A Combinatorial Model for
for the
Macdonald Polynomials

J. Haglund, Proc. Nat. Acad. Sci. USA, 2004

A Combinatorial Formula for
Macdonald Polynomials

J. Haglund, M. Haiman, N. Loehr, 2004

arXiv.org/abs/math.CO/0409538

condensed version to also appear in PNAS

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Permutation Statistics and q-Analogues

In combinatorics a *statistic* on a finite set S is a mapping from $S \to \mathbb{N}$ given by an explicit combinatorial rule.

Ex. Given $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, define

$$\mathrm{inv}\pi = |\{(i,j) : i < j \quad \text{and} \quad \pi_i > \pi_j\}|$$

and

$$\mathrm{maj}\pi = \sum_{\pi_i > \pi_{i+1}} i.$$

If $\pi = 31542$,

$$\mathrm{inv}\pi = 2 + 2 + 1 = 5$$

and

$$\mathrm{maj}\pi = 1 + 3 + 4 = 8.$$

Let

$$(n)_q = (1-q^n)/(1-q)$$
$$= 1 + q + \ldots + q^{n-1}$$

and

$$(n!)_q = \prod_{i=1}^{n} (i)_q$$
$$= (1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{n-1})$$

be the q-analogues of n and $n!$. Then

$$\sum_{\pi \in S_n} q^{\mathrm{inv}\pi} = (n!)_q = \sum_{\pi \in S_n} q^{\mathrm{maj}\pi}.$$

Symmetric Functions

A *symmetric function* is a polynomial $f(x_1, x_2, \ldots, x_n)$ which satisfies

$$f(x_{\pi_1}, \ldots, x_{\pi_n}) = f(x_1, \ldots, x_n),$$

i.e. $\pi f = f$, for all $\pi \in S_n$.

Examples

- The *monomial symmetric functions* $m_\lambda(X)$

$$\begin{aligned} m_{(2,1)}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1^2 x_3 \\ &\quad + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2. \end{aligned}$$

- The elementary symmetric functions $e_k(X)$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3.$$

- The power-sums $p_k(X) = \sum_i x_i^k$.

• The Schur functions $s_\lambda(X)$, which are important in the representation theory of the symmetric group:

$$s_\lambda(X) = \sum_{\beta \vdash n} K_{\lambda,\beta}\, m_\beta(X)$$

where $K_{\lambda,\beta}$ equals the number of ways of filling the Ferrers shape of λ with elements of the multiset $\{1^{\beta_1} 2^{\beta_2} \cdots\}$, weakly increasing across rows and strictly increasing down columns. For example $K_{(4,2),(2,2,1,1)} = 4$

$$\begin{array}{llll} 2 & 3 & & \\ 1 & 1 & 2 & 4 \end{array} \qquad \begin{array}{llll} 2 & 2 & & \\ 1 & 1 & 3 & 4 \end{array}$$

$$\begin{array}{llll} 2 & 4 & & \\ 1 & 1 & 2 & 3 \end{array} \qquad \begin{array}{llll} 3 & 4 & & \\ 1 & 1 & 2 & 2 \end{array}$$

Selberg's Integral For $k, a, b \in \mathbb{C}$,

$$\int_{(0,1)^n} |\prod_{1 \le i < j \le n} (x_i - x_j)|^{2k} \prod_{i=1}^{n} x_i^{a-1}(1-x_i)^{b-1} dx_1 \cdots dx_n$$

$$= \prod_{i=1}^{n} \frac{\Gamma(a+(i-1)k)\Gamma(b+(i-1)k)}{\Gamma(a+b+(n+i-2)k)} \times \frac{\Gamma(ik+1)}{\Gamma(k+1)}.$$

• Kadell (1988) conjectured there existed symmetric functions which could be inserted into the integrand to generalize Selberg's integral in a certain interesting way.

• Macdonald (1988) showed how to define such, by means of orthogonality with respect to a generalization of the Hall scalar product, or aternatively by means of a complicated constant term identity.

Macdonald's generalization: There exist symmetric functions $P_\lambda(X;q,t)$ such that if $t=q^k$ for some $k\in\mathbb{N}$,

$$\frac{1}{n!}\int_{(0,1)^n} P_\lambda(X;q,t)$$

$$\prod_{1\le i<j\le n}\prod_{r=0}^{k-1}(x_i-q^r x_j)(x_i-q^{-r}x_j)$$

$$\prod_{i=1}^{n} x_i^{a-1}(x_i;q)_{b-1}d_q x_1\cdots d_q x_n$$

$$= q^F\prod_{i=1}^{n}\frac{\Gamma_q(\lambda_i+a+(i-1)k)}{\Gamma_q(\lambda_i+a+b+(n+i-2)k)}$$

$$\times\Gamma_q(b+(i-1)k)$$

$$\times\prod_{1\le i<j\le n}\frac{\Gamma_q(\lambda_i-\lambda_j+(j-i+1)k)}{\Gamma_q(\lambda_i-\lambda_j+(j-i)k)}$$

where $k \in \mathbb{N}$,

$$F = k\eta(\lambda)$$
$$+ kan(n-1)/2 + k^2 n(n-1)(n-2)/3,$$

$t = q^k$,

$$\Gamma_q(z) = (1-q)^{1-z}(q;q)_\infty/(q^z;q)_\infty$$

is the q-gamma function with

$$(x;q)_\infty = \prod_{i \geq 0}(1 - xq^i),$$

and

$$\int_0^1 f(x)d_q x = \sum_{i=0}^{\infty} f(q^i)(q^i - q^{i+1})$$

is the q-integral.

Plethysm: If $F(X)$ is a symmetric function, then $F[(w-1)X]$ is defined by expressing $F(X)$ as a polynomial in the $p_k(X) = \sum_i x_i^k$'s and then replacing each $p_k(X)$ by $(w^k - 1)p_k(X)$. Macdonald's construction of the P_λ can be reformulated as follows.

Theorem. *(Macdonald) Given a partition μ, there exists a unique symmetric polynomial $\tilde{H}_\mu[X; q, t]$ characterized by the following:*

(i) $\tilde{H}_\mu[X(q-1); q, t] \in \mathbb{Q}(q,t)\{m_\lambda : \lambda \leq \mu'\}$

(ii) $\tilde{H}_\mu[X(t-1); q, t] \in \mathbb{Q}(q,t)\{m_\lambda : \lambda \leq \mu\}$

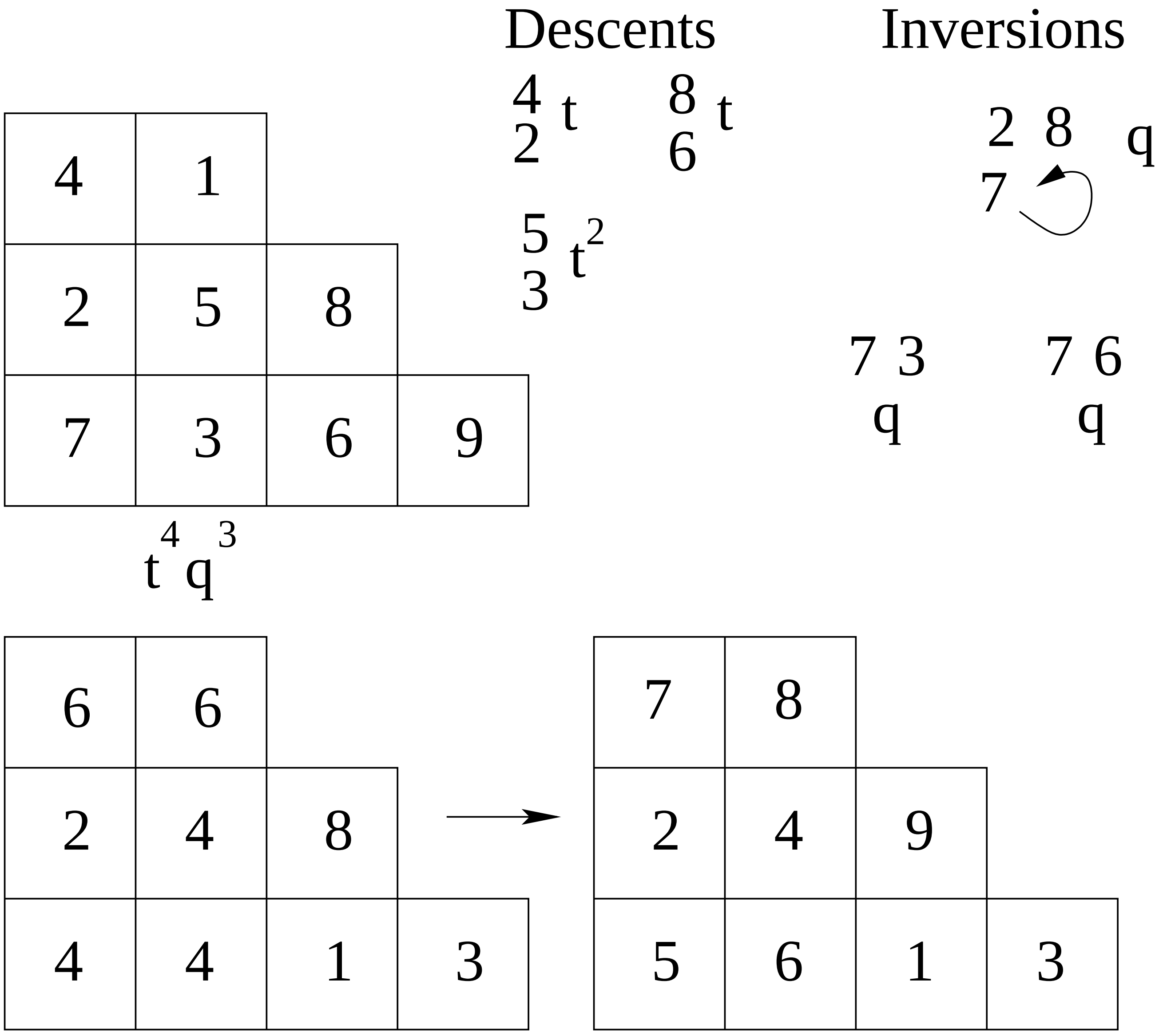
(iii) $\tilde{H}_\mu[1; q, t] = 1.$

where we use the "dominance" partial order on partitions;

$$\lambda \leq \mu \iff \lambda_1 + \ldots + \lambda_i \leq \mu_1 + \ldots + \mu_i$$

for $1 \leq i \leq n$.

The Combinatorics of $\tilde{\mathbf{H}}_\mu[\mathbf{X};\mathbf{q},\mathbf{t}]$



Reading Word: 6 6 2 4 8 4 4 1 3

Definition. *Let $Inv(\sigma, \mu)$ denote the set of inversion triples, and and $Des(\sigma, \mu)$ the set of descents. Set*

$$inv(\sigma, \mu) = |Inv(\sigma, \mu)|,$$

$$maj(\sigma, \mu) = \sum_{w \in \mathrm{Des}(\sigma,\mu)} 1 + leg(w)$$

Remark: $\mathrm{maj}(\sigma, 1^n) = \mathrm{maj}(\sigma)$ and $\mathrm{inv}(\sigma, n) = \mathrm{inv}(\sigma)$.

Definition. *Let $X = (x_1, x_2, \ldots, x_n)$ and set*

$$C_\mu[X; q, t] = \sum_{\sigma : \mu \to \mathbb{Z}^+, \sigma_i \leq n} q^{\mathrm{inv}(\sigma,\mu)} t^{\mathrm{maj}(\sigma,\mu)} x^\sigma,$$

where $x^\sigma = \prod_i x_{\sigma_i}$.

Theorem. *(H., Haiman, Loehr; 2004) (Conjectured by H., PNAS (2004))*

$$\tilde{H}_\mu[X; q, t] = C_\mu[X; q, t].$$

The Proof

Proposition. *Let $\mathcal{A}$ be an alphabet of positive and negative letters, with any fixed total ordering. Then*

$$C_\mu[X(w-1);q,t] = \sum_{\sigma:\mu\to\mathcal{A}} (-1)^{\#\mathrm{neg}} w^{\#\mathrm{pos}} q^{\mathrm{inv}} t^{\mathrm{maj}} x^{|\sigma|}$$

where #neg, #pos are the # of negative and positive letters, respectively.

$$\begin{array}{|c|c|c|c|}
\cline{1-2}
\overline{3} & \overline{2} \\
\cline{1-3}
\overline{2} & 1 & 4 \\
\hline
\overline{1} & 3 & 4 & 2 \\
\hline
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|}
\cline{1-2}
7 & 5 \\
\cline{1-3}
4 & 1 & 8 \\
\hline
2 & 6 & 9 & 3 \\
\hline
\end{array}$$

$$(-1)^4 t^4 w^5 q^4 x_1^2 x_2^3 x_3^2 x_4^2$$

$$1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$$

Involution 1

(a) We say two squares "attack" each other if they form a potential inversion pair. Find last attacking pair of 1's, $\overline{1}$'s in the reading word (if none, find last attacking pair of 2's, etc.).

(b) Switch the sign of first element (in reading word order) of attacking pair.

(c) Use ordering $1 < \overline{1} < \cdots < n < \overline{n}$.

• The Descent set is fixed, so the t-weight is fixed. The q-weight is also fixed

• The fixed points are those super fillings without attacking pairs, so at most one $1, \overline{1}$ in any row, at most one $2, \overline{2}$ in any row, so if the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ is nonzero, we must have $\lambda_1 \leq \mu'_1$, $\lambda_1 + \lambda_2 \leq \mu'_1 + \mu'_2$, etc., where μ' is the conjugate partition obtained by rotating the Ferrers graph of μ.

Involution 2

(a) Find lst 1 or $\overline{1}$ in reading word, not in the bottom row. (if none, find first 2 or $\overline{2}$ not in the bottom two rows, etc.).

(b) Switch the sign of this element.

(c) Use the ordering

$$1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}.$$

• The q and t weights are preserved.
• The fixed points have $1, \overline{1}$'s only in the bottom row, $2, \overline{2}$'s only in the bottom two rows, etc. Thus if coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ is nonzero, we must have $\lambda_1 \leq \mu_1$, $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$, etc.

The fact that $C_\mu[1; q, t] = 1$ is easy to show, and the proof is complete.

Using the well-known relation $\tilde{H}_\mu[Z;q,t] = \tilde{H}_{\mu'}[Z;t,q]$, involution (1) from the proof of our theorem gives a interpretation for

$$t^{\eta(\mu)+n}\tilde{H}_{\mu'}[X(t^{-1}-1);t^{-1},q] = J_\mu[X;q,t]$$

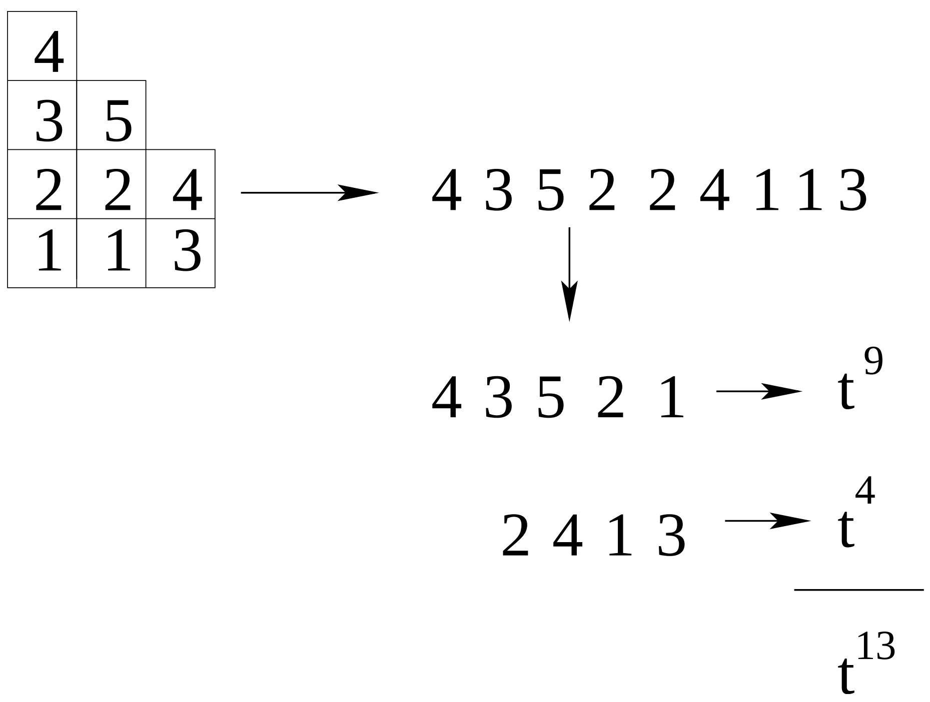
in terms of super fillings of μ'. By grouping super fillings σ according to the value of $|\sigma|$ we get

Corollary. *For any partition μ,*

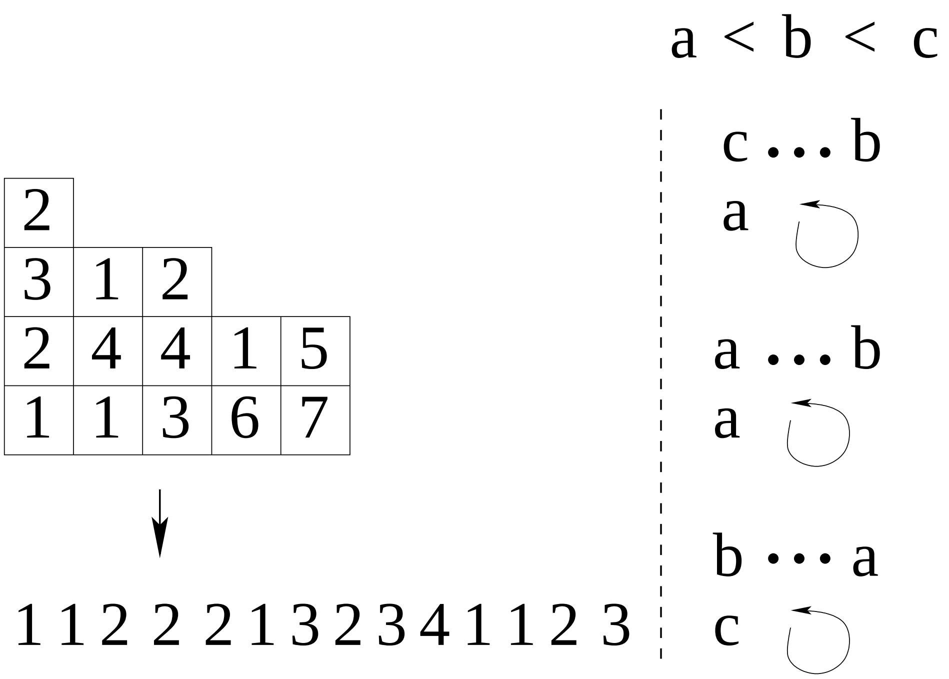
$$J_\mu[X;q,t] = \sum_{\text{nonattacking fillings } (T,\mu')} x^T q^{\mathrm{maj}(T,\mu')} t^{\eta(\mu)-\mathrm{inv}(T,\mu')}$$
$$\times \prod_{\substack{w\in\mu' \\ T(w)=T(\mathrm{South}(w))}} (1-q^{1+\mathrm{leg}(w)}t^{1+\mathrm{arm}(w)})$$
$$\times \prod_{\substack{w\in\mu' \\ T(w)\neq T(\mathrm{South}(w))}} (1-t).$$

Theorem. *(Lascoux - Schützenberger)* 1978 *(cocharge formula for Hall-Littlewood polynomials)*

$$\tilde{H}_\mu[X;0,t] = \sum_\lambda s_\lambda \sum_{T \in SSYT(\lambda,\mu)} t^{\mathrm{cocharge}(T)}.$$



Proof: Difficult, using recurrences.



New Proof:

$$\tilde{H}_\mu[X;0,t] = \sum_{\sigma:\mathrm{inv}(\sigma,\mu)=0} t^{\mathrm{maj}(\sigma,\mu)} x^\sigma .$$

Now use well-known properties of the RSK algorithm to get the Schur expansion:

$$\sum_\lambda \Big(\sum_{P \in SSYT(\lambda,\mu)} t^{\mathrm{cocharge}(P)} \Big)$$

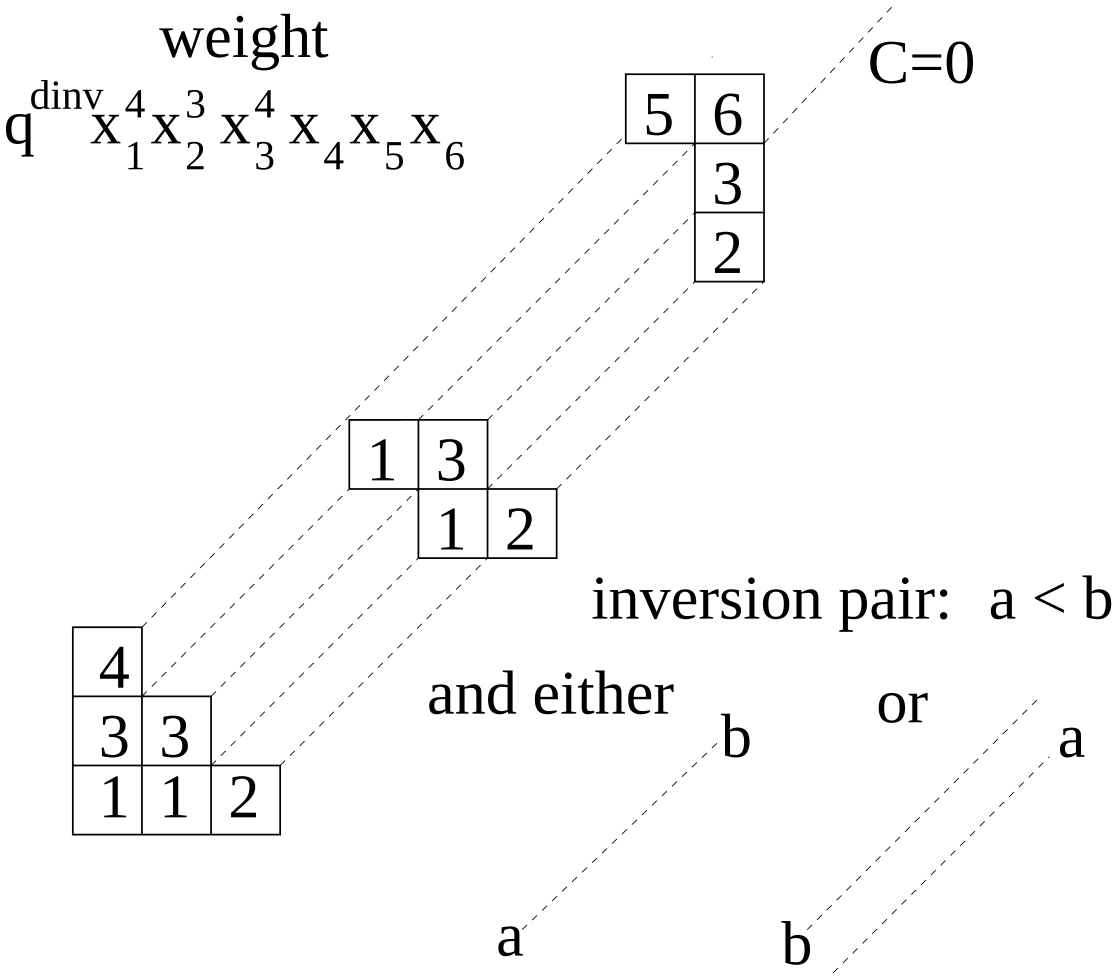
$$\Big(\sum_{Q \in SSYT(\lambda)} x^Q \Big)$$

LLT Polynomials

In 1997, Lascoux, Leclerc and Thibon introduced ribbon tableaux generating functions, or spin generating functions, commonly known as LLT polynomials, which depend on $x_1, \ldots x_n$ and a parameter q. They proved (non-combinatorially) these polynomials are symmetric in the x_i and conjectured they are Schur positive.

Theorem. *(H., Haiman, Loehr, Remmel, Ulyanov; 2003) (to appear in Duke Math. J.) The LLT polynomial equals a power of q times the sum, over all tuples $\mathbf{T}$ of SSYT of skew shape, of $q^{\mathrm{dinv}(\mathbf{T})}$, where dinv($\mathbf{T}$) is the total number of inversion pairs, described below.*

• Schilling, Schimizono and White have a similar result, with dinv replaced by the number of "inversion triples", more complicated to describe.



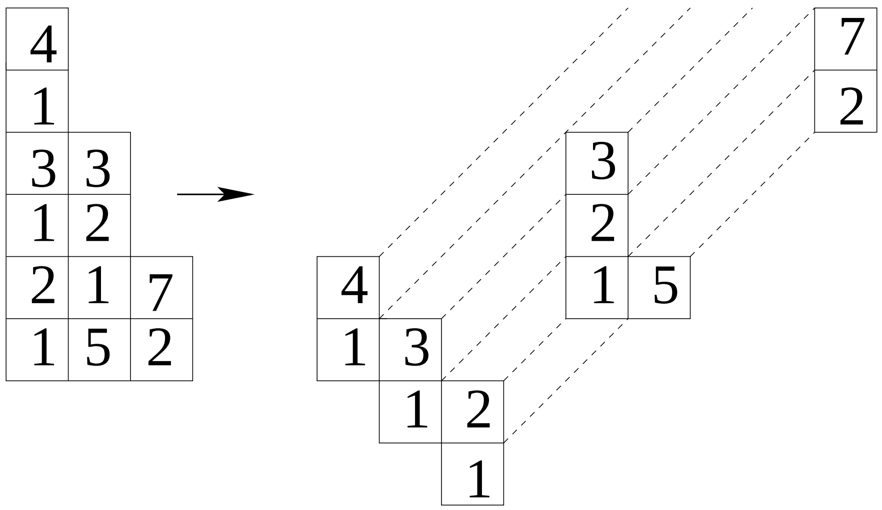
• In H., Haiman, Loehr we show how $\tilde{H}_\mu[X;q,t]$ can be expressed as a sum of LLT polynomials times nonnegative powers of t. We also give a new, combinatorial proof of LLT symmetry.

Theorem. *If we fix a descent set D, then*

$$\sum_{\sigma:\mathrm{Des}(\sigma)=D} t^{\mathrm{maj}(\sigma,\mu)} q^{\mathrm{inv}(\sigma,\mu)} x^{\sigma}$$

is a fixed power of t times a fixed power of q times an LLT product of ribbons.

Proof: There is a bijection between fillings with a fixed descent set D and tuples of SSYT of ribbons:



Now use the fact that

$$\mathrm{inv}(\sigma,\mu) = \mathrm{dinv} - \sum_{w\in\mathrm{Des}(\sigma)} \mathrm{arm}(w).$$

We define the (q,t)-Kostka coefficients as follows:

$$\tilde{H}_\mu[X;q,t] = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t) s_\lambda[X].$$

Macdonald conjectured these coefficients were in $\mathbb{N}[q,t]$. He proved $\tilde{K}_{\lambda,\mu}(1,1) = K_{\lambda,1^n}$ and asked if

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_T q^{a(\mu,T)} t^{b(\mu,T)}$$

for some statistics a, b on partitions μ and standard tableaux T.

By the previous result, understanding the Schur coefficients is equivalent to understanding the Schur coefficients of LLT products of ribbons. (LLT products of tuples of partition diagrams are known to be Schur positive. A combinatorial interpretation is known for tuples of length 2).

Theorem. *(HHL) If μ has at most two columns, i.e. $\mu_1 \leq 2$, then*

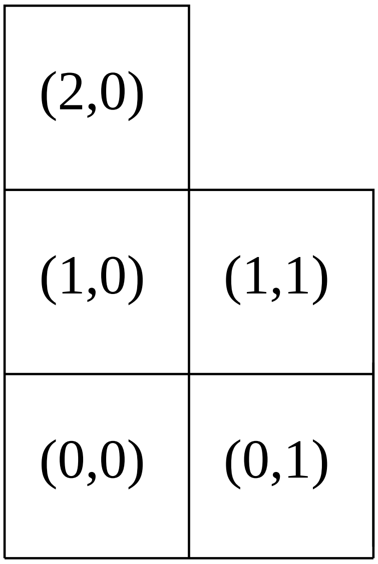
$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{\sigma} t^{\mathrm{maj}(\sigma,\mu)} q^{\mathrm{inv}(\sigma,\mu)} x^{\sigma},$$

summed over all fillings σ of μ for which the reading word $\sigma_1\sigma_2\cdots\sigma_n$ is Yamanouchi, i.e. each final segment $\sigma_k\sigma_{k+1}\cdots\sigma_n$ has partition content.

Proof: This also follows from van Leeuwen's (2000) theorem on LLT Schur coefficients for 2-tuples, but our proof is simpler, avoiding domino tableaux.

• H. (PNAS 2004) contains a conjectured combinatorial formula for three column shapes, which reduces to above when μ has 2 columns.

The $n!$ Conjecture



$\mu = (2,2,1)$

$$\Delta(\mu) = \begin{vmatrix} 1 & y_1 & x_1 & x_1 y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2 y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3 y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4 y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5 y_5 & x_5^2 \end{vmatrix}$$

For $\mu \vdash n$ let $V(\mu)$ denote the linear span over $\mathbb{Q}$ of all partial derivatives of all orders of $\Delta(\mu)$. $V(\mu)$ decomposes as a direct sum of its bihomogeneous subspaces $V^{i,j}(\mu)$ of degree i in the x-variables and j in the y-variables. There is an S_n-action on $V^{i,j}(\mu)$ given by

$$\pi f = f(x_{\pi_1}, \ldots, x_{\pi_n}, y_{\pi_1}, \ldots, y_{\pi_n})$$

called the *diagonal action.*

The *Frobenius Series* is the symmetric function

$$\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{ij},$$

where m_{ij} is the multiplicity of the irreducible S_n-character χ^λ in the diagonal action on $V^{i,j}(\mu)$.

Theorem. *(Haiman; JAMS 2001) (The "$n!$ Conjecture", advanced by Garsia and Haiman in the early 1990's). The Frobenius Series of $V(\mu)$ is given by the modified Macdonald polynomial $\tilde{H}_\mu[X;q,t]$. In particular, the dimension of $V(\mu)$ is $n!$, where $n = \sum_i \mu_i$.*

Corollary.

$$\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$$

The proof uses algebraic geometry and commutative algebra. It doesn't yield any combinatorial interpretation for the $\tilde{K}_{\lambda,\mu}(q,t)$.

Since the coefficient of m_{1^n} in the expansion of the Frobenius series into monomials equals the Hilbert series, the above theorem and our formula together imply

Corollary. *(HHL) The Hilbert Series of $V(\mu)$ is given by*

$$\sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma,\mu)} q^{\mathrm{inv}(\sigma,\mu)}.$$

What led to the statistics?

Garsia and Haiman pioneered the study of the space of *diagonal harmonics* DH_n, which is

$$\{f : \sum_{i=1}^{n} \partial x_i^h \partial y_i^k f = 0, \forall h + k > 0\}.$$

The space DH_n decomposes as a direct sum of subspaces of bihomogeneous degree (i, j); $DH_n = \bigoplus_{i,j} DH_n^{i,j}$. The *Hilbert Series* is the sum

$$\sum_{i,j \geq 0} q^i t^j \dim(DH_n^{i,j}).$$

Example: If $n = 2$, a basis for the space is $1, x_2 - x_1, y_2 - y_1$, and the Hilbert Series is $1 + q + t$.

The Frobenius Series is the sum

$$\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{i,j}$$

where $m_{i,j}$ is the multiplicity of χ^λ in the character of $DH_n^{i,j}$ under the diagonal action of S_n.

Example: If $n = 2$, the Frobenius series is

$$s_2(X) + s_{1^2}(X)(q + t).$$

Let ∇ be a linear operator on the basis $\tilde{H}_\mu(X; q, t)$ given by

$$\nabla \tilde{H}_\mu(X; q, t) = t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_\mu(X; q, t).$$

Theorem. *(Haiman, Inventiones* 2002*) The Frobenius Series of DH_n is given by $\nabla e_n(X)$.*

Corollary. *The dimension of the space DH_n of diagonal harmonics, as a vector space over $\mathbb{Q}$, is $(n + 1)^{n-1}$.*

Corollary. *The dimension of the space DH_n^ϵ of diagonal harmonic alternants, corresponding to the sign character χ^{1^n}, is the nth Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Garsia and Haiman (1995) introduced the q,t-Catalan sequence, $C_n(q,t)$, defined by

$$C_n(q,t) = \langle \nabla e_n, s_{1^n} \rangle.$$

Using some of Macdonald's original results, $C_n(q,t)$ can be expressed as a complicated sum of rational functions in q,t. They posed the problem of finding combinatorial statistics to describe $C_n(q,t)$.

Example. *From the rational function definition of $C_n(q,t)$, for $n=2$ we have*

$$C_2(q,t) = \frac{t^2}{t-q} + \frac{q^2}{q-t} = \frac{t^2-q^2}{t-q} = t+q.$$

After simplification the terms in $C_3(q,t)$ are

$$\mu = 3; \quad \frac{q^6}{q^2-t)}$$

$$\mu = 21; \quad \frac{t^2q^2(1+q+t)}{(q-t^2)(t-q^2)}$$

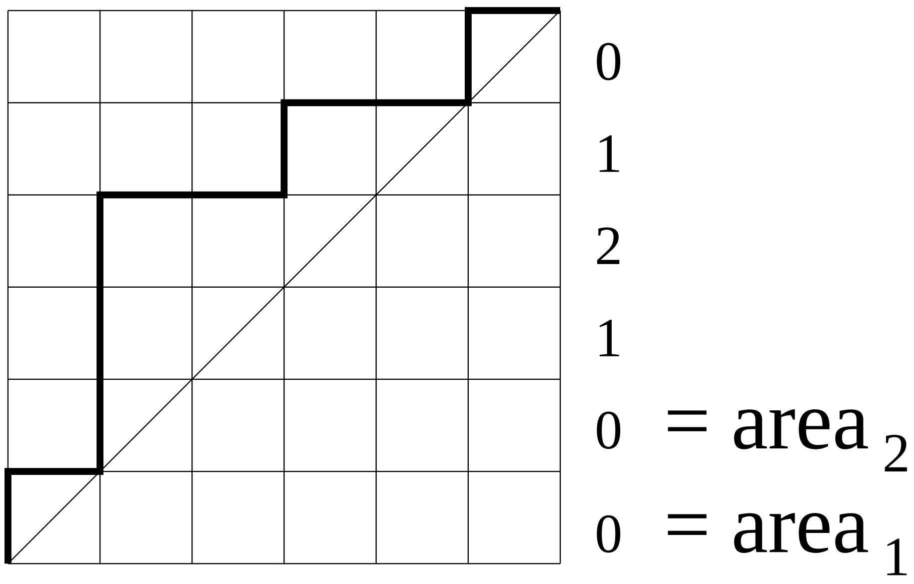
$$\mu = 1^3; \quad \frac{t^6}{(t^2-q)(t-q)}$$

So

$$C_3(q,t) =$$

$$\frac{q^6(t^2-q)+t^2q^2(1+q+t)(q-t)+t^6(t-q^2)}{(q^2-t)(t^2-q)(q-t)}$$

$$= q^3+q^2t+qt^2+qt+t^3.$$



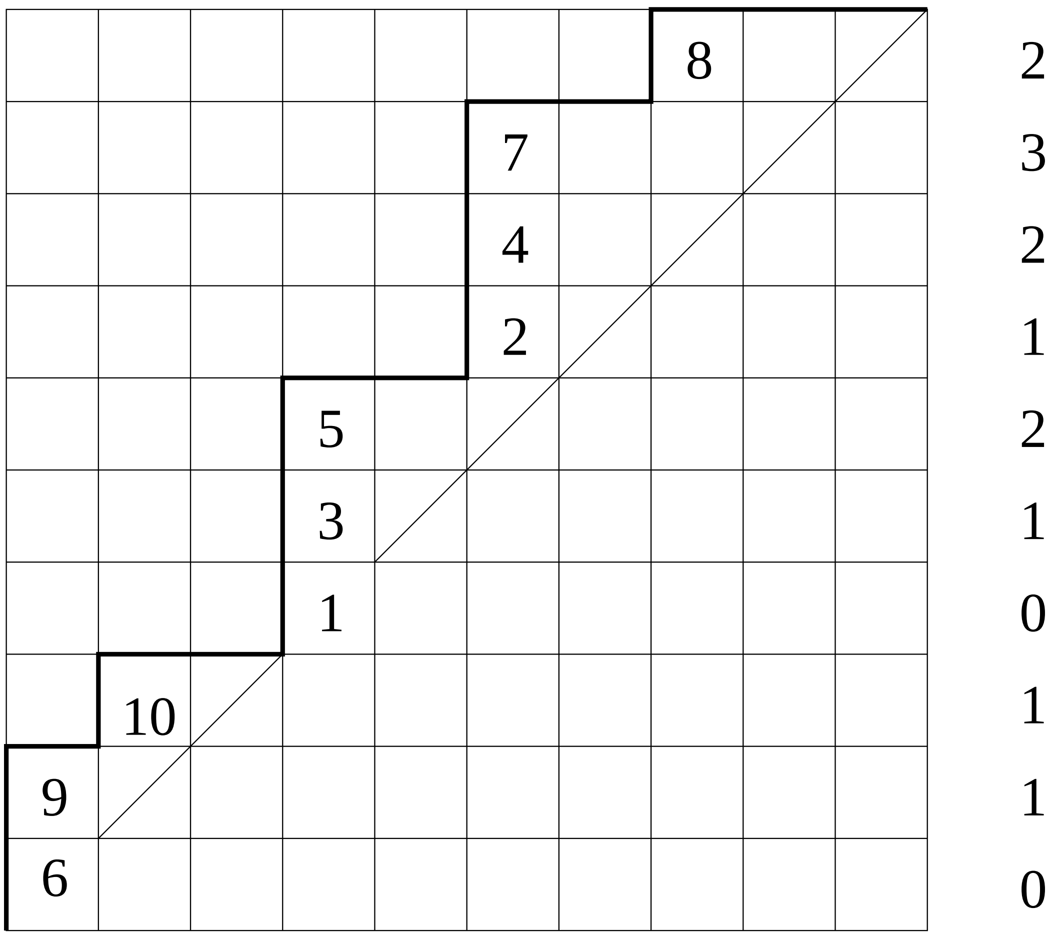
area = 4, dinv = 7

Theorem. *(Garsia, H.,* 2001*, PNAS) (First conjectured by H. (Adv. Math.) in a different form, and later independently by Haiman)*

$$C_n(q,t) = \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)},$$

where $dinv(\pi) = \#\{(i,j), i < j\}$ *with*

$$area_i = area_j \quad or \quad area_i = area_j - 1.$$



$$q^{6}\,t^{13}$$

$$\text{dinv} = \#\{(i,j), i<j\} \text{ satisfying either}$$

$$\text{area}_i = \text{area}_j \quad \text{and} \quad \text{car}_i < \text{car}_j, \quad \text{or}$$

$$\text{area}_i = \text{area}_j - 1 \quad \text{and} \quad \text{car}_i > \text{car}_j$$

Conjecture. *(H., Loehr) The Hilbert Series of DH_n is given by*

$$\sum_{\sigma} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\sigma)},$$

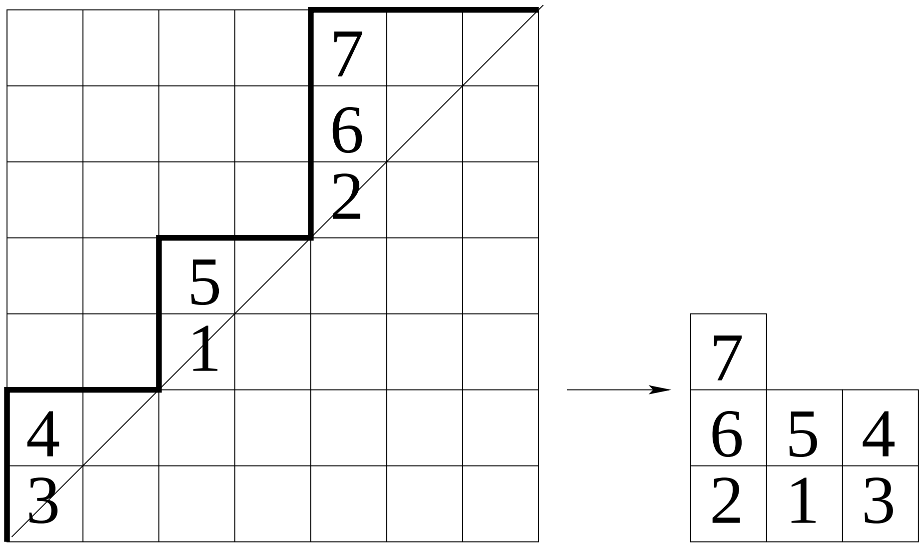
where the sum is over all parking functions on n cars.

Conjecture. *(H., Haiman, Loehr, Remmel, Ulyanov,* 2003, *to appear in Duke Math. J.)*

$$\nabla e_n = \sum_{\sigma} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\sigma)} x^{\sigma},$$

where the sum is over all "word parking functions" with n cars.

Open Problem. *Prove $C_n(q,t) = C_n(t,q)$ bijectively, and similarly for the Hilbert series, etc.*



- $q = 1$: Macdonald's work implies that

$$\tilde{H}_\mu[X;1,t] = \sum_\sigma t^{\mathrm{maj}(\sigma,\mu)} x^\sigma.$$

- descents everywhere:

$$\mathrm{inv}(\sigma,\mu) = \mathrm{dinv} - \sum_{i>1} \binom{\mu_i}{2}$$

- no descents : $\quad \mathrm{inv}(\sigma,\mu) = \mathrm{dinv}$
- in general:

$$\mathrm{inv}(\sigma,\mu) = \mathrm{dinv} - \sum_{w \in Des} \mathrm{arm}(w)$$