha = q$, we have:

$$V_\mu[Xq - X;q,t] = \sum_{\tilde{\sigma}:\mu \to \mathbb{A}_\pm} q^{amaj_\mu(std(\tilde{\sigma}))} t^{acomaj_\mu(std(\tilde{\sigma}))} q^{pos(\tilde{\sigma})} (-1)^{neg(\tilde{\sigma})} x^{|\tilde{\sigma}|},$$

and we will convert this to a sum over *nearly non-attacking* fillings of μ, where we view a filling starting in the first row, left to right, bottom to top. This is how we view the filling:



Nearly-non-attacking means that within any row of the filling, there will be at most one element from $\{i, \bar{i}\}$, with a few exceptions depending on $\sigma_{k-2}, \sigma_{k-1}, \sigma_k, \sigma_{k+1}$. We call σ_i and σ_j an *attacking pair* if they are in the same row and $|\sigma_i| = |\sigma_j|$. We now construct the sign-reversing, weight-preserving involution.

Case 1: Attacking $1's$ in First Row

Consider the filling $\tilde{\sigma}$ of μ. Suppose there exist attacking 1's in the first row. Further, of all such possible attacking pairs of 1's, consider the first pair (i.e., the two 1's that are furthest left). Then, $|\sigma_i| = |\sigma_j| = 1$, $i < j$, and i is minimal. Further, assume $i < k-2$. We claim that changing the sign (i.e. 1 changed to $\bar{1}$ and $\bar{1}$ changed to 1) of σ_i is a sign-reversing, weight-preserving involution, call the new filling $\tilde{\sigma'}$. To see this, first note that the sign clearly changes and the x power stays the same. Further, the value of χ_j will not change for any j besides possibly $i-1$ and i since σ_j and σ_{j+1} are unchanged outside of those values for j. Additionally, since $i < k-2$, $\chi_\mu(\tilde{\sigma})$ is unchanged as well. Hence, $acomaj_\mu(std(\tilde{\sigma}) = acomaj_\mu(std(\tilde{\sigma'}))$, and the t weight is preserved.

Now, we show the q weight is preserved. First, suppose $\sigma_i = 1$, then upon standardization, σ_i will be the smallest value, and hence $\chi_{i-1}(\tilde{\sigma}) = 1$, and $\chi_i(\tilde{\sigma}) = 0$. However, when switching the sign $\sigma_i' = \bar{1}$ is now the largest element after standardization, so $\chi_{i-1}(\tilde{\sigma'}) = 0$, and $\chi_i(\tilde{\sigma'}) = 1$. Thus, $amaj_\mu(std(\tilde{\sigma})) = amaj_\mu(std(\tilde{\sigma'})) - 1$. However, $pos(\tilde{\sigma}) = pos(\tilde{\sigma}) + 1$, and thus, the $q - weight$

is preserved. Hence, this is a sign-reversing involution. Note that the argument for if $\sigma_i = \bar{1}$ is identical.

In fact, this same argument works if we have $i = k-2$, but we have to be careful when checking $\chi_\mu(\tilde{\sigma})$. When we have $\sigma_{k-2}, \sigma_{k-1}, \sigma_k, \sigma_{k+1}$ versus $\sigma'_{k-2}, \sigma'_{k-1}, \sigma'_k, \sigma'_{k+1}$, if $\sigma_{k-2} = 1$, then it is the smallest element of this set under standardization, and thus $\sigma'_{k-2} = \bar{1}$ is now the largest upon standardization. As an example, if the relative order was 1324, it will be 4213 in $\tilde{\sigma'}$. We can check that $\chi_\mu(\tilde{\sigma}) = \chi_\mu(\tilde{\sigma'})$ in all six of these cases using Table 4.2. Hence, all statistics are fixed, and this is a sign-reversing, weight-preserving involution. Thus, we have eliminated all fillings where there is a $|\sigma_j| = 1$ for $j < k-1$, so there is at most one 1 or $\bar{1}$ in the first row. In fact, the above argument actually used no knowledge of the position of the second 1 or $\bar{1}$, and can be used to show that we can eliminate all fillings with a 1 or $\bar{1}$ in the first $k-2$ values. Hence, if there is a 1 or $\bar{1}$ in one of the first $k-2$ positions, we have a second filling with the opposite weight and sign cancelling it out. Thus, if there is a 1 or $\bar{1}$ in the first row, we can assume it is σ_{k-1}.

Case 2: Attacking 1's in the Second Row

With the previous eliminated cases in mind, we now consider when $|\sigma_k| = |\sigma_{k+1}| = 1$. This leads to our *first allowed exception to non-attacking.* If there are no $1's$ or $\bar{1}'s$ in the first row then this filling will not violate the triangularity conditions with respect to the $1's$ and $\bar{1}'s$ because we are allowed $n-k+1$ $1's$ and $\bar{1}'s$ and if there are none in the first row, there can be a maximum of $n-k+1$ in the filling. Now, if there is a 1 or $\bar{1}$ in the first row, then it must be σ_{k-1} after eliminating fillings from Case 1. We claim that switching the sign of σ_{k-1} is a sign-reversing, weight-preserving involution. As in Case 1, the only values of $\chi_i(\tilde{\sigma}))$ that could change in $\chi_i(\tilde{\sigma'})$ are $i = k-2$ or $k-1$, and in fact, just as before, if $\sigma_{k-1} = 1$, then $\sigma'_{k-1} = \bar{1}$, and we are switching from the smallest value to largest value upon standardization, so χ_{k-2} changes from 1 to 0, and χ_{k-1} changes from 0 to 1. Now, I claim that $\chi_\mu(\tilde{\sigma}) = \chi_\mu(\tilde{\sigma'})$, and this would imply all statistics are preserved, ensuring a sign-reversing involution. Again, this is a case by case basis that needs to be checked. There are 8 total possible cases. As an example, supposed $\sigma_{k-2} = X$, $\sigma_{k-1} = 1$, $\sigma_k = \bar{1}$, and $\sigma_{k+1} = 1$, then $\sigma'_{k-2} = X$, $\sigma'_{k-1} = \bar{1}$, $\sigma'_k = \bar{1}$, and $\sigma'_{k+1} = 1$, and upon standardization the relative orders are 3142

and 2431 respectively (note that X is not 1 or $\bar{1}$, so we can deduce its relative order). When we check all 8 of these cases (really only 4 cases as they come in pairs), the value of χ_μ remains the same. Thus, we can eliminate these fillings as well. Now, we either have a filling with no 1's in the first row, and zero, one, or two one's in the second row, or a filling with one 1 in the first row (at σ_{k-1}), and zero or one 1 in the second row.

With these cases in mind, it is now irrelevant what the value of $\sigma_{k+2}, \ldots, \sigma_n$ are, as we have at most $n-k+1$ $1's$ and $\bar{1}'s$ regardless. Thus, all of our remaining fillings are nearly non-attacking with respect to 1 and $\bar{1}$. Next, we take these cases and consider what happens with attacking 2's.

Case 3: Attacking 2's

By assumption, we have narrowed it down to fillings that are nearly non-attacking with respect to 1's. Suppose we have attacking 2's in the first row. First, if there is a one in the first row, it must be in position σ_{k-1}. Thus, if there are attacking 2's, the left most must be in position at most $k-3$, say σ_i with $i \leq k-3$. In particular, this means that after standardization, it will be either the smallest, or largest element among the first $k-2$ numbers. In this case, we claim the exact same involution works as in Case 1. I.e., switch the sign of σ_i and all χ_j values are fixed besides χ_{i-1} and χ_i, and both χ_{i-1} and χ_i switch values using the same argument from Case 1. Further, by assumption, $i \leq k-3$, so the value of χ_μ will be unchanged. Thus, the weights will be preserved, and this is a sign-reversing, weight-preserving involution.

Now, suppose there are no 1's in the first row, and there are attacking 2's. If the leftmost 2 is σ_i and $i \leq k-3$, the same argument applies, and we can reverse its sign, and the proof follows. However, if $i = k-2$, we leave this filling and consider it nearly non-attacking with respect to 1's and 2's. We can do this because we know there are at most $n-k+1$ $1's$ and $\bar{1}'s$ above the first row, and at most only 2 $1's$, $\bar{1}'s$, $2's$, and $\bar{2}'s$ in the first row. This gives us a total of at most $n-k+3 = \mu'_1 + \mu'_2$, satisfying triangularity with respect to 1's and 2's.

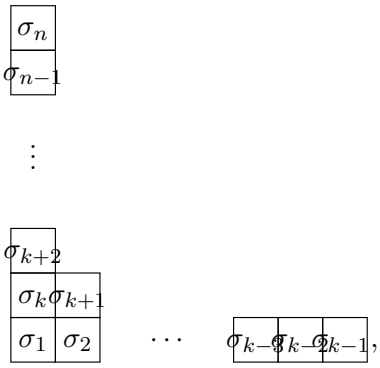
We now note that we need not consider attacking $2's$ (and in case 4 any attacking $j's$) in the second row at this point. Since our case analysis shows we can assume no attacking 1's or 2's in the first

row or just nearly non-attacking case of 2's in the first row, if the filling does have attacking 2's in the second row, we cannot violate triangularity conditions with respect to 1's and 2's regardless of the remaining filling (in fact we will not need to consider these cases for other $j's$ either) for the same triangularity argument. Like in the previous case, suppose there is only a single 2 in the first row. If it is in position σ_{k-2} or σ_{k-1}, we do nothing. If it is in position $i < k-2$, we can change the sign and have a sign-reversing involution as before. We specified these conditions specifically to avoid potential issues with evaluating χ_μ. However, this implies we have fillings with no $2's$ before position $k-2$. Thus, we have triangularity with respect to 1's and 2's.

Case 4: Attacking $j's$

The remaining case will be similar to the previous case. We are in the position with nearly non-attacking fillings with respect to $1's$ and $2's$. Consider the first pair of attacking $3's$ in the first row in positions i and j, with $i < j$. If $j <= k-2$, then we can proceed directly as before and change the sign of σ_i, as this will not change χ. Now, suppose $j = k-1$, so we have exactly one pair of attacking $3's$ in the first row. If $i < k-3$, then we can change the sign of σ_i and have the desired sign-reversing involution. If $i = k-3$, and $|\sigma_{k-2}| = 2$, then we leave this case and add it to the nearly non-attacking category (note this case is possible after removing attacking 2's in the previous case). Similarly, if $i = k-2$, we leave this case and add it to the nearly non-attacking category. Why are we skipping these cases? If we try to change the sign of σ_{k-2}, we may change the value of χ_μ, and if we try to change $\sigma_i's$ sign, we may not necessarily change the value of χ_i if $|\sigma_{i+1}| < |\sigma_i|$, which means the statistics may not match. Similarly, if there is only one 3 in the first row, it is not σ_{k-2} or σ_{k-1}, and $|\sigma_i| > |\sigma_{i+1}|$, then we can change the sign of σ_i and obtain the correct statistics. We continue with this procedure, and we are left with fillings of the following form after considering

$j's$ and $\bar{j}'s$:



where the segment of σ from σ_i through σ_{k-1} is of one of the following forms:

$$|\sigma_i| > |\sigma_{i+1}| > \cdots > |\sigma_{k-3}| > |\sigma_{k-2}| = |\sigma_{k-1}|$$

$$|\sigma_i| > |\sigma_{i+1}| > \cdots > |\sigma_{k-3}| > |\sigma_{k-2}| > |\sigma_{k-1}|$$

$$|\sigma_i| > |\sigma_{i+1}| > \cdots > |\sigma_{k-3}| > |\sigma_{k-2}| < |\sigma_{k-1}|$$

In other words, in terms of absolute value, we have a strictly decreasing sequence, except possibly σ_{k-2} and σ_{k-1}. We can do this because any time we have $|\sigma_l| < |\sigma_{l+1}|$ where the value of σ_l is the first occurrence and $|\sigma_p| > |\sigma_l|$ for $p < l$, we can switch its sign like above (aside from when $l = k-2$).

Our remaining cases are those that have a strictly decreasing sequence like the three possibilities above. But this implies that we have either 1 or 2 $a's$ for $|a| <= j$ in the row, where having two $a's$ only occurs when $|\sigma_{k-1}| = a$. We allow this as a nearly non-attacking filling, and this satisfies triangularity as we have at most $n-k+j-1$ numbers from $\{1, \bar{1}, \ldots, j, \bar{j}\}$, and we are allowed $\mu'_1 + \mu'_2 + \cdots + \mu'_j = n-k+1+2+1+\cdots+1 = n-k+j-1$. Thus, we are left with fillings that are nearly non-attacking and satisfies $C1$.

Condensed and Easier Proof to Follow Showing C1

First, let $\tilde{\sigma} = \sigma_1\sigma_2\cdots\sigma_n$ be the filling. Note that the first $k-1$ terms correspond to the first row. Find the smallest $i < k-1$ where $|\sigma_i| = 1$, if it exists. This will either be the largest or smallest element after standardizing. Change the sign of σ_i, and this is a sign-reversing involution as above.

If no i exists, we next check to see if $|\sigma_{k-1}| = |\sigma_k| = |\sigma_{k+1}| = 1$. If this is true, change the sign on σ_{k-1}, and this is a sign-reversing involution as above.

If none of this is true, we move on to finding the first i such that $|\sigma_i| = 2$ and $i < k-2$. If such an i exists, we change the sign of σ_i and note that by eliminating the previous cases, we are guaranteed that σ_i is either the smallest or largest element after standardizing among the first $i+1$ elements, and thus preserve the statistics.

If we still have not changed $\tilde{\sigma}$, we repeat this for 3, and find the first $i < k-2$ such that $|\sigma_i| = 3$. We then check to see if $|\sigma_{i+1}| >= 3$. If this is the case, we change the sign of σ_i, and it is a sign-reversing involution because we are guaranteed σ_i is the smallest or largest number after standardizing among the first $i+1$ elements. Further, note that the possible fringe cases only occur if $|\sigma_{i+1}| = 1$ or 2 which only happens if $i = k-3$. Now repeat the process for 3 with all larger numbers.

With the above analysis, we will map $\tilde{\sigma}$ to a filling with the same weight but opposite sign in all cases aside from the following:

$$|\sigma_1| > |\sigma_2| > \cdots > |\sigma_{k-3}| > |\sigma_{k-2}|$$

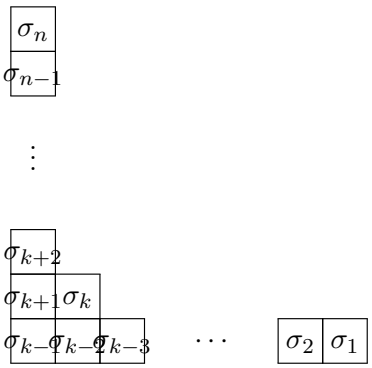
and σ_{k-1} is arbitrary. We already know that the number of 1's and $\bar{1}'s$ is at most μ'_1. With the remaining $\tilde{\sigma}$, because of the strictly decreasing sequence, we are checking for the maximum number of elements from $\{1, \bar{1}, 2, \bar{2}, \ldots, j, \bar{j}\}$. We can have at most j in the first row (the final j), 2 in the second row, and $n-k-1$ above the second row. This gives a maximum of $n-k-1+j+2 = n-k+j+1 = \mu'_1 + \cdots \mu'_j$, satisfying the triangularity for all j as desired. Thus, we are left with nearly non-attacking fillings and have proven $C1$.

4.3.2. Proof of C2

This proof will be modelled after the proof for $C1$ in the previous section. We use the following ordering, which is the opposite of the ordering for the $C1$ proof:

$$\bar{1} < \bar{2} < \cdots < \bar{n} < n < n-1 < \cdots < 2 < 1$$

We also consider fillings of μ from bottom to top, right to left for easier understanding. If $\tilde{\sigma} = \sigma_1 \cdots \sigma_n$, then the filling is:



Our goal is to find a sign-reversing, weight-preserving involution so that the remaining fillings are those where all of the $1's$ are in the first row, $2's$ in the first two rows, and in general, $j's$ in the first j rows. Here is the process.

If there is a 1 or $\bar{1}$ in position σ_i for $i \geq k+1$, pick the largest i. We change the sign of σ_i. If $\sigma_i = 1$, then after standardizing, it is the largest element, hence $\chi_{i-1}(\tilde{\sigma}) = 0$ and $\chi_i(\tilde{\sigma}) = 1$. After changing the sign, in $\tilde{\sigma'}$, $\chi_{i-1}(\tilde{\sigma'}) = 1$ and $\chi_i(\tilde{\sigma}) = 0$, as $\sigma_i' = \bar{1}$ is now the smallest number after standardizing. Hence, changing from 1 to $\bar{1}$ gains 1 t in $acomaj_\mu$, but loses 1 t from $pos(\tilde{\sigma})$. Thus, the t statistic is fixed. Note that the value of χ_μ is also fixed. There are six cases to check for the possible relative orders of $\sigma_{k-2}, \sigma_{k-1}, \sigma_k, \sigma_{k+1}$ when $|\sigma_{k+1}| = 1$ as in the case for $C1$. For example, if the relative order is 2314, then the involution changes the relative order to 3421, and χ_μ on these is the same using Table 4.2. The q-statistic and x weight are also clearly fixed in this case. Thus, this is a sign-reversing involution that preserves the statistics. This eliminates all cases for $1's$ outside the first row besides $|\sigma_k| = 1$.

To satisfy the triangularity condition, we are allowed $k-1$ total $1's$ in a filling. If either $|\sigma_{k-2}|$ or $|\sigma_{k-1}|$ is not 1, we allow this as a permissible filling with respect to 1's, and move this filling to the next step. Otherwise, consider the fillings where $|\sigma_{k-2}| = |\sigma_{k-1}| = |\sigma_k| = 1$ and $|\sigma_{k+1}| \neq 1$. We claim that changing the sign of σ_{k-1} is a weight preserving involution. The key observation is that

in these cases, if $\sigma_{k-1} = 1$, then $\chi_\mu(\tilde{\sigma}) = 0$, and if $\sigma_{k-1} = \bar{1}$, then $\chi_\mu(\tilde{\sigma}) = 1$. Hence, changing the sign changes the value of χ_μ. Thus, changing from 1 to $\bar{1}$ increases $acomaj_\mu$ by 1 and the number of positive values decreases by 1, so the t statistic is preserved. Further, $\chi_{k-2}(\tilde{\sigma}) = 0$, but $\chi(\tilde{\sigma'}) = 1$, and the value of $\chi_{k-1}(\tilde{\sigma})$ is unchanged. Hence, χ_{k-2} increases the q-power by $k-2$, but χ_μ decreases the q-power by $k-2$. With this map the q-power is fixed. Thus, we have a sign-reversing involution such that the remaining fillings have at most $k-1$ $1's$ and $\bar{1}$, and, critical to the remaining of the proof, the maximum i such that $|\sigma_i| = 1$ is k.

With the above fillings eliminated, we consider $2's$ and $\bar{2}'s$. Find the last i such that $|\sigma_i| = 2$ and $i \geq k+2$. If one exists, change the sign, which will change this from the smallest to largest or largest to smallest upon standardization among all σ_j for $j > k$ (this is guaranteed when $i = k+2$ as the previous involution prevents $|\sigma_{k+1}| = 1$). Now, changing the sign preserves weights for the same reason as with the $1's$ and $\bar{1}'s$. Hence, after removing the fillings from this involution, this guarantees all $2's$ are in the bottom two rows.

The same logic applied for larger $j's$. Consider the last j or $\bar{j}$ above the first j rows in the filling. If it is in position i, then upon standardization, σ_i is either the largest or smallest number among $\sigma_{i-1}, \sigma_i, \ldots, \sigma_n$ (note because of the iterative process, we have guaranteed $|\sigma_{i-1}| >= j$). Thus, changing the sign changes the values of χ_{i-1} and χ_i, and the overall t weight will be unchanged. The q-weight will also be fixed and value of χ_{mu} will also be fixed. Thus this is a sign-reversing involution in total, and we have the desired triangularity from $C2$. This completes the proof of Theorem 4.1.2.

4.4. Implications of the Combinatorial Formula for Augmented Hooks

Now that we have proven the main theorem, we can now consider an important corollary in regard to Butler's Conjecture.

Corollary 4.4.1. *Let $\mu = (k, 1^{n-k})$ a hook shape and $\nu = (k-1, 2, 1^{n-k-1})$ an augmented hook shape. Then the Schur coefficients of $\tilde{H}_\nu(X; q, t)$ and $\tilde{H}_\mu(X; q, t)$ satisfy Butler's conjecture.*

Proof. Recall the following Schur expansions for $\tilde{H}_\nu(X;q,t)$ and $\tilde{H}_\mu(X;q,t)$:

$$\tilde{H}_\mu(X;q,t) = \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} q^{maj_\mu(T)} t^{comaj_\mu(T)} \right) s_\lambda$$

and

$$\tilde{H}_\nu(X;q,t) = \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} q^{amaj_\mu(T)} t^{acomaj_\mu(T)} \right) s_\lambda.$$

We need to compare maj_μ with $amaj_\mu$ and $comaj_\mu$ with $acomaj_\mu$. Recall the following four formulas:

$$\begin{aligned}
maj_\mu(T) &= \sum_{1 \le i \le k-1} i\chi_i(T) \\
comaj_\mu(T) &= \sum_{k \le i \le n-1} (n-i)\chi_i(T) \\
amaj_\nu(T) &= \sum_{1 \le i \le k-1} i\chi_i(T) - (k-2)\chi_\nu(T^k) \\
acomaj_\nu(T) &= \sum_{k \le i \le n-1} (n-i)\chi_i(T) + \chi_\nu(T^k).
\end{aligned}$$

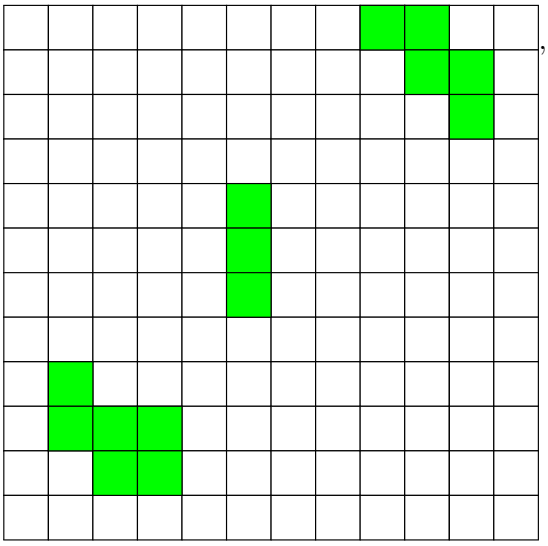
In particular, these statistics are exactly the same unless $\chi_\nu(T^k) = 1$. In this case for the given T, the weight of T changes by $q^{-(k-2)}t$, the exact amount from Butler's conjecture. Further, the number that changes depend on the result of jdt on $k-2$, $k-1$, k, and $k+1$ in a given T. The number of such cases is exactly the same as the number of $SYT(\lambda)$ with 2 above 1 because of the definition of χ_ν. In particular, if the result of jdt is not the shape 22, we check if k is above $k-1$ in jdt. If it is 22, then we check the exact opposite. This verifies the number conjectured by Butler, proving the corollary. □

CHAPTER 5

e-EXPANSION OF LLT POLYNOMIALS

Our goal is to give a combinatorial interpretation of coefficients of a special family of LLT polynomials. In 1997, Lascoux, Leclere, and Thibon introduced what are now known as LLT polynomials. LLT polynomials are defined in terms of a tuple of skew partitions.

Here is an example of a tuple of 3 skew shapes in the 12×12 grid.

$Y_\lambda =$  ,

In order to form a term in the LLT polynomial, we fill each skew shape so that we obtain a semistandard Young tableaux. For a given filling, the x term is the product of all $x_i^{k_i}$ where i occurs k_i times in the filling of the skew shapes. We also get a q weight from the filling. We define an algebraic statistic called an inversion when two numbers i and j are on the same diagonal of the grid, the smaller number is above and to the right of the bigger number, and the numbers are in different skew shapes. We also have an inversion if two numbers i and j are on successive diagonals and in different shapes of the diagram, and the bigger number is on higher diagonal and to the right of the smaller number. For a given filling, we obtain the weight $q^{|inv|}$.

Here is the formal definition of LLT polynomials.

Definition 5.0.1 (LLT Polynomials)**.** Given a tuple of skew shapes $\gamma = (\lambda^{(1)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \ldots, \lambda^{(k)}/\mu^{(k)})$, where $\lambda^{(i)}$ is a partition, and $\mu^{(i)}$ is a subpartition of $\lambda^{(i)}$, define the LLT polynomial as:

$$LLT_\gamma(X, q) = \sum_{\substack{\text{SSYT fillings } \sigma \\ \text{of shape } \gamma}} X^\sigma q^{inv(\sigma,\gamma)}.$$

Note that in the previous chapter, we saw a similar characterization of $LLT_\gamma(X, q)$. In particular,

$$LLT_\gamma(X, q) = \sum_{T \in SYT(\gamma)} q^{inv(T)} F_{Des(T)}(x).$$

We will consider some special cases of LLT polynomials. First, if all parts of γ consists of single boxes, we call this a *unicellular* LLT polynomials. Next, if each part of γ is a single column, we call this a *vertical strip* LLT polynomial. With these definitions of LLT polynomials in mind, we will consider some recent, interesting results found in Abreu and Nigro (2021), Alexandersson and Panova (2018), Alexandersson and Sulzgruber (2022), and D'Adderio (2020).

5.1. Chromatic Polynomial and its Extensions

We now consider the chromatic polynomial of a graph and its extensions. Let G be a graph. We say a coloring of G, denoted $\kappa(G)$, is proper if whenever u and v are connected, the color of u is not the same as the color of v. We us this to define the chromatic polynomial for a graph G. Let $\chi_k(G)$ be the number of colorings of the graph G with exactly k colors. Then,

$$\chi(G) = \sum_{i=1} \chi_G(k) x^k$$

is the classic chromatic polynomial. Stanley used this to define a new symmetric function in Stanley (1995). Formally, let $\kappa : V(G) \to \mathbb{N}$ be the map associated to a coloring of the vertices of G. Let $x_\kappa = \prod_{v \in V(G)} x_{\kappa(v)}$. Now, define the chromatic symmetric function of a graph G to be

$$X_G(X) = \sum_\kappa x_\kappa$$

where the sum is over all proper colorings κ of G. It turns out that this polynomial is in fact symmetric, and a great deal of research concerning these polynomials has occurred over the years. Eventually, in 2012, Shareshian and Wachs introduced a new extension that has spawned extensive research in recent years. The full details are in Shareshian and Wachs (2016). We now define this new extension. We will associate an algebraic statistic to a coloring of G. Let E be the set of edges in G, and let κ be a coloring. Then, define

$$asc(\kappa) = |\{(i,j) \in E, i < j, \kappa(i) < \kappa(j)\}|,$$

we consider this to be the number of ascents of the coloring. Then, we have the following q-analogue of the chromatic symmetric function:

$$X_G(X,q) = \sum_{\kappa} q^{asc(\kappa)} x_\kappa$$

the *chromatic quasisymmetric function*. These functions are not symmetric in general, hence the 'quasi' designation. However, there is a large family of graphs such that the above is a symmetric function. We will introduce a few key concepts to develop these special graphs.

Definition 5.1.1. Let P be a finite Poset with respect to some order. We form the *incomparability graph, inc(P),* by letting the elements of P to be the vertices of the graph, and we have an edge between any two elements that are incomparable with respect to the order.

With the above in mind, we will define a special order to form a Poset.

Definition 5.1.2. Let $[a_i + 1]$ for $i \in [n]$ such that $a_i < a_j$ be a set of n intervals of unit length in $\mathbb{R}$. We form the *natural unit interval order Poset* P on the set $[n]$ by letting $i <_P j$ if $a_i + 1 < a_j$.

The above definition may seem strange and unmotivated, but it turns out that there is a bijection between the set of the natural unit interval Posets and the set of regular semisimple Hessenberg varieties of type A_{n-1} Shareshian and Wachs (2016). We now define the concept of an inversion with respect to a graph G.

Definition 5.1.3. Let $G = ([n], E)$ be a graph on $[n]$ with edge set E. Let $\sigma \in S_n$. If $(\sigma(i), \sigma(j)) \in E$, then we say this is a $G - inversion$ if $i < j$ and $\sigma_i > \sigma_j$. We let $inv_G(\sigma)$ be the number of $G - inversions$ of σ.

We can also define descents with respect to a Poset.

Definition 5.1.4. Let P be a Poset on $[n]$ and $\sigma \in S_n$. We define a $P-descent$ i of σ if $\sigma_i >_P \sigma_{i+1}$. We let $des_P(\sigma)$ be the set of $P - descents$ of σ.

Both of these are extensions of the usual inversion and descent statistics on permutations. In fact, if G is the complete graph, $inv_G(\sigma) = inv(\sigma)$, and if P is the usual order on integers, then $des_P(\sigma) = des(\sigma)$.

The following is a major theorem in Shareshian and Wachs (2016).

Theorem 5.1.5 (Theorem 4.5 Shareshian and Wachs (2016))**.** *Let G be the incomparability graph of a natural unit interval order Poset. Then, the chromatic quasisymmetric function $X_G(X, q)$ is symmetric.*

After proving this theorem and examining the expansions of these polynomials into different bases, the authors conjectured the following.

Conjecture 5.1.6 (Conjecture 5.1 Shareshian and Wachs (2016))**.** *Let G be the incomparability graph of a natural unit interval order Poset. Then, the chromatic quasisymmetric function $X_G(X, q)$ is e-positive.*

We will also consider a special family of graphs whose chromatic quasisymmetric function is Schur positive. To do this, we will use a Poset extension to fillings of tableau.

Definition 5.1.7. Let P be a Poset on $[n]$ and $\lambda \vdash n$. We consider fillings of the Young diagrams of shape λ to be a $P - tableau$ if

- Each element of $[n]$ appears exactly once.

- If y appears immediately to the right of x in the filling, then $y >_P x$.
- If y appears immediately above x in the fillings, then $y \not<_P x$.

We let T_P to be the set of all $P - tableau$. For $T \in T_P$ and $G = inc(P)$, then we define the $G - inversion$ of T to be an edge in G such that $i < j$ and i appears above j in T. We let $inv_G(T)$ to be the set of inversions in T and $\lambda(T)$ to be the shape of T.

With the above definition in mind, we have the following Schur expansion:

Theorem 5.1.8 (Theorem 6.4 Shareshian and Wachs (2016))**.** *Let G be the incomparability graph of a natural unit interval order Poset. Then, we have the following*

$$X_G(X, q) = \sum_{T \in T_P} q^{inv_G(T)} s_{\lambda(T)}.$$

5.2. Dyck Paths and Unicellular LLT Polynomials

We now consider a combinatorial formulation of natural unit interval graphs and their connections to chromatic quasisymmetric functions as found in Alexandersson and Panova (2018).

Definition 5.2.1. A *circular unit arc digraph* is a directed graph with vertices $[n]$ and edge set:

$$(i - a_i) \to i, \quad (i - a_i + 1) \to i, \quad \dots \quad , \quad (i - a_i + (a_i + 1)) \to i$$

for all $i \in [n]$ and $\mathbf{a} = (a_1, \dots, a_n)$ satisfies:

- $0 \leq a_i \leq n - 1$
- $a_{i+1} \leq a_i + 1$

where the indices are all taken mod n. Denote this graph as Γ_a.

It turns out that if $a_1 = 0$, then Γ_a is the unit interval graph. Further, in this case, we call $\mathbf{a}$ the *area sequence* of the graph. This allows us to naturally relate the graphs to Dyck Paths. Below is

an example of an area sequence and its corresponding Dyck Path.

Example 5.2.2. Let $\mathbf{a} = (0, 1, 1, 0, 1, 2, 2)$. Then, the following is the corresponding Dyck Path.

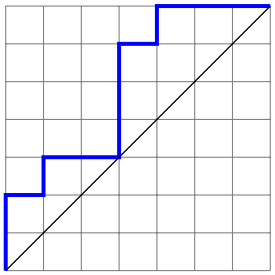


Figure 5.1: Dyck Path for $n = 7$

Further, note that because of this bijection, the number of unit interval graphs on n vertices is the nth Catalan number, C_n. See Haglund (2007) for more details on the Catalan numbers.

Above, it was claimed that Γ_a is the unit interval graph associated to $\mathbf{a}$. We now show the how this Poset arises from $\mathbf{a}$ as in Alexandersson and Panova (2018). For an area sequence $\mathbf{a}$ and its corresponding Dyck Path, we construct a Poset P_a as follows. Put the identity permutation on the diagonal of the Dyck Path. Using the path, if the cell in the column of i and the row j is above the Dyck Path, then $i < j$ in P_a. With this Poset in mind, $inc(P_a) = \Gamma_a$. With this in mind, the authors in Alexandersson and Panova (2018) let

$$X_{\Gamma_a}(X; q) = \sum_{\kappa:\Gamma_a \to \mathbb{N}} x_\kappa q^{asc_a(\kappa)}$$

where the sum is over proper colorings of Γ_a. This agrees with the definition of the chromatic quasisymmetric function in Shareshian and Wachs (2016). The authors in Alexandersson and Panova (2018) extend this definition so the colorings are not necessarily proper. They denote this extension $G_a(X; q)$. Why is this case interesting? It turns out this is exactly the classic unicellular LLT polynomial. With this in mind, we have the following interesting relationship between the unicellular LLT polynomials and chromatic quasisymmetric functions.

Proposition 5.2.3 (Lemma 6.1 Alexandersson and Panova (2018))**.** *Let* $\boldsymbol{a}$ *be an area sequence for*

a unit interval graph. Then

$$(q-1)^{-n}G_a[X(q-1);q] = X_{\Gamma_a}(X;q).$$

This relationship has spawned extensive research. We will see some interesting results when we let $q \to q+1$.

5.3. Schröeder Paths and Vertical Strip LLT Polynomials

With the previous section on Dyck Paths and the Unicellular LLT Polynomials in mind, the authors in Alexandersson and Sulzgruber (2022) extend this to cover other cases.

Definition 5.3.1. Let n indicate the unit north path on a grid, e indicate the unit east path on a grid, and d indicate a diagonal path from (i,j) to $(i+1,j+1)$. We define a Schröeder Path from $(0,0)$ to (n,n) as any sequence of n, e, and d as any path that stays above the main diagonal and has no d step from (i,i) to $(i+1,i+1)$- i.e., along the main diagonal. We call the set of all these possible paths SP_n.

Notice that the set of Dyck paths is a subset of Schröeder Paths.

Definition 5.3.2. Let $\Gamma = (V,E)$ be a unit interval graph and let $S \subseteq E$ be a subset of the edges. We call $\Gamma* = (V,E,S)$ a *decorated* unit interval graph with *strict edges* S. Let $P \in SP_n$ be a Schröeder Path. We associate a decorated unit interval graph Γ_P to P as follows: if $u,v \in [n]$ and $u<v$, then there is a non-strict edge uv in Γ_P if and only if there is a cell in column u and row v below the path P. If the cell (u,v) is the endpoint of a d step, then uv is a strict edge in Γ_P.

Now, we extend the definition of a coloring to these graphs.

Definition 5.3.3. If Γ_P is a decorated unit interval graph of the Schröeder Path P, then a *coloring* of Γ_P is $\kappa : [n] \to \mathbb{N}^+$ such that $\kappa(u) < \kappa(v)$ for all $U,v \in [n]$ such that $u<v$ and uv is a strict edge. We also have an *ascent* statistic where an ascent uv is a non-strict edge with is $u<v$ and $\kappa(u) < \kappa(v)$. We let $asc(\kappa)$ be the number of ascents of a coloring.

Finally, we define the following functions for $P \in SP_n$:

$$G_P(X;q) = \sum_{\kappa} q^{asc(\kappa)} x_\kappa$$

where the sum is over all coloring of Γ_P. It turns out that this is exactly the same as vertical strip LLT polynomials, shown in Alexandersson and Panova (2018). With this construction of the vertical strip LLT polynomials, the authors in Alexandersson and Sulzgruber (2022) are able to show a positive e-expansion of G_P. First, we define an orientation of the graphs.

Definition 5.3.4. Let $\Gamma = (V, E)$ be a graph. Then a function $\theta : E \to V^2$ is an orientation of Γ where θ assigns a direction to each edge in E (i.e., either $\vec{uv}$ or $\vec{vu}$). The *natural orientation* is the orientation that assigns $\vec{uv}$ when $u < v$. For a decorated unit interval graph $\Gamma_P = (V, E, S)$, we let $\mathcal{O}(P)$ be the set of orientations of Γ_P such that the restriction of θ to S is the natural orientation. Finally, for $\theta \in \mathcal{O}(P)$ and $\vec{uv} \in \theta$, we have an ascent if $u < v$ and $uv \notin S$ and $asc(\theta)$ is the number of ascents for a given θ.

This gives us the following result in Alexandersson and Sulzgruber (2022).

Corollary 5.3.5 (Corollary 2.10 Alexandersson and Sulzgruber (2022))**.** *Let $P \in SP_n$ and define*

$$\hat{G_P}(X; q+1) = \sum_{\theta \in \mathcal{O}(P)} q^{asc(\theta)} e_{\lambda(\theta)}(X)$$

then, $G_P = \hat{G_P}$, giving a positive, combinatorial e-expansion of $G_P(X; q+1)$.

With this e-expansion in mind, the authors prove a new Schur expansion of the vertical strip LLT polynomials as well. In particular,

Corollary 5.3.6 (Corollary 6.2 Alexandersson and Sulzgruber (2022))**.** *Let $P \in SP_n$ and $K_{\lambda,\mu}$ the usual Kostka number, then*

$$G_P(X;q) = \sum_{\mu \vdash n} \sum_{\theta \in \mathcal{O}(P)} (q-1)^{asc(\theta)} K_{\mu', \lambda(\theta)} s_\mu(X).$$

The previous corollaries hold for vertical strip LLT polynomials, and because unicellular LLT polynomials are a subset of these, we also get the next result on the Chromatic Quasisymmetric Functions.

Corollary 5.3.7 (Corollary 6.19 Alexandersson and Sulzgruber (2022))**.** *For all Dyck paths $P \in \mathcal{D}$,*

$$X_P(X;q) = \sum_{\theta \in \mathcal{O}(P)} (q-1)^{asc(\theta)-n} e_{\lambda(\theta)}[X(q-1)].$$

The previously results from Alexandersson and Sulzgruber (2022) were mainly proven through various bijections on paths and showing special relations satisfied by the polynomials. A few years prior D'Adderio actually proved the same result using the Dyck Path algebra and various algebraic operators. See D'Adderio (2020) for full details of the proof.

5.4. Indifference Graphs and a Combinatorial Formula for LLT Polynomials

We now describe the results found in Abreu and Nigro (2021) concerning chromatic quasisymmetric functions and LLT Polynomials. Their work deals with spanning forests of a graph, which is beyond the scope of what we need. We will now define this family of graphs known as *indifference graphs*. These graphs naturally relate to previous notions we have discussed.

Definition 5.4.1. Let $V = [n]$ and $G = (V, E)$. Then, G is an *indifference graph* if for any $(i, j) \in E$ with $i < j$, (i, k) and (k, j) are also edges in E for every $i < k < j$.

Indifference graphs are directly related to Dyck paths and Hessenberg functions.

Definition 5.4.2. A *Hessenberg Function* $\mathbf{m} : [n] \to [n]$ is a non-decreasing function such that $\mathbf{m}(i) \geq i$ for all i. TO each Hessenberg function $\mathbf{m}$, we associate an indifference graph $G_{\mathbf{m}}$ where $V = [n]$ and $E = \{(i, j) : i < j \leq \mathbf{m}(i)\}$.

Note that these are directly related to Dyck paths. In particular, in the $n \times n$ grid, let the height of the ith column be $\mathbf{m}(i)$, then we get a Dyck path. Further, if a cell in column i and row j is below the corresponding Dyck path and above the main diagonal (so $i < j$), then (i, j) is an edge in $G_{\mathbf{m}}$. In other words, the edges in $G_{\mathbf{m}}$ correspond to the cells below the Dyck path and above the main

diagonal. Now, let σ be a permutation. We let $\sigma \leq \mathbf{m}$ if $\sigma(i) \leq \mathbf{m}(i)$ for all i. Let $S_{n,\mathbf{m}}$ be the set of permutations in S_n such that $\sigma \leq \mathbf{m}$.

Suppose a permutation is written in cycle notation. We use the convention that the beginning of each cycle is the smallest number in the cycle. Further, we order the cycles from smallest first number to largest first number. With this in mind, we let σ^c be the permutation formed by first writing σ in cycle notation and then removing the parentheses. We now define an inversion in σ associated to a graph G.

Definition 5.4.3. Let $\sigma \in S_n$ and let $G = ([n], E)$ be a graph. We say (i, j) is a G-inversion if $i < j$, $\sigma(i) > \sigma(j)$ and $(\sigma(j), \sigma(i)) \in E(G)$. Let inv_G be the number of G-inversions.

We can also reinterpret this in terms of a Hessenberg function $\mathbf{m}$. Let $\sigma \in S_{n,\mathbf{m}}$ and σ^c its corresponding permutation from the process above. Now, let $inv_{\mathbf{m}}(\sigma)$ be the number of $\mathbf{m}$-inversions of σ, i.e.

$$inv_{\mathbf{m}}(\sigma) = |\{(i, j) : i < j \leq \mathbf{m}(i); \sigma^{-1}(i) > \sigma^{-1}(j)\}|.$$

Then we let the $\mathbf{m}$-weight of a permutation be defined as

$$wt_{\mathbf{m}}(\sigma) = inv_{\mathbf{m}}(\sigma^c)$$

This leads to a main result in Abreu and Nigro (2021).

Theorem 5.4.4 (Theorem 1.2 Abreu and Nigro (2021))**.** *If $G_{\boldsymbol{m}}$ is an indifference graph associated to Hessenberg function $\boldsymbol{m}$, then*

$$LLT_{G_{\boldsymbol{m}}}(X; q) = \sum_{\sigma \leq \boldsymbol{m}} (q-1)^{n-\ell(\lambda(\sigma))} q^{inv_{G_{\boldsymbol{m}}}(\sigma)} e_{\lambda(\sigma)}.$$

As a corollary, $LLT_{G_{\mathbf{m}}}(X; q+1)$ is e-positive. Now, this is just a different way of showing the e-positivity of unicellular LLT polynomials as we have previously seen in Alexandersson and Panova

(2018), Alexandersson and Sulzgruber (2022), and D'Adderio (2020). In the next section, we will use this theorem and the **m**-inversions of permutations to prove a multivariate expansion of certain coefficients of LLT polynomials.

CHAPTER 6

SCHUR COEFFICIENTS OF MULTIVARIATE LLT POLYNOMIALS

In this chapter, we will define a multivariate version of LLT polynomials for certain graphs. To begin, we will assume our graph $G = K_n$, the complete graph. Note that the complete graph has corresponding Dyck path equal to n north steps followed by n east steps. The corresponding Hessenberg function is $\mathbf{m}(i) = n$. In this case, if $\sigma \in S_n$ in cycle notation, then $\sigma \leq \mathbf{m}$ and inv_{K_n} is the usual inversion statistic on permutations not in cycle notation. Noting this, Theorem 1.2 in Abreu and Nigro (2021) says

$$LLT_{K_n}(X;q) = \sum_{\sigma \in S_n} (q-1)^{n-\ell(\lambda(\sigma))} q^{inv(\sigma^c)} e_{\lambda(\sigma)}.$$

6.1. A Determinant Formula for the LLT Expansion

We will now derive a modified version of the formula of Abreu and Nigro. First, we consider a subset of permutations in cycle notation. Take any permutation not written in cycle notation with first element equal to 1. Starting from the right side of the permutation and scanning left, find all numbers that are the smallest number encountered to that point. This set will define the first element of every cycle. As an example, consider the permutation 18254367. This permutation is mapped to (18)(254)(3)(6)(7) in cycle notation. We will call these *primary cycle permutations*, denoted $\hat{S}_n$. We will now convert the LLT formula from a sum over all permutations to a sum over primary cycle permutations. To do this, we will first describe a process for assigning weights to any permutation with its cycles written so the first elements are in ascending order.

Suppose $\sigma = (1\sigma_2 \cdots \sigma_k) \ldots (\sigma_j \cdots \sigma_{n-1})(\sigma_n)$. Note that σ_n is always in its own cycle. We will now assign a weight to each σ_i. First, if σ_i is the first number in a cycle, it gets a weight of 1. If σ_i is not the first number in a cycle, it gets a weight of $(q-1)q^{inv(\sigma_i)}$ where $inv(\sigma_i)$ is the total number of σ_j to the right of σ_i with $\sigma_i > \sigma_j$. To the permutation σ, its weight is the product of all the σ_i weights, denoted $wt(\sigma)$. Using the previous example which is a primary permutation, (18)(254)(3)(6)(7), its

weight is $(q-1)^3q^9$. If instead we consider the non-primary cycle permutation (18)(254)(367), its weight is $(q-1)^5q^9$. Thus, going from the primary to non-primary simply increases the power of $(q-1)$. With this in mind, we now convert the LLT expansion into a sum over primary permutations. To do this, we will associate a determinant to a primary cycle permutation.

Definition 6.1.1. Suppose $\sigma = (1\sigma_2\cdots\sigma_k)\ldots(\sigma_j\cdots\sigma_{n-1})(\sigma_n)$ is a primary cycle permutation with cycles structure $\alpha = (\alpha_1,\ldots,\alpha_j,1)$. We define $det(\sigma)$ as follows:

$$det\begin{pmatrix} e_{\alpha_1} & (-1)(q-1)e_{\alpha_1+\alpha_2} & (-1)^2(q-1)^2e_{\alpha_1+\alpha_2+\alpha_3} & \cdots & (-1)^{\ell(\lambda(\sigma))-1}(q-1)^{\ell(\lambda(\sigma))-1}e_n \\ 1 & e_{\alpha_2} & (-1)(q-1)e_{\alpha_2+\alpha_3} & \cdots & (-1)^{n-2}(q-1)^{n-2}e_{n-\alpha_1} \\ 0 & 1 & e_{\alpha_3} & \cdots & (-1)^{\ell(\lambda(\sigma))-3}(q-1)^{\ell(\lambda(\sigma))-3}e_{n-\alpha_1-\alpha_2} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & (-1)(q-1)e_{\alpha_j+1} \\ 0 & 0 & \cdots & \cdots & e_1 \end{pmatrix}$$

i.e. place the e_{α_i} on the main diagonal, 1's below the main diagonal, and then for row i and column j with $i<j$, we have $(-1)^{j-i}(q-1)^{j-i}e_{\alpha_i+\cdots+\alpha_j}$.

With this definition of weights in mind, we have the following result.

Proposition 6.1.2. *For all $G = K_n$, we have the following LLT formula:*

$$LLT_{K_n}(X;q) = \sum_{\sigma\in\hat{S}_n} wt(\sigma)det(\sigma)$$

where $\hat{S}_n$ is the set of primary cycle permutations.

Proof. The proof follows from the definition of the determinant and the entries in the σ matrix. For ease of notation, consider our primary cycle permutation from before (18)(254)(3)(6)(7) whose

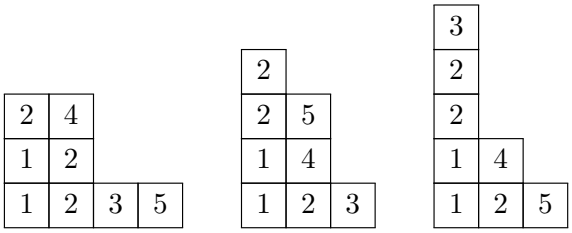
corresponding matrix is:

$$det\begin{pmatrix} e_2 & (-1)(q-1)e_5 & (-1)^2(q-1)^2e_6 & (-1)^3(q-1)^3e_7 & (-1)^{4-1}(q-1)^4e_8 \\ 1 & e_3 & (-1)(q-1)e_4 & (-1)^2(q-1)^2e_5 & (-1)^3(q-1)^3e_6 \\ 0 & 1 & e_1 & (-1)(q-1)e_2 & (-1)^2(q-1)^2e_3 \\ 0 & 0 & 1 & e_1 & (-1)(q-1)e_2 \\ 0 & 0 & 0 & 1 & e_1 \end{pmatrix}.$$

When computing terms in the matrix, the nonzero terms correspond precisely to combining certain cycles of $(18)(254)(3)(6)(7)$. For instance, if we want to compute the weight of $(18)(254)(367)$ from the weight of $(18)(254)(3)(6)(7)$, we replace $e_1e_1e_1$ by $e_3(q-1)^2$. Looking at out matrix, this correspond to the entry in row 3 and column 5. Now to obtain a nonzero weight from this, we must have the 1 in row 5, column 4, which then implies we need the 1 in row 4, column 3 term. Now, we still need terms from the first 2 rows and first 2 columns. We either get e_2 and e_3, or we get 1 and $(-1)(q-1)e_5$. The first corresponds to the weight for $(18)(254)(367)$, and the second corresponds to the weight for $(18254)(367)$. Thus, each term in this matrix corresponds to combining certain adjacent cycles and then accounting for extra $(q-1)$'s and modified e terms. Finally, the $(-1)^j$ offsets the sign of the permutation in the determinant product. □

Our goal now is to convert the LLT formula into a sum over Schur functions. In particular, we want to find some combinatorial object and algebraic statistics that correspond to the Schur coefficients. Our first observation is about expressing Schur coefficients in the product of elementary symmetric functions. Using the Pieri rules, if we are multiplying $e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_j}$, we obtain s_λ if we are able to place a vertical strip of size α_1, then of size α_2 up until α_j so that the resulting diagram is of shape λ. Consider our example permutation $(18)(254)(3)(6)(7)$, when multiplying $e_2e_3e_1e_1e_1$, we place a vertical strip of size 2, then 3, then 1, then 1, and then 1 so we always have a Young diagram. For

instance, we could have the following diagrams and more:



(6.1)

where the 1's correspond to the vertical strip of size 2, the 2's correspond to the vertical strip of size 3, and the 3, 4, and 5 correspond to the vertical strips of size 1. The first gives a term of $s_{4,2,2}$, the second a term of $s_{3,2,2,1}$, and the last a term of $s_{3,2,1,1,1}$. We will now use this idea to associate a Standard Young Tableaux to multiplication of cycles of permutations.

6.1.1. Multivariate LLT Formula

We are going to create a multivariate version of this LLT formula. In particular, we will replace q with z_{ij} for certain i, j. When we do this, if we specialize each z_{ij} to q, we will recover the original LLT formula. Now, recall the above formula

$$LLT_{K_n}(X;q) = \sum_{\sigma \in S_n} (q-1)^{n-\ell(\lambda(\sigma))} q^{inv(\sigma^c)} e_{\lambda(\sigma)}.$$

We will replace this formula with the following:

$$LLT_{K_n}(X;q) = \sum_{\lambda \vdash n} \left(\sum_{T \in SYT(\lambda)} \left(\sum_{\sigma \in \Sigma_T} wt(\sigma) \right) \right) s_\lambda$$

where Σ_T is the set of permutations that can multiply and sort to the standard Young Tableau T. This formula comes from the Pieri rules. In other words, a given permutation can multiply to multiple different tableau shapes like our example for $(18)(254)(3)(6)(7)$, and for each of these possible tableau shapes, we get the corresponding weight $(q-1)^{n-\ell(\lambda(\sigma))} q^{inv(\sigma^c)}$. We now refine this formula further. For each shape, we will create a multivariate version of this weight such that if we specialize each z_{ij} to q, we get the original formula. A full solution to this problem is still open. We present weighting systems and sorting standardizing algorithms for $\lambda = (n-k, 1^k)$ and

$\lambda = (n-k-2, 2, 1^{k-1})$.

6.2. Multivariate Hook Shape Schur Coefficients

In this section, we derive a formula for the Schur coefficients of the multivariate LLT formula for hook shapes. We begin by describing a weighting system and sorting algorithm. In other words, we will find a concise formula for each $\lambda = (n-k, 1^k)$ in

$$\sum_{T \in SYT(\lambda)} \left(\sum_{\sigma \in \Sigma_T} wt(\sigma) \right) s_\lambda.$$

6.2.1. Hook Shape Weights

For any hook shape $\lambda = (n-k, 1^k)$, let Σ_λ be the multiset of permutations that multiply to a tableau of shape λ. We note that this is in fact a multiset as the same permutation may multiply to many different tableau of the same shape. As an example, consider $(13)(2)(4)(5)$. This permutation may multiply to a tableau of shape $(2,1,1,1)$ as either

5		or	4		or	5	
4			2			2	
3			3			3	
1	2		1	5		1	4

and all 3 appear as terms when expanding $e_2e_1e_1e_1$ into Schur functions. When this happens, the weight associated to the permutation will be the same. So, let $\sigma \in \Sigma_\lambda = (1\sigma_2 \cdots \sigma_k) \ldots (\sigma_j \cdots \sigma_n)$ such that, we assign a weight to each σ_i. First, if σ_i is the first number in a cycle, it has a weight of 1. Next, if σ_i is in a cycle and σ_j is any number to the right of σ_i with $\sigma_i > \sigma_j$ we assign $z_{\sigma_i\sigma_j}$. Additionally, find the first σ_k to the left of σ_i in the same cycle such that $\sigma_k < \sigma_i$, and assign a weight of $(z_{\sigma_i\sigma_k} - 1)$. To σ_i, its weight is the product of all of these terms. To the permutation σ, its weight is the product of all the σ_i weights.

Continuing with the example $\sigma = (18)(254)(3)(6)(7)$, we have

$$wt_z(\sigma) = z_{82}z_{85}z_{84}z_{83}z_{86}z_{87}(z_{81}-1)z_{54}z_{53}(z_{52}-1)z_{43}(z_{42}-1)$$

Here is the important observation: if we let all $z_{ij} = q$, then

$$wt_z(\sigma) = (q-1)^{n-\ell(\lambda(\sigma))}q^{inv(\sigma^c)}e_{\lambda(\sigma)}$$

as in the LLT formula.

6.2.2. Hook Shape Sorting

Our goal is to use this formula to compute certain Schur coefficient associated to hook shapes λ. Now suppose $\sigma \in \Sigma_\lambda$ with cycle structure $(\alpha_1, \ldots, \alpha_j)$. When this permutation multiplies to the shape λ, we have j vertical strips of size α_1 through α_j. Fill these vertical strips so that within each cycle, the numbers are sorted. After doing this, we likely have a non-standard tableau. If it is non-standard, it will be because the first column is not sorted. If we sort the first column, we now have a standard tableau. Here is an example.

Example 6.2.1. Let $\sigma = (18)(254)(3)(6)(7)$ and let $\lambda = (3, 1^5)$. Then, we multiply the vertical strips of size 2, 3, 1, 1, and 1. First, we sort (254) to (245). Then, one possible way to get $(3, 1^5)$ is:

7		
3		
5		
4		
8		
1	2	6

which then sorts to:

8		
7		
5		
4		
3		
1	2	6

As before, This permutation contributes a weight of

$$z_{81}z_{82}z_{85}z_{84}z_{83}(z_{81}-1)z_{54}z_{53}(z_{52}-1)z_{43}(z_{42}-1).$$

6.2.3. Hook Inversions

Here, we define inversions on a hook-shape tableau. Let T be a hook shape tableau of shape λ. We say (i,j) form an inversion pair if $i > j$ and i is in a row above j in T. The total number of these inversions is denoted $inv_\lambda(T)$.

Example 6.2.2. Let $\lambda = (3, 1^5)$ and let

$$T = \begin{array}{|c|c|c|}\hline 8 & & \\ \hline 7 & & \\ \hline 5 & & \\ \hline 4 & & \\ \hline 3 & & \\ \hline 1 & 2 & 6 \\ \hline \end{array} \quad .$$

Then, $inv_\lambda(T) = 7 + 6 + 4 + 3 + 2 = 22$.

6.2.4. Multivariate Hook Coefficient Theorem

We now state the main theorem for the multivariate hook coefficients.

Theorem 6.2.3. *Let $\lambda = (n-k, 1^k)$, let $T \in SYT(\lambda)$, and let Σ_T be the permutations that multiply*

to T. Then,

$$\sum_{\sigma \in \Sigma_T} wt_z(\sigma) = \prod_{\substack{i,j \\ (i,j) \text{ an inversion pair in } T}} z_{ij}.$$

An immediate corollary is

Corollary 6.2.4. *If $\lambda = (n-k, 1^k)$, the multivariate LLT Schur coefficient is:*

$$\sum_{T \in SYT(\lambda)} \prod_{\substack{i,j \\ (i,j) \text{ an inversion pair in } T}} z_{ij}.$$

Proof. We will prove this theorem using induction. We first prove some simple base cases.

First, if $\lambda = (n)$, the only permutation that multiplies to this shape is the identity $(1)\cdots(n)$, which has a weight of 1. This coincides with the formula of Abreu and Nigro.

Second, we will prove the case $\lambda = (1^n)$. Note that the conjectured formula for this coefficient is:

$$\prod_{1 \leq j < i \leq n} z_{ij}$$

i.e., the product of all possible inversions. Further, for any σ, we can multiply to (1^n) simply by stacking vertical strips on top of each other. We now prove this case by induction. Let $T/(n)$ be the single column standard tableau for 1 through n. Let $\Sigma_{T/(n)}$ be the set of permutations of $n-1$ that multiply and sort to the single column tablea (note as above that this is all of S_{n-1}. Then, we have

$$\sum_{\sigma \in \Sigma_{T/(n)}} wt_z(\sigma) = \prod_{1 \leq j < i \leq n-1} z_{ij}$$

. For any $\sigma \in \Sigma_{T/(n)}$, we can insert n into any position as follows. Suppose $\sigma = (1 \cdots \sigma_{i_1}) \cdots (\sigma_j \cdots \sigma_{n-1})$.

Then, for this given permutation, we can place n as follows:

$$(1 \cdots \sigma_{i_1}) \cdots (\sigma_j \cdots \sigma_{n-1})(n)$$
$$(1 \cdots \sigma_{i_1}) \cdots (\sigma_j \cdots \sigma_{n-1} n)$$
$$(1 \cdots \sigma_{i_1}) \cdots (\sigma_j \cdots n \sigma_{n-1})$$
$$\vdots$$
$$(1n \cdots \sigma_{i_1}) \cdots (\sigma_j \cdots n \sigma_{n-1})$$

Inserting n in this way gives the following weights:

$$wt_z(\sigma) + wt_z(\sigma)(z_{n\sigma_{n-1}} - 1) + wt_z(\sigma) z_{n\sigma_{n-1}}(z_{n\sigma_{n-2}} - 1) + \cdots + wt_z(\sigma)(\prod_{1<i<n} z_{ni})(z_{n1} - 1)$$

Let us denote this sum using the shuffle notation $\tilde{\sigma} \in \sigma \sqcup n$. Then,

$$\sum_{\tilde{\sigma} \in \sigma \sqcup n} wt_z(\tilde{\sigma}) = wt_z(\sigma) \prod_{1 \leq i \leq n-1} z_{ni}.$$

Summing over all permutations in $\Sigma_{T/(n)}$, we get:

$$\sum_{\sigma \in \Sigma_{T/(n)}} \left(\sum_{\tilde{\sigma} \in \sigma \sqcup n} wt_z(\tilde{\sigma}) \right) = wt_z(T/(n)) \prod_{1 \leq i \leq n-1} z_{ni}.$$

Thus, we get:

$$\sum_{\sigma \in \Sigma_T} = \prod_{1 \leq j < i \leq n-1} z_{ij} \prod_{1 \leq i \leq n-1} z_{ni} = \prod_{1 \leq j < i \leq n} z_{ij}$$

as desired. We now prove the formula for a general hook shape using an almost identical proof as the single column formula.

There are two cases we must consider. The first case is trivial. Let $\lambda = (n-k, 1^k)$ and let $T \in SYT(\lambda)$. Either n is at the end of the first row or at the top of the first column. In the former case, if $\sigma \in \Sigma_T$, then $\sigma = (1 \cdots \sigma_i) \cdots (\sigma_j \cdots \sigma_{n-1})(n)$. This is because the only way for this permutation to sort n to the end of the first row is if it starts there. These are the only permutations

where this is true. Thus,

$$\sum_{\sigma \in \Sigma_T} wt_z(\sigma) = \sum_{\tilde{\sigma} \in \Sigma_{T/(n)}} = wt(T/(n)),$$

and since n forms no inversion in T, we have $wt_z(T) = wt_z(T/(n))$ and the formula holds by induction.

For the case where n is in the first column, we will use the same proof as the single column case. In particular, start with $\sigma \in \Sigma_{T/(n)}$ and shuffle n into these permutations. For each shuffle, n is placed in the first column which will then sort it to the top. Nothing else changes when sorting. Thus,

$$\sum_{\tilde{\sigma} \in \sigma \sqcup n} wt_z(\tilde{\sigma}) = wt_z(\sigma) \prod_{1 \leq i \leq n-1} z_{ni}.$$

and summing over all permutations in $\Sigma_{T/(n)}$, we get:

$$\sum_{\sigma \in \Sigma_{T/(n)}} \left(\sum_{\tilde{\sigma} \in \sigma \sqcup n} wt_z(\tilde{\sigma}) \right) = wt_z(T/(n)) \prod_{1 \leq i \leq n-1} z_{ni}.$$

This is precisely $wt_z(T)$ as desired. □

We will now turn to a formula for augmented hook shapes and a slightly altered weighting system and sorting algorithm.

6.3. Multivariate Augmented Hook Shape Schur Coefficients

In this section, we derive a formula for the Schur coefficients of the multivariate LLT formula for augmented hooks. The weighting system will be nearly identical to the hook case and depend on the sorting algorithm. We also must define inversions for an augmented hook shape.

6.3.1. Augmented Hook Inversions

6.3.2. Hook Inversions

Here, we define inversions on an augmented hook-shape tableau. Let T be am augmented hook shape tableau of shape $\lambda = (n-k-1, 2, 1^{k-1})$. We say (i, j) form an inversion pair if $i > j$ and j

is weakly southeast of i in T, or $j = 1$, i is in the $(2, 2)$ square, and 2 is in the first row. The total number of these inversions is denoted $inv_\lambda(T)$.

Example 6.3.1. Let $\lambda = (3, 2, 1^3)$ and let

$$T = \begin{array}{|c|c|c|}\hline 8 \\ \hline 7 \\ \hline 4 \\ \hline 3 & 5 \\ \hline 1 & 2 & 6 \\ \hline \end{array} \quad .$$

Then, $inv_\lambda(T) = 7 + 6 + 2 + 3 + 2 = 20$, where $(5, 1)$ is an inversion since 2 is in the first row.

6.3.3. Augmented Hook Sorting Algorithm

When multiplying permutations, for numbers in the same vertical strip, we sort the numbers from smallest to largest. When finished with the multiplication of vertical strips, we want to get a standard young tableau. We use Jeu-De-Taquin starting with the smallest number and ending with the largest. Here are two examples of the sorting algorithm. We will see that the sorting is highly dependent on the location of n.

Example 6.3.2. Consider the permutation $(18)(254)(3)(6)(7)$ and its multiplication and cycle sorting to:

$$\begin{array}{|c|c|c|}\hline 7 \\ \hline 3 \\ \hline 5 \\ \hline 8 & 4 \\ \hline 1 & 2 & 6 \\ \hline \end{array} \quad .$$

This tableau is then sorted to:

$$\begin{array}{|c|c|c|}\hline 8 \\ \hline 7 \\ \hline 5 \\ \hline 3 & 4 \\ \hline 1 & 2 & 6 \\ \hline \end{array}$$

which only really depended on 3 being above the 8. Here, 3 was not included in the cycle (254).

If instead it was included and the permutation was (18)(2543)(6)(7), the multiplication and cycle sorting would give:

7		
5		
4		
8	3	
1	2	6

which then sorts to:

7		
5		
4		
3	8	
1	2	6

.

We note a few things. First, the bottom row will always be sorted. In particular, and element in the bottom row is the smallest number in its cycle, and these are already sorted left to right. Further, At no point will a number in the bottom row move upwards, since 1 is always in the bottom left corner, and any number in the $(2, 2)$ position is bigger than the leading number of any cycle before it.

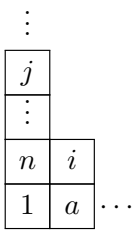
Let $\sigma = (1 \cdots \sigma_i) \cdots (\sigma_j) \cdots \sigma_n)$. The largest number in a permutation n can be in 3 different locations in a sorted tableau:

- The end of the first row, which means $\sigma_n = n$ and is in its own cylce.
- The $(2, 2)$ position, which means n is in a 2-cycle or $\sigma_n = n$ is in its own cycle.
- The top of the first column.

Lemma 6.3.3. *Let σ multiply to an unsorted tableau T. Then, if n is in the bottom row, it is at the end and will not move during the sorting algorithm. If n is in the $(2, 2)$ position, it will also remain in its position. If n is above the second row, then it will sort to the top of the first column. Finally, if n is in the $(2, 1)$ position and i is in the $(2, 2)$ position, then n sorts to the $(2, 2)$ position*

if and only if i is the smallest number above the first row, else n sorts to the top of the first column.

Proof. The first 3 cases are obvious by the definition of the sorting algorithm. What remains to be shown is the scenario where:



. If i is minimal among elements above the first row, then i and n switch places during the first step of the algorithm (note that $i > a$ as well). If i is not minimal, then there is an element j in the first column that is. The first step of the algorithm will continually move j down the first column until it is in the 3rd row and then switches with n. Once n is in the third row, it is guaranteed to move to the top of the first column. □

6.3.4. Assigning Weights

Let T be a filling of a partition $\lambda \vdash n$ where λ is an augmented hook. Let Σ_T be the set of all permutations such that $\sigma \in \Sigma_T$ multiplies and sorts to T. We assign a weight to σ in the following way. First, if we have a cycle $(x_1 x_2 \cdots x_k)$ in σ, note that x_1 is the smallest number in the cycle and smaller than all numbers in cycles to the right. We assign a weight of 1 to x_1. For x_i for $i \in \{2, \ldots, k\}$, scan to the left of x_i for the first number smaller than x_i, say it is x_j. This contributes a factor of $(z_{x_i x_j} - 1)$. Now, scan to the right of x_i in the rest of σ. For any number m to the right of x_i that is smaller than x_i, we get a factor of $z_{x_i m}$. However, suppose (x_i, m) is not an inversion in T, then we switch $z_{x_i m}$ to $z_{m x_1}$.

Lemma 6.3.4. *In the scenario, where we need to switch $z_{x_i m}$ to $z_{m x_1}$ if (x_i, m) is not an inversion pair in T, then (m, x_1) is an inversion pair in T.*

Proof. First, we are considering a standard Young tableau of an augmented hook and x_i is part of a cycle and not the smallest number. Thus, it is guaranteed to be above the first row after multiplication. Further, $m < x_i$, so in T, if x_i and m are in the same column, this is an inversion.

Thus, if it is not an inversion, x_i and m must be in separate columns, and in particular, x_i must be in the $(2,2)$ square. As m is to the left and potentially above x_i, m is in the first column above the first row. But this implies (m, x_1) is an inversion since $x_1 < m$ and must be in T below x_i. □

Now, for $\sigma \in \Sigma_T$, we say $wt_z(\sigma) = \prod_{i \in \sigma} wt(i)$, i.e. the product of the weights assigned to each number in sigma.

6.3.5. Augmented Hook Theorem

With this algorithm and weight assignment, we have:

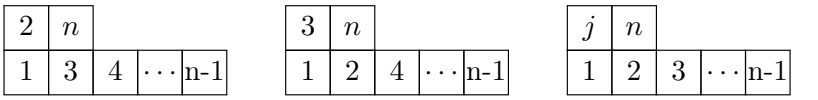
Theorem 6.3.5. *Let $\lambda \vdash n$ be an augmented hook, T a standard young tableau of shape λ, and Σ_T the permutations that multiply and sort to T. We have:*

$$\sum_{\sigma \in \Sigma_T} wt(\sigma) = \prod_{\substack{i,j \\ (i,j) \text{ an inversion pair in } T}} z_{ij}. \tag{6.2}$$

Before the details of the proof, here are a few notes and observations. All cases contained inside of the $(3,2,1)$ shape were verified by hand or by computer and represent base cases. If the tableau, T, under consideration has the largest number n at the end of the first row, then the statement follows by induction. In particular, n forms no inversions and the algorithm maps a permutation to the tableau if and only if n is the last number in the permutation and not part of a cycle. In this case, n contributes no weight, and there is a weight preserving bijection sending permutations that map to T to permutations that map to $T \backslash (n)$. Now, the proof is broken into several cases. First, we prove the 2-row augmented hook case ($\lambda = (n-2, 1)$), then we prove the 2-column augmented hook case ($\lambda = (2^2, 1^{n-4})$), and finally we prove the general statement.

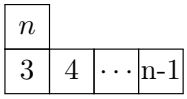
Case 1: 2-row Augmented Hook

For the proof of this shape, we consider 3 separate cases.

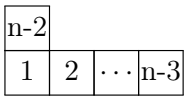


Case a Case b Case c, $4 \leq j \leq n-1$

For case a, note that any permutation mapped to this tableau must be of the form $(1)(2)\tilde{\sigma}$ or $(12)\tilde{\sigma}$, where $\tilde{\sigma}$ is some permutation that maps to the tableau:



We note that any such $\tilde{\sigma}$ is in bijection with a σ in S_{n-2} that maps to the tableau:



and by hook case, summing over all such $\tilde{\sigma}$ gives us the weight of the hook shape, which is

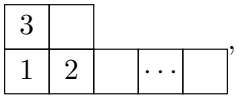
$$\prod_{3 \leq i \leq n-1} z_{ni}. \tag{6.3}$$

Further since each permutation starts with $(1)(2)$ or (12), we get a weight of $(1 + (z_{21} - 1)) = z_{21}$. Hence, the total weight is:

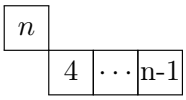
$$z_{21} \prod_{3 \leq i \leq n-1} z_{ni} \tag{6.4}$$

as desired.

Now, for case b, we have 6 different ways a permutation can begin. They are $(1)(2)(3)\tilde{\sigma}$, $(1)(23)\tilde{\sigma}$, $(13)\tilde{\sigma}$, $(1)(2n)(3)\tilde{\sigma}$, $(1n)(2)(3)\tilde{\sigma}$, or $(1n)(23)\tilde{\sigma}$. For the first two permutations, $\tilde{\sigma}$ must fill the remaining portion of the diagram:



with n in the second row. In other words, we must get the skew tableau:



from $\tilde{\sigma}$. Any such $\tilde{\sigma}$ is in bijection with a permutation in S_{n-3} that gives the tableau:

n-3				
1	2	3	$\cdots$	n-4

,

and thus by the hook case, summing over all such $\tilde{\sigma}$ gives us the weight of the hook shape, which is

$$\prod_{4 \leq i \leq n-1} z_{ni}. \tag{6.5}$$

Further since each permutation starts with (1)(2)(3) or (1)(23), we get a weight of $1+(z_{32}-1) = z_{32}$. Hence, the total weight is:

$$z_{32} \prod_{4 \leq i \leq n-1} z_{ni} \tag{6.6}$$

from the first two cases. The second set of permutations is very similar, but now $\tilde{\sigma}$ must fill the remaining portion of the diagram:

3				
1			$\cdots$	

,

with n in the second row. In other words, we must get the skew tableau:

n			
2	4	$\cdots$	n-1

from $\tilde{\sigma}$. Any such $\tilde{\sigma}$ is in bijection with a permutation in S_{n-2} that gives the tableau:

n-2				
1	2	3	$\cdots$	n-3

,

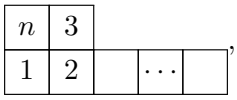
and thus by the hook case, summing over all such $\tilde{\sigma}$ gives us the weight of the hook shape, which is

$$z_{n2} \prod_{4 \leq i \leq n-1} z_{ni}. \tag{6.7}$$

Further since each permutation starts with (13), we get a weight of $(z_{31}-1)z_{32}$. Hence, the total weight is:

$$(z_{31}-1)z_{32}z_{n2}\prod_{4\leq i\leq n-1}z_{ni} \tag{6.8}$$

from this set of permutations. The next set of permutations is very simple. Now $\tilde{\sigma}$ must fill the remaining portion of the diagram:

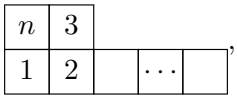


in other words, we must get the skew tableau:

from $\tilde{\sigma}$. However, there is only one way to do this, and it gives a weight of 1. Since the permutation began with $(1)(2n)(3)$, we get a weight of

$$(z_{n2}-1)z_{32}\prod_{4\leq i\leq n-1}z_{ni} \tag{6.9}$$

from this case- note that z_{n3} was replaced by z_{32} because $(n,3)$ is not an inversion pair in T. The final case and two sets of permutations is again very simple. Now $\tilde{\sigma}$ must fill the remaining portion of the diagram:



in other words, we must get the skew tableau:

from $\tilde{\sigma}$. However, there is only one way to do this, and it gives a weight of 1. Since the permutation began with $(1n)(2)(3)$ or $(1n)(23)$, we get a weight of

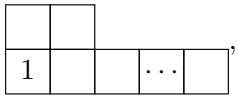
$$((z_{n1}-1)z_{n2}z_{31}+(z_{n1}-1)z_{n2}z_{31}(z_{32}-1))\prod_{4\leq i\leq n-1}z_{ni}=(z_{n1}-1)z_{n2}z_{31}z_{32}\prod_{4\leq i\leq n-1}z_{ni} \tag{6.10}$$

from this case. Hence, in total we get a weight of:

$$\begin{aligned}
&z_{32} \prod_{4\le i\le n-1} z_{ni} \\
&+ (z_{31}-1) z_{32} z_{n2} \prod_{4\le i\le n-1} z_{ni} \\
&+ (z_{n2}-1) z_{32} \prod_{4\le i\le n-1} z_{ni} \\
&+ (z_{n1}-1) z_{n2} z_{32} z_{31} \prod_{4\le i\le n-1} z_{ni} \\
&= z_{n1} z_{n2} z_{32} z_{31} \prod_{4\le i\le n-1} z_{ni}
\end{aligned} \tag{6.11}$$

which is the desired weight.

For case c, we have a fairly similar argument to case b. We have 3 different ways a permutation can begin and obtain the desired tableau. They are $(1)\tilde{\sigma}$, $(1j)\tilde{\sigma}$, and $(1n)\tilde{\sigma}$. For the first set of permutations, $\tilde{\sigma}$ must fill the remaining portion of the diagram:



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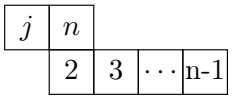
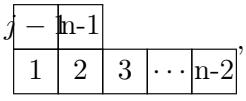
in other words, we must get the skew tableau:



from $\tilde{\sigma}$. Any such $\tilde{\sigma}$ is in bijection with a permutation in S_{n-1} that gives the tableau:



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and thus by induction, summing over all such $\tilde{\sigma}$ gives us the weight of the augmented hook shape, which is

$$\prod_{2\le i\le j-1} z_{ji} \prod_{\substack{2\le i\le n-1 \\ i\neq j}} z_{ni}, \tag{6.12}$$

the total weight from this first case. The second set of permutations is very similar, but now $\tilde{\sigma}$ must fill the remaining portion of the diagram:

j				
1			$\cdots$	

,

in other words, we must get the skew tableau:

n			
2	3	$\cdots$	n-1

from $\tilde{\sigma}$. Any such $\tilde{\sigma}$ is in bijection with a permutation in S_{n-2} that gives the tableau:

n-2				
1	2	3	$\cdots$	n-3

,

and thus by the hook case, summing over all such $\tilde{\sigma}$ gives us the weight of the hook shape, which is

$$\prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni}. \tag{6.13}$$

Further since each permutation starts with $(1j)$, we get a weight of $(z_{j1}-1)\prod_{i=2}^{j-1} z_{ji}$. Hence, the total weight is:

$$(z_{j1}-1)\prod_{i=2}^{j-1} z_{ji} \times \prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni} \tag{6.14}$$

from the second case. For the third set of permutations, $\tilde{\sigma}$ must fill the remaining portion of the diagram:

n				
1			$\cdots$	

,

with j in the second row. In other words, we must get the skew tableau:

j			
2	3	$\cdots$	n-1

from $\tilde{\sigma}$. Any such $\tilde{\sigma}$ is in bijection with a permutation in S_{n-2} that gives the tableau:

j-1				
1	2	3	$\cdots$	n-2

,

and thus by the hook case, summing over all such $\tilde{\sigma}$ gives us the weight of the hook shape, which is

$$\prod_{2\leq i\leq j-1} z_{ji}. \tag{6.15}$$

Further since each permutation starts with $(1n)$, we get a factor of

$$(z_{n1}-1)z_{j1}\prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni}. \tag{6.16}$$

Thus, for this case, we get a weight of

$$(z_{n1}-1)z_{j1}\prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni}\times\prod_{2\leq i\leq j-1} z_{ji}. \tag{6.17}$$

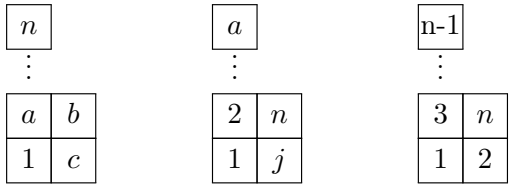
Hence, for case c, we get a total weight of:

$$\begin{aligned}
&\prod_{2\leq i\leq j-1} z_{ji}\prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni}\\
&+(z_{j1}-1)\prod_{2\leq i\leq j-1} z_{ji}\times\prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni}\\
&+(z_{n1}-1)z_{j1}\prod_{\substack{2\leq i\leq n-1\\ i\neq j}} z_{ni}\times\prod_{2\leq i\leq j-1} z_{ji}\\
&=\prod_{\substack{1\leq i\leq n-1\\ i\neq j}} z_{ni}\times\prod_{1\leq i\leq j-1} z_{ji}
\end{aligned} \tag{6.18}$$

which is the desired weight. This completes the proof for the two row augmented hook case.

Case 2: 2-column Augmented Hook

This case will reflect the general proof more fully. We will consider 3 separate cases which are:



For case a, let T be the given tableau. We argue by induction and shuffling n into each of the permutations that gives $T/(n)$. Note a few observations. If a permutation multiplies to the shape $(2^2, 1^{n-4})$, then we get T in the following scenarios: first if the second column contains b and c (note b will necessarily be on top of c), or if $a > c$, and the bottom $(2, 2)$ tableau is of the form

b	a
1	c

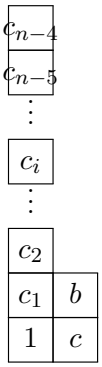
before sorting. Note, this second case only happens when $c = 2$, $a = 3$, and $b \neq n$. Now, suppose σ multiplies to $T/(n)$. Then, observe that $\sigma = (1 \cdots \sigma_i) \cdots (\sigma_j \cdots \sigma_{n-1})$. Now, consider $\tilde{\sigma} = \sigma(n)$ or $\tilde{\sigma} = (1 \cdots \sigma_i) \cdots (\sigma_j \cdots \sigma_{n-1} n)$ such that $\tilde{\sigma}$ multiplies to T. In particular, we have that these two cases of $\tilde{\sigma}$ contribute a weight of $wt_z(\sigma) z_{n\sigma_{n-1}}$ and n sorts to the top of the first column.

Suppose σ has k cycles, and the numbers that begin each cycle are 1, j_2, j_3,..., $j_k = \sigma_j$. Now, we shuffle n into σ, and we claim for each shuffle, there is exactly one way to multiply the permutation to get the tableau T. Suppose n is now in a cycle of length m, then in the multiplication, we can always have the last box in the vertical strip occur in the first column. Then, when we sort the vertical strip from smallest element to largest element, n is guaranteed to be in the first column. Thus, for the first shuffle, $\tilde{\sigma}$ has a weight of $wt_z(\sigma)(z_{n\sigma_{n-2}} - 1) z_{\sigma_{n-1}}$.

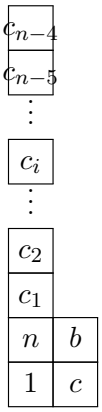
Now, suppose n shuffles within a cycle from $(\cdots xyn \cdots)$ to $(\cdots xny \cdots)$. Note that from the weight assignment as we shuffle, n replaces $(z_{ny} - 1)$ with $(z_{nx} - 1) z_{ny}$ and the rest of the weight is unchanged. Thus, as we shuffle n, we get $wt_z(\sigma)$ as a constant factor, and also get a factor of

$(z_{ni} - 1) \prod_{l \text{ to right of n}} z_{nl}$ where i immediately precedes n.

Now, we may have an issue when n is multiplied and ends up in the second row. First, if n is multiplied and ends in the second row and second column, then n will not be sorted to the top of the first column. Thus, our main concern is when n is multiplied and is the second number in the first column. Assuming a is the smallest number besides 1 in the first column, we have two possibilities. First, in σ, a is in the first column before sorting. In this case, before sorting, we have the following from σ:

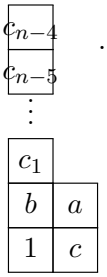


where $c_i = a$. Then in $\tilde{\sigma}$, we have:



and since $c_i = a$ is the smallest number in the first column and less than b, a will sort to position $(2, 1)$ and n will sort to the top of the first column. The other case is if, in σ, a is in the second

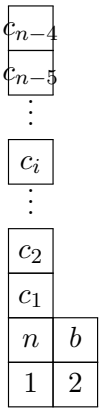
column before sorting. In this case, before sorting, we have the following from σ:



.

In this case, c must be 2. This follows because b is bigger than a, and if a is 2, then it cannot be in the second row of the second column from any permutation. This then implies that $a = 3$. Further, we also know that we either have $\sigma = (1b\cdots)\tilde{\sigma}$ or $\sigma = (1)(2b\cdots)\tilde{\sigma}$. If $\sigma = (1b\cdots)\tilde{\sigma}$, then when we shuffle n, since cycles are sorted, n will never appear in the second row of the first column. Thus, n will be above the second row, and the sorting algorithm will map n to the top of the first column and the rest of the tableau will be the same as σ.

However, we have the case $(1n)(2b\cdots)(3\cdots)$ from shuffling to consider. In this case, $\tilde{\sigma}$ maps to T after sorting from the following tableau:



where $c_i = 3$ and n will sort to the top of the first row. Thus, from shuffling n into σ that maps to $T/(n)$, each shuffle give exactly one permutation that maps to T. Further, if we take the sum over all of the shuffles for a given σ, the weight is now

$$\sum_{\tilde{\sigma}\text{: n shuffled into } \sigma} wt_z(\tilde{\sigma}) = wt_z(\sigma) \prod_{i=1}^{n-1} z_{ni}. \tag{6.19}$$

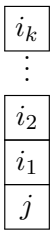
Thus, when we sum over all permutations that map to T, we get :

$$\sum_{\tilde{\sigma} \text{ map to } T} wt_z(\tilde{\sigma}) = \sum_{\sigma \text{ map to } T/(n)} \sum_{\tilde{\sigma}\text{: n shuffled into } \sigma} wt_z(\tilde{\sigma}) = wt_z(T/(n)) \prod_{i=1}^{n-1} z_{ni} \tag{6.20}$$

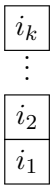
as desired for case a.

Now, we consider case b. We note the following possibilities for the permutations that multiply and are sorted to T. First, any permutation $\tilde{\sigma}$ that multiplies to $T/(n)$ also gives a permutation that multiplies to T by attaching n at the end, giving $\sigma = \tilde{\sigma}(n)$ or $\tilde{\sigma} = (1 \cdots \sigma_i) \cdots (\sigma_j \cdots \sigma_{n-1} n)$. Further, the weight of such a permutation is either $wt_z(\tilde{\sigma})$ or $wt_z(\tilde{\sigma})(z_{n\sigma_{n-1}} - 1)$.

Let $J = \{j+1, \ldots, n-1\}$, and let $\emptyset \subsetneq I \subseteq J$. Then, any permutation that multiplies to T is of the form $\sigma_1(j \cdots i)\sigma_2(n)$ where $(j \cdots i)\sigma_2$ only contains numbers from I, or $\sigma_1(jn)\sigma_2$ where σ_2 only contains numbers from I. Consider all $(j \cdots i)\sigma_2$ for a fixed σ_1. The sum over all $(j \cdots)\sigma_2$ gives the weight of the tableau, $T_{I,j}$:



where $\{i_1, \ldots, i_k\} = I$. In other words, we know that all $(j \cdots)\sigma_2$ multiply to a tableau of this form, and any such suffix to σ_1 gives the proper T. Similarly, consider all $(jn)\sigma_2$ for a fixed σ_1 that multiplies to T. The sum over all $(jn)\sigma_2$ gives the weight of the tableau, T_I:



where $\{i_1, \ldots, i_k\} = I$, multiplied by $(z_{nj} - 1)\prod_{l=1}^{k} z_{i_l j}$ from the weight of (jn) and n to the right of all the i_l, replacing z_{ni_l} with $z_{i_l j}$. Thus, for a fixed σ_1 and I, we get a total contribution of weight

equal to

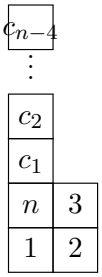
$$\begin{aligned} &wt(\sigma_1)(wt_z(T_{I,j}) + wt_z(T_I)(z_{nj} - 1)\prod_{l=1}^{k} z_{i_k j}) \\ &= wt_z(\sigma_1)wt_z(T_{I,j})z_{nj}. \end{aligned} \tag{6.21}$$

However, note that the above analysis also tells us that for a fixed σ_1 and a suffix that multiplies to $T/(n)$, the weight is $wt_z(\sigma_1)wt_z(T_{I,j})$. Thus, by summing over all σ_1, we get:

$$\begin{aligned} &\sum_{\sigma_1} wt_z(\sigma)wt_z(T_{I,j})z_{nj} \\ &\qquad = z_{nj}\sum_{\sigma_1} wt_z(\sigma)wt(T_{I,j}) \\ &\qquad = z_{nj}wt_z(T/(n)), \end{aligned} \tag{6.22}$$

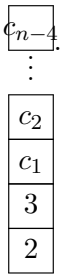
and this is precisely the weight of T as desired.

Now, case c is very similar to case b, but now we have to factor in all permutations of the form $\sigma = (1n)\tilde{\sigma}$, where $\tilde{\sigma}$ multiplies and sorts the vertical strips so that 3 is in the augmented square before sorting, i.e., we have the skew tableau:



after sorting vertical strips. However, we note that this is equivalent to multiplying and sorting to

the vertical strip tableau:



This is true because if 2 and 3 are in the same cycle, then the augmented square must be in the vertical strip containing them, else 3 is not sorted to the correct square, and if 2 and 3 are not in the same cycle, the augmented square is not part of the vertical strip containing 2, and the next cycle begins with $(3\dots)$ and there is a one to one correspondence with placing 3 in the augmented square with placing 3 on top of the strip containing 2. Then, summing over all $\tilde{\sigma}$ gives the weight of

$T_2 =$

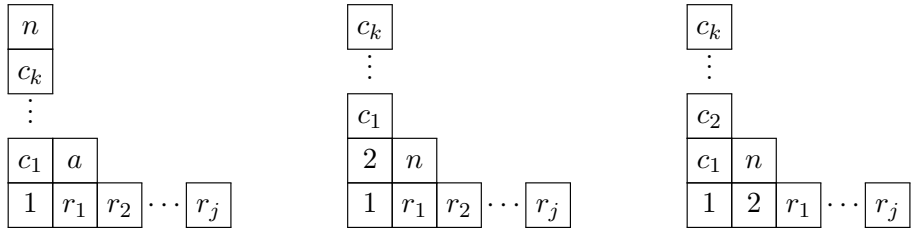
multiplied by a factor of: $(z_{n1}-1)z_{n2}\prod_{i=3}^{n-1} z_{i1}$. Hence, in total, we get:

$$\begin{aligned} & z_{n2}wt_z(T/(n)) \\ & + (z_{n1}-1)z_{n2}\prod_{i=3}^{n-1} z_{i1}wt_z(T_2) \\ & = z_{n1}z_{n2}wt_z(T/(n)), \end{aligned} \tag{6.23}$$

and this is precisely the weight of T as desired. This proves the two-column augmented hook case, and now we show the general augmented hook case. We will use inductive arguments that rely on the two-column and two-row augmented hooks, and $(3,2,1)$ augmented hooks as the base cases.

Case 3: General Augmented Hook

We will consider multiple cases for the general theorem. They are:



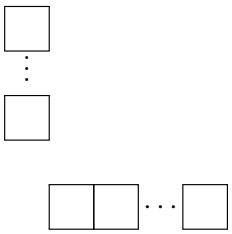
where we note that the case where n is the last number in the first row immediately follows by induction.

For case a, we shuffle n into all of the permutations that multiply to $T/(n)$. We claim for each shuffle, there is exactly one way to multiply and sort to T. In fact, it is an identical argument to the two column augmented hook case. In particular, If σ multiplies to $T/(n)$ and $\tilde{\sigma} = \sigma_1 n \sigma_2$ where $\sigma_1\sigma_2 = \sigma$ and n is contained in the last cycle of σ_1, we can always have the last box in the vertical strip containing n to be in the first column. In fact, if the last box in the vertical strip containing n does not end in the first column, then this permutation is not mapped to T by the sorting algorithm. If n is above the $(2,1)$ square in the first column, then the sorting gives T. If n is in the $(2,1)$ square, then we need to know what happens to c_1 and a when $\tilde{\sigma}$ is multiplied. This argument is identical to the one in the $(2,2,1^{n-4})$ case. This shows that n can be shuffled into every permutation that multiplies to $T/(n)$ in a unique way. Thus, we get a weight of:

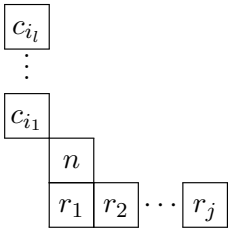
$$\begin{aligned}
&\sum_{\sigma \in \Sigma_{T/(n)}} \sum_{\substack{\tilde{\sigma} \\ \text{n shuffled into } \sigma}} wt_z(\tilde{\sigma}) \\
&= \sum_{\sigma \in \Sigma_{T/(n)}} wt_z(\sigma) \prod_{i=1}^{n-1} z_{ni} \\
&= wt_z(T/(n)) \prod_{i=1}^{n-1} z_{ni},
\end{aligned}$$

which is the desired weight.

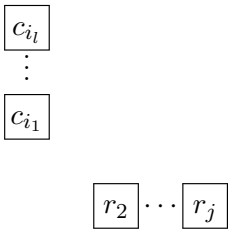
For case b, we again have a similar argument to the $(2, 2, 1^{n-4})$ case. Note that if we take any permutation that multiplies to $T/(n)$ and insert n into the permutation to obtain T, the only places we can do this are $(r_i n)$ where r_i was in a cycle by itself, or at the end of the permutation. Take any $\sigma = \sigma_1\sigma_2$ that multiplies to $T/(n)$, where σ_2 starts with r_1 and has the numbers $r_1, \ldots, r_j, c_{i_1}, c_{i_2}, \ldots, c_{i_l}$ such that $c_{i_1} < \cdots < c_{i_l}$ and $c_{i_p} = c_i$ for some $i = 1, \ldots, k$. For a fixed σ_1, consider all of the σ_2 that multiply to $T/(n)$. In particular, since σ_2 starts with r_1, it multiplies to the skew shape:



where $|\sigma_1|$ boxes have been removed from the first column (note that if $|\sigma_1| = 1$, then the boxes in the first column begins in the second row). Further, note that if we want to obtain T from this permutation, n does not appear in σ_1. Hence, for the tableau T, we are interested in all permutations $\sigma_1\tilde{\sigma}_2$ that multiply to T where $tilde\sigma_2$ is obtained from σ_2 by inserting n. In particular, all $\tilde{\sigma}_2$ that multiply and are sorted to



We have two cases to consider. First, consider any permutation that begins with $(r_1 n)$. Then, the remaining $\tilde{\sigma}_2$ multiplies and sorts to the skew tableau:



Now, suppose $c_{i_1} < r_2$. Then, we claim that all such permutations are in bijection with permutations

that multiply and sort to the tableau

$$T_{c_{i_1}} = \begin{array}{|c|c|c|c|c|}\hline c_{i_l} \\ \hline \vdots \\ \hline c_{i_2} \\ \hline c_{i_1} & r_2 & \cdots & r_j \\ \hline \end{array}$$

which is straightforward to verify. By the hook case, and summing over all $\tilde{\sigma}_2$, we get a total weight of:

$$(z_{nr_1} - 1) \prod_{v=2}^{j} z_{nr_v} \prod_{w=1}^{l} z_{c_{i_w} r_1} wt_z(T_{c_{i_1}}).$$

If the permutation does not begin with $(r_1 n)$, and $c_{i_1} < r_2$, then we claim all permutations are in bijection with a permutation that multiplies and sorts to

$$\begin{array}{|c|c|c|c|}\hline c_{i_l} \\ \hline \vdots \\ \hline c_{i_1} & n \\ \hline r_1 & r_2 & \cdots & r_j \\ \hline \end{array}$$

, which follows because at least 2 numbers appear in the first column before n. Now, by induction on the augmented hook case, the total weight here is the weight of this augmented hook, which is the same as the weight of the augmented hook without n, call it T_{r_1} multiplied by all of the inversions with n, which is:

$$wt_z(T_{r_1}) \prod_{w=2}^{j} z_{nr_w}.$$

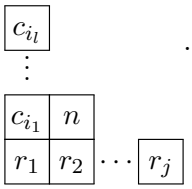
Thus, for $c_{i_1} < r_2$, and a fixed σ_1, and summing over all $\tilde{\sigma}_2$, we get a weight of:

$$\begin{aligned} &(z_{nr_1} - 1) \prod_{v=2}^{j} z_{nr_v} \prod_{w=1}^{l} z_{c_{i_w} r_1} wt_z(T_{c_{i_1}}) \\ &+ wt_z(T_{r_1}) \prod_{w=2}^{j} z_{nr_w} \\ &= \prod_{v=1}^{j} z_{nr_v} wt_z(T_{r_1}). \end{aligned} \tag{6.24}$$

In particular, this means that for a fixed σ_1, when we sum over all σ_2 and insert n into all of its possible spots, we get a factor of $\prod_{i=1}^{j} z_{nr_i}$ multiplied by the weight of the hook case. This is exactly the factor that we want, i.e.:

$$\begin{aligned}
&\sum_{\sigma_1} wt_z(\sigma_1) \sum_{\tilde{\sigma}_2} wt_z(\tilde{\sigma}_2) \\
&= \sum_{\sigma_1} wt_z(\sigma_1) \prod_{i=1}^{j} z_{nr_i} \sum_{\sigma_2} wt_z(\sigma_2) \\
&= \prod_{i=1}^{j} z_{nr_i} \sum_{\sigma_1} wt_z(\sigma_1) \sum_{\sigma_2} wt_z(\sigma_2) \\
&= \prod_{i=1}^{j} z_{nr_i} wt_z(T/(n))
\end{aligned}$$

which is exactly the weight of T as claimed. Now, this was assuming $c_{i_1} < r_2$. If not, then any permutation $\tilde{\sigma}$ is now in bijection with a permutation that multiplies to:



.

This follows from the sorting algorithm. In particular, it is clear that this is true if any number in the first column appears before n. If n appears first, then the only way this tableau sorts to the correct one is if c_{i_1} is multiplied to the augmented square. For each permutation, there is only one way for this to happen, and there is always one way as c_{i_1} is either sorted to the first number in its cycle or sorted to the second number. The first case happens if only $c's$ are in its cycle, and the second case happens if the first number in the cycle is an r. Thus, we always have one way for this to happen. Now, by induction, the weight of this augmented hook is equal to the weight of the hook

T_{r_1} multiplied by the factor from n, namely $\prod_{i=1}^{j} z_{nr_i}$. Thus, for this case we also get the equation:

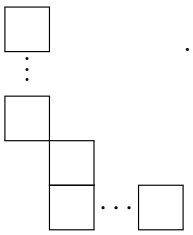
$$\begin{aligned}
&\sum_{\sigma_1} wt_z(\sigma_1) \sum_{\tilde{\sigma}_2} wt_z(\tilde{\sigma}_2) \\
&= \sum_{\sigma_1} wt_z(\sigma_1) \prod_{i=1}^{j} z_{nr_i} \sum_{\sigma_2} wt_z(\sigma_2) \\
&= \prod_{i=1}^{j} z_{nr_i} \sum_{\sigma_1} wt_z(\sigma_1) \sum_{\sigma_2} wt_z((r_1\sigma_2) \\
&= \prod_{i=1}^{j} z_{nr_i} wt_z(T/(n))
\end{aligned}$$

as desired. This finishes case b. The argument for case c will be extremely similar but involve an extra factor (as was true for the two column augmented hook).

For case c, we get the previous weight from b:

$$\prod_{i=1}^{j} z_{nr_i} wt(T/(n))$$

plus the weight of all permutations that start with $(1n)$ and are sorted to the tableau in case c. In particular, we are filling the skew tableau:



Now, note that if c_1 is not in the augmented square after multiplication and sorting vertical strips, then we do not obtain T. Thus, c_1 must be placed in the augmented square. We claim this means

the total weight contribution of all σ in $(1n)\sigma$ is equal to the weight of the tableau:

$$T_2 = \begin{array}{|c|c|c|c|}\hline c_k \\ \hline \vdots \\ \hline c_1 \\ \hline 2 & r_1 & \cdots & r_j \\ \hline \end{array} \quad .$$

This is exactly the same a previous argument. Any σ that multiplies to the skew shape must have c_1 in a cycle that includes the augmented square in the multiplication. Thus, if we consider any permutation that multiplies and sorts to T_2, this can be mapped to a σ since we can always place c_1 in the augmented square. Conversely, for a given σ, instead of using the augmented square, we place c_1 in the first column, and we sort to T_2 as desired. Thus, the total weight from these permutations is:

$$(z_{n1} - 1) \prod_{i=1}^{j} z_{nr_i} \prod_{i=1}^{k} z_{c_i 1} wt_z(T_2).$$

Hence, in total from Case c, we obtain:

$$\begin{aligned} &\prod_{i=1}^{j} z_{nr_i} wt_z(T/(n)) \\ &+ (z_{n1} - 1) \prod_{i=1}^{j} z_{nr_i} \prod_{i=1}^{k} z_{c_i 1} wt_z(T_2) \\ &= z_{n1} \prod_{i=1}^{j} z_{nr_i} wt_z(T/(n)) \end{aligned}$$

as desired. This completes the proof of the augmented hook case.

6.4. Consequence of the Multivariate Formulas

The work in this section needs some motivation to justify the extensive computations and proofs. We studied this formula for the following reason: it allows us to track inversions. In particular, we assumed the graph we were working with was K_n, which corresponds to the Dyck path that covers the entire $n \times n$ grid. But what if we want to study the formula of Abreu and Nigro for other

graphs? Recall their formula for indifference graphs:

$$LLT_{G_{\mathbf{m}}}(X;q) = \sum_{\sigma \leq \mathbf{m}} (q-1)^{n-\ell(\lambda(\sigma))} q^{inv_{G_{\mathbf{m}}}(\sigma)} e_{\lambda(\sigma)}.$$

Then with our results, we can compute the multivariate Schur coefficient for the complete graph as:

Theorem 6.4.1. *Let $\lambda = (n-k, 1^k)$ or $\lambda = (n-k-1, 2, 1^{k-1})$, then in $LLT_{K_n}(X; z_{ij})$, the coefficient of s_λ is*

$$\sum_{T \in SYT(\lambda)} \prod_{\substack{i,j \\ (i,j) \text{ an inversion pair in } T}} z_{ij}.$$

We now observe something very curious. Suppose we have an indifference graph $G_{\mathbf{m}}$, if $\sigma \in \Sigma_T$ where T is a hook or an augmented hook, but $\sigma \not\leq \mathbf{m}$, then there is some cycle in σ with $\sigma_i = j$ and $\sigma_{i+1} = k$ with $k > \mathbf{m}(i)$. In other words, we have some number that is mapped to another number that violates the Hessenberg function. Here is the important observation: this permutation σ has a factor of $(z_{kj} - 1)$. This means if we let $z_{kj} = 1$, this permutation actually gives a weight of zero. This leads to the following Corollary.

Corollary 6.4.2. *Let $\lambda = (n-k, 1^k)$ or $\lambda = (n-k-1, 2, 1^{k-1})$, let m be the Hessenberg function with corresponding Dyck path D. Let and (i,j) under the path D be a potential inversion pair, then in $LLT_{G_{\boldsymbol{m}}}(X; z_{ij})$, the coefficient of s_λ is*

$$\sum_{T \in SYT(\lambda)} \prod_{\substack{i,j \\ (i,j) \text{ an inversion pair in } T}} z_{ij}$$

where an inversion pair in T must also be a potential inversion corresponding to D.

Proof. Using the theorem for the complete graph, if we take any (i,j) that is no longer a potential inversion corresponding to the path D and evaluate $z_{ij} = 1$, we are left with the above formula. In other words, we can write the sum over all possible permutations that multiply to a hook or an augmented hook, evaluate $z_{ij} = 1$ for the now invalid inversions, and we are left with the only valid

inversions. Thus, we can evaluate formula for the coefficients for K_n

$$\sum_{T \in SYT(\lambda)} \prod_{\substack{i,j \\ (i,j) \text{ an inversion pair in T}}} z_{ij}$$

at $z_{ij} = 1$ for any non-potential inversion and obtain the formula as desired. □

This corollary gives us an easy way to compute the Schur coefficients for these multivariate LLT polynomials in an easy way. This method also may provide a potential path to finding a combinatorial interpretation for all of the Schur coefficients of unicellular LLT polynomials. There is an intimate connection between the inversions in SYT and the Schur coefficients. Finding similar weighting methods and sorting algorithms like those for the hook and augmented hook shapes may be a path towards solving more general cases.

BIBLIOGRAPHY

Alex Abreu and Antonio Nigro. A symmetric function of increasing forests. *Forum of Mathematics, Sigma*, 9:1–21, 2021.

Per Alexandersson and Greta Panova. Llt polynomials, chromatic quasisymmetric functions and graphs with cycles. *Discrete Mathematics*, 341(12):3453–3482, 2018.

Per Alexandersson and Robin Sulzgruber. A combinatorial expansion of vertical-strip llt polynomials in the basis of elementary symmetric functions. *Advances in Mathematics*, 400:108256, 2022.

Sami Assaf. Toward the schur expansion of macdonald polynomials. *Electronic Journal of Combinatorics*, 25:1–20, 2018.

Jonah Blasiak. Haglund's conjecture on 3-column macdonald polynomials. *Mathematische Zeitschrift*, 283:601–628, 2016.

Lynne Butler. *Subgroup Lattices and Symmetric Functions.* Memoirs of the American Mathematical Society; no 539. American Mathematical Society, 1994.

Michele D'Adderio. e-positivity of vertical strip llt polynomials. *Journal of Combinatorial Theory, Series A*, 172:105212, 2020.

Susanna Fishel. Statistics for special q,t-kostka polynomials. *Proceedings of the American Mathematical Society*, 125:2961–2969, 1995.

H.O. Foulkes. A survey of some combinatorial aspects of symmetric functions. *Permutations*, 1974.

Curtis Greene, Albert Nijenhuis, and Herbert Wilf. A probabilistic proof of a formula for the number of young tableaux of a given shape. *Advances in Mathematics*, 31:104–109, 1979.

Jim Haglund. A combinatorial model for the macdonald polynomials. *Proceedings of the National Academy of Sciences*, 101:16127–16131, 2004.

Jim Haglund. The q, t-catalan numbers and the space of diagonal harmonics : with an appendix on the combinatorics of macdonald polynomials. American Mathematical Society, 2007.

Jim Haglund, Mark Haiman, and Nicholas Loehr. A combinatorial formula for macdonald polynomials. *Journal of the American Mathematical Society*, 18:735–761, 2005a.

Jim Haglund, Mark Haiman, and Nick Loehr. A combinatorial formula for macdonald polynomials. *Journal of the American Mathematical Society*, 18:735–761, 2005b.

Mark Haiman. Hilbert schemes, polygraphs, and the macdonald positivity conjecture. *Journal of*

the American Mathematical Society, 14, 2000.

S.V. Kerov, A.N. Kirillov, and N.Y. Reshetikhin. Combinatorics, bethe ansatz, and representations of the symmetric group. *Journal of Soviet Mathematics*, 41:916–924, 1988.

A.N. Kirillov and N.Y. Reshetikhin. Bethe ansatz and combinatorics of young tableaux. *Journal of Soviet Mathematics*, 41:65–115, 1988.

Luc Lapointe and Jennifer Morse. *Tableaux statistics for two part Macdonald polynomials*. World Scientific Publishing, River Edge, NJ, 2003.

Nicholas Loehr. Variants of the rsk algorithm adapted tocombinatorial macdonald polynomials. *Journal of Combinatorial Theory, Series A*, 146:129–164, 2017.

Ian Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, 2nd edition, 2015.

Gilbert de Beauregard Robinson. On representations of the symmetric group. *American Journal of Mathematics*, 60:745–760, 1938.

Bruce Sagan. *The Symmetric Group : Representations, Combinatorial Algorithms, and Symmetric Functions*. Springer Science+Business Media, 2nd edition, 2013.

Craige Schensted. Longest increasing and decreasing subsequences. *Canadian Journal of Mathematics*, 13:179–191, 1961.

Craige Schensted. Permutations, matrices, and generalized young tableaux. *Pacific Journal of Mathematics*, 34:709–727, 1970.

John Shareshian and Michelle Wachs. Chromatic quasisymmetric functions. *Advances in Mathematics*, 295:497–551, 2016.

Richard Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111(1):166–194, 1995.

Richard Stanley. *Enumerative Combinatorics: Volume 2*. Cambridge University Press, USA, 1st edition, 1997.

Richard Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, USA, 2nd edition, 2011.

John Stembridge. Some particular entries of the two-parameter kostka matrix. *Proceedings of the American Mathematical Society*, 121:469–490, 1994.

Mike Zabrocki. A macdonald vertex operator and standard tableaux statistics for the two-column

(q;t)-kostka coeficients. *Electronic Journal of Combinatorics*, 5:1–46, 1998.