The (q, t)-Catalan Numbers and the Space of Diagonal Harmonics

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Permutation Statistics and q-Analogues

In combinatorics a *statistic* on a finite set S is a mapping from $S \to \mathbb{N}$ given by an explicit combinatorial rule.

Ex. Given $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, define

$$\mathrm{inv}\pi = |\{(i,j) : i < j \quad \text{and} \quad \pi_i > \pi_j\}|$$

and

$$\mathrm{maj}\pi = \sum_{\pi_i > \pi_{i+1}} i.$$

If $\pi = 31542$,

$$\mathrm{inv}\pi = 2 + 2 + 1 = 5$$

and

$$\mathrm{maj}\pi = 1 + 3 + 4 = 8.$$

Let

$$(n)_q = (1 - q^n)/(1 - q)$$
$$= 1 + q + \ldots + q^{n-1}$$

and

$$(n!)_q = \prod_{i=1}^{n} (i)_q$$
$$= (1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{n-1})$$

be the q-analogues of n and $n!$. Then

$$\sum_{\pi \in S_n} q^{\mathrm{inv}\pi} = (n!)_q = \sum_{\pi \in S_n} q^{\mathrm{maj}\pi}.$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{N}$ for $1 \le i \le n$ be a *partition* and let $|\lambda| = \sum_i \lambda_i$. Define

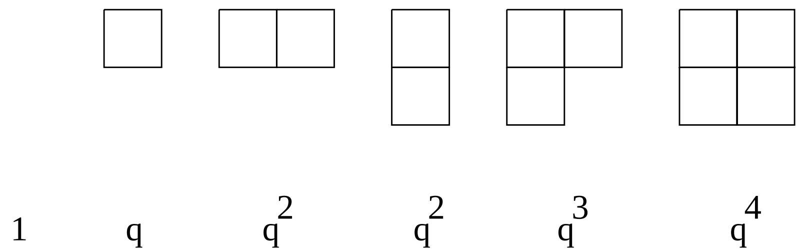
$$\binom{n}{k}_q = \frac{(n!)_q}{(k!)_q((n-k)!)_q}.$$

Theorem. *For $n, k \in \mathbb{N}$,*

$$\binom{n+k}{k}_q = \sum_{(\lambda_1,\ldots,\lambda_n)\le(k,k,\ldots,k)} q^{|\lambda|}.$$

Note: We denote the conjugate partition by λ'.

Example $n = k = 2$. The Ferrers shapes are



The Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Recurrence:

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

Over 70 interpretations in Stanley's Enumerative Combinatorics Volume 2, including

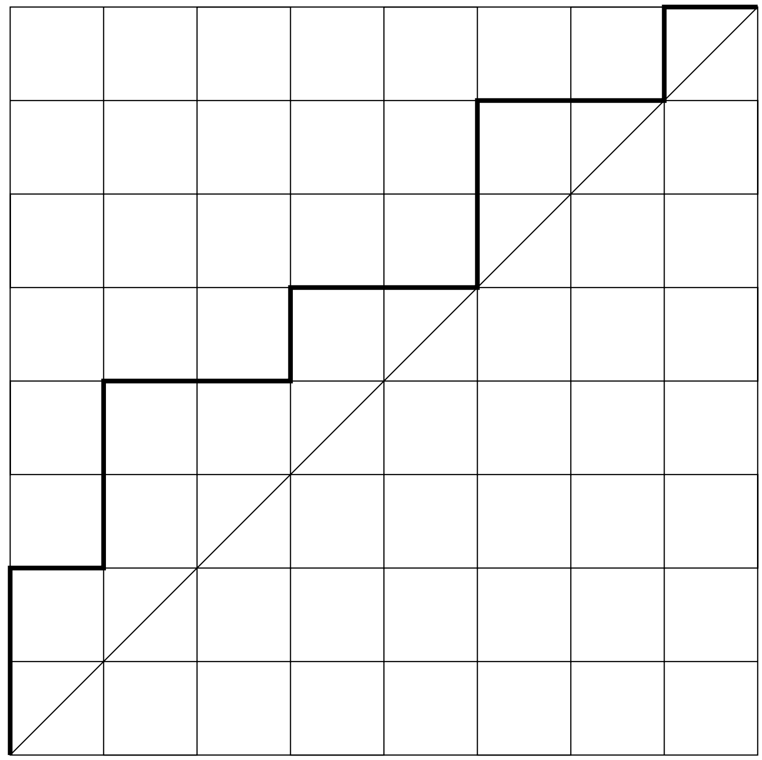
• The number of "standard tableaux" of shape (n, n):

123	124	125	134	135
456	356	346	256	246

• The number of "Catalan words", i.e. mulitset permutations of $\{0^n 1^n\}$ where in any initial segment, the number of zeros is at least as big as the number of ones.

000111 001011 001101 010011 010101

• The number of "Catalan paths" from $(0, 0)$ to (n, n), i.e. lattice paths consisting of N and E steps which never go below the main diagonal.



A Catalan Path

Theorem. *(MacMahon)*

$$\sum_{\text{Catalan words } \sigma} q^{\mathrm{maj}(\sigma)} = \frac{1}{(n+1)_q} \binom{2n}{n}_q .$$

The *Carlitz-Riordan q-Catalan*
Let $\mathcal{D}_n$ denote the set of Catalan paths, and set

$$C_n(q) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{area}(\sigma)}$$

where $\mathrm{area}(\sigma)$ is the number of squares below the path and strictly above the diagonal.

Proposition.

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q).$$

Symmetric Functions

A *symmetric function* is a polynomial $f(x_1, x_2, \ldots, x_n)$ which satisfies

$$f(x_{\pi_1}, \ldots, x_{\pi_n}) = f(x_1, \ldots, x_n),$$

i.e. $\pi f = f$, for all $\pi \in S_n$.

Examples

- The *monomial symmetric functions* $m_\lambda(X)$

$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2.$$

- The elementary symmetric functions $e_k(X)$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3.$$

- The power-sums $p_k(X) = \sum_i x_i^k$.

• The Schur functions $s_\lambda(X)$, which are important in the representation theory of the symmetric group:

$$s_\lambda(X) = \sum_{\beta \vdash n} K_{\lambda,\beta}\, m_\beta(X)$$

where $K_{\lambda,\beta}$ equals the number of ways of filling the Ferrers shape of λ with elements of the multiset $\{1^{\beta_1} 2^{\beta_2} \cdots\}$, weakly increasing across rows and strictly increasing down columns. For example $K_{(4,2),(2,2,1,1)} = 3$

$$\begin{matrix} 1 & 1 & 2 & 4 \\ 2 & 3 & & \end{matrix}$$

$$\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 4 & & \end{matrix} \qquad \begin{matrix} 1 & 1 & 2 & 2 \\ 3 & 4 & & \end{matrix}$$

Selberg's Integral For $k, a, b \in \mathbb{C}$,

$$\int_{(0,1)^n} |\prod_{1 \leq i < j \leq n} (x_i - x_j)|^{2k}$$

$$\prod_{i=1}^{n} x_i^{a-1}(1 - x_i)^{b-1} dx_1 \cdots dx_n$$

$$= \prod_{i=1}^{n} \frac{\Gamma(a + (i-1)k)\Gamma(b + (i-1)k)}{\Gamma(a + b + (n + i - 2)k)}$$

$$\times \frac{\Gamma(ik + 1)}{\Gamma(k + 1)}.$$

Macdonald's Generalization: There exist symmetric functions $P_\lambda(X;q,t)$ such that

$$\frac{1}{n!}\int_{(0,1)^n} P_\lambda(X;q,t)$$

$$\prod_{1\le i<j\le n}\prod_{r=0}^{k-1}(x_i - q^r x_j)(x_i - q^{-r} x_j)$$

$$\prod_{i=1}^{n} x_i^{a-1}(x_i;q)_{b-1} d_q x_1 \cdots d_q x_n$$

$$= q^F \prod_{i=1}^{n} \frac{\Gamma_q(\lambda_i + a + (i-1)k)}{\Gamma_q(\lambda_i + a + b + (n+i-2)k)}$$

$$\times \Gamma_q(b + (i-1)k)$$

$$\times \prod_{1\le i<j\le n} \frac{\Gamma_q(\lambda_i - \lambda_j + (j-i+1)k)}{\Gamma_q(\lambda_i - \lambda_j + (j-i)k)}$$

where $k \in \mathbb{N}$,

$$F = k\eta(\lambda)$$

$$+kan(n-1)/2+k^2n(n-1)(n-2)/3,$$

$$t = q^k,$$

$$\Gamma_q(z) = (1-q)^{1-z}(q;q)_\infty/(q^z;q)_\infty$$

is the q-gamma function with

$$(x;q)_\infty = \prod_{i\geq 0}(1 - xq^i),$$

and

$$\int_0^1 f(x)d_qx = \sum_{i=0}^{\infty} f(q^i)(q^i-q^{i+1})$$

is the q-integral.

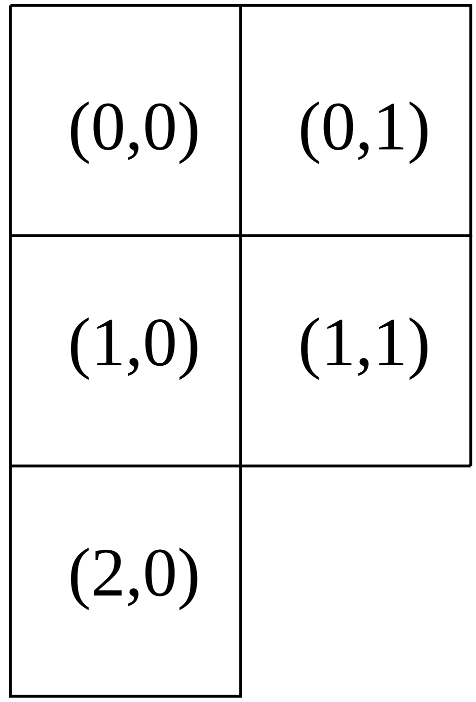
Plethysm: If $F(X)$ is a symmetric function, then $F[(1-t)X]$ is defined by expressing $F(X)$ as a polynomial in the $p_k(X) = \sum_i x_i^k$'s and then replacing each $p_k(X)$ by $(1-t^k)p_k(X)$.

Macdonald expanded scalar multiples of his $P_\lambda(q,t)$ in terms of the basis $s_\lambda[(1-t)X]$ and called the coefficients $K_{\lambda,\mu}(q,t)$. He conjectured these coefficients were in $\mathbb{N}[q,t]$. He proved $K_{\lambda,\mu}(1,1) = K_{\lambda,1^n}$ and asked if

$$K_{\lambda,\mu}(q,t) = \sum_T q^{a(\mu,T)} t^{b(\mu,T)}$$

for some statistics a, b on partitions μ and standard tableaux T.

μ



$$\Delta(\mu) = \begin{vmatrix} 1 & y_1 & x_1 & x_1 y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2 y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3 y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4 y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5 y_5 & x_5^2 \end{vmatrix}$$

For $\mu \vdash n$ let $V(\mu)$ denote the linear span over $\mathbb{Q}$ of all partial derivatives of all orders of $\Delta(\mu)$.

$\pi($ a basis element $)$
$=$ linear combo. of basis elements.

$$\pi(\text{ a basis }) = \text{ matrix } M(\pi).$$

$$M(\pi) * M(\beta) = M(\pi * \beta).$$

The *character* χ is the *trace* of $M(\pi)$, which is independent of the basis. Furthermore $\exists$ a basis for which

$$M = \begin{bmatrix} \begin{matrix} \ast\ast\ast\ast\ast\ast \\ \ast\ast\ast\ast\ast\ast \\ \ast\ast\ast\ast\ast\ast \\ \ast\ast\ast\ast\ast\ast \\ \ast\ast\ast\ast\ast\ast \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \ast\ast\ast\ast \\ \ast\ast\ast\ast \\ \ast\ast\ast\ast \\ \ast\ast\ast\ast \end{matrix} & 0 \\ 0 & & \ddots \end{bmatrix}$$

$V(\mu)$ decomposes as a direct sum of its bihomogeneous subspaces $V^{i,j}(\mu)$ of degree i in the x-variables and j in the y-variables. There is an S_n-action on $V^{i,j}(\mu)$ given by

$$\pi f = f(x_{\pi_1}, \ldots, x_{\pi_n}, y_{\pi_1}, \ldots, y_{\pi_n})$$

called the *diagonal action*. The *Frobenius Series* is the symmetric function

$$\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{ij},$$

where m_{ij} is the multiplicity of the irreducible S_n-character χ^λ in the diagonal action on $V^{i,j}(\mu)$.

Conjecture. *(Garsia, Haiman; PNAS 1993) The Frobenius Series of* $V(\mu)$ *is given by the modified Macdonald polynomial*

$$\tilde{H}_\mu(X;q,t) = \sum_{\lambda \vdash n} t^{\eta(\mu)} K_{\lambda,\mu}(q,1/t) s_\lambda(X),$$

where $\eta(\mu) = \sum_i (i-1)\lambda_i$.

Garsia and Haiman also pioneered the study of the space of *diagonal harmonics* $\mathcal{R}_n$, which is

$$\{f : \sum_{i=1}^{n} \partial x_i^h \partial y_i^k f = 0, \forall h+k > 0\}.$$

This is known to be isomorphic to the quotient ring

$$\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I,$$

where I is the ideal generated by the set of all "polarized power sums" $\sum_{i=1}^{n} x_i^h y_i^k, \forall h+k > 0$. The $V(\mu)$ are S_n-submodules of R_n.

Conjecture. *(Haiman) The dimension of the space of diagonal harmonics, as a vector space over $\mathbb{Q}$, is $(n+1)^{n-1}$.*

The space $\mathcal{R}_n$ decomposes as a direct sum of subspaces of bihomogeneous degree (i, j); $\mathcal{R}_n = \bigoplus_{i,j} \mathcal{R}_n^{i,j}$. The *Hilbert Series* is the sum

$$\sum_{i,j \geq 0} q^i t^j \dim(\mathcal{R}_n^{i,j}).$$

Example: If $n = 2$, a basis for

the space is $1, x_2 - x_1, y_2 - y_1$, and the Hilbert Series is $1 + q + t$.

The Frobenius Series is the sum

$$\sum_{\lambda \vdash n} s_\lambda(X) \sum_{i,j \geq 0} q^i t^j m_{i,j}$$

where $m_{i,j}$ is the multiplicity of χ^λ in the character of $R_n^{i,j}$ under the diagonal action of S_n. For $n = 2$ this is

$$s_2(X) + s_{1^2}(X)(q + t).$$

Let ∇ be a linear operator on the basis $\tilde{H}_\mu(X; q, t)$ given by

$$\nabla \tilde{H}_\mu(X; q, t) = t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_\mu(X; q, t).$$

Conjecture. *(Garsia, Haiman) The Frobenius Series of R_n is given by $\nabla e_n(X)$.*

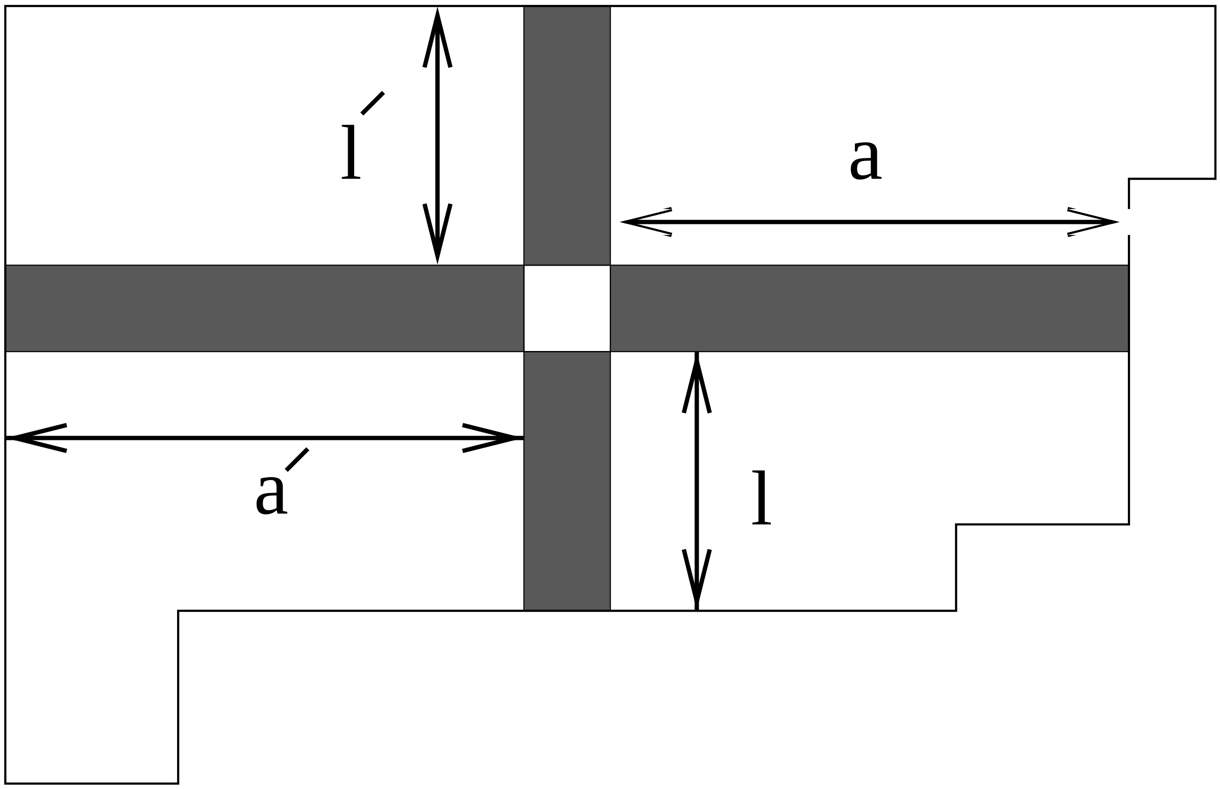
A polynomial f is *alternating* if $\pi f = (-1)^{\mathrm{inv}\pi} f$ for all $\pi \in S_n$. A special case of the above conjecture is that the coefficient of $S_{1^n}(X)$ in $\nabla e_n(X)$, corresponding to the "sign" character χ^{1^n}, is the Hilbert Series of the subspace R_n^ϵ of alternates. When $q, t \to 1$ in $\nabla e_n(X)$ they showed this coefficient equals the nth Catalan number, which would then equal $\dim(R_n^\epsilon)$. By results of Macdonald, this coefficient has an expression as a rational function in q, t.

Definition. *(q, t-Catalan)* *Let*

$$C_n(q,t) = (1-q)(1-t)\sum_{\mu \vdash n} t^{2\eta(\mu)} q^{2\eta(\mu')}$$

$$\times \frac{\prod'(1-q^{a'}t^{l'})\sum q^{a'}t^{l'}}{\prod(q^a - t^{l+1})(t^l - q^{a+1})},$$

where the products are over the squares of μ*, and the arm* a*, coarm* a'*, leg* l*, and coleg* l' *of a square are as below.*



l´
a
a´
l

Conjecture. *(Garsia, Haiman; 1992)* $C_n(q,t)$ *is a polynomial in* q *and* t *with nonnegative coefficients.*

For $n = 2$ the terms in $C_2(q,t)$ are:

$$\mu = 2; \quad \frac{q^2(1-t)(1-q)(1-q)(1+q)}{(1-q^2)(q-t)(1-q)(1-t)}$$

$$\mu = 1^2; \quad \frac{t^2(1-t)(1-q)(1-t)(1+t)}{(1-t^2)(t-q)(1-t)(1-q)}$$

So

$$C_2(q,t) = \frac{t^2}{t-q} + \frac{q^2}{q-t} = \frac{t^2-q^2}{t-q} = t+q.$$

After simplification the terms in $C_3(q,t)$ are

$$\mu = 3; \quad \frac{q^6}{q^2 - t)}$$

$$\mu = 21; \quad \frac{t^2 q^2 (1+q+t)}{(q-t^2)(t-q^2)}$$

$$\mu = 1^3; \quad \frac{t^6}{(t^2-q)(t-q)}$$

So

$$C_3(q,t) =$$

$$\frac{q^6(t^2-q) + t^2q^2(1+q+t)(q-t) + t^6(t-q^2)}{(q^2-t)(t^2-q)(q-t)}$$

$$= q^3 + q^2 t + qt^2 + qt + t^3.$$

Theorem. *(Garsia, Haiman)*

$$q^{\binom{n}{2}} C_n(q, 1/q) = \frac{1}{(n+1)_q} \binom{2n}{n}_q .$$

Theorem. *(Garsia, Haiman)*

$$C_n(q, 1) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{area}(\sigma)}.$$

Problem: Is there a pair of statistics (qstat, tstat) on Catalan paths such that

$$C_n(q, t) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{qstat}(\sigma)} t^{\mathrm{tstat}(\sigma)} ?$$

Theorem. *(Haiman; JAMS 2001) If $\mu \vdash n$, the Frobenius Series of $V(\mu)$ is the modified Macdonald polynomial $\tilde{H}_\mu(X; q, t)$.*

Pf: Algebraic Geometry and Commutative Algebra.

Corollaries. *For all $\lambda, \mu \vdash n$,*

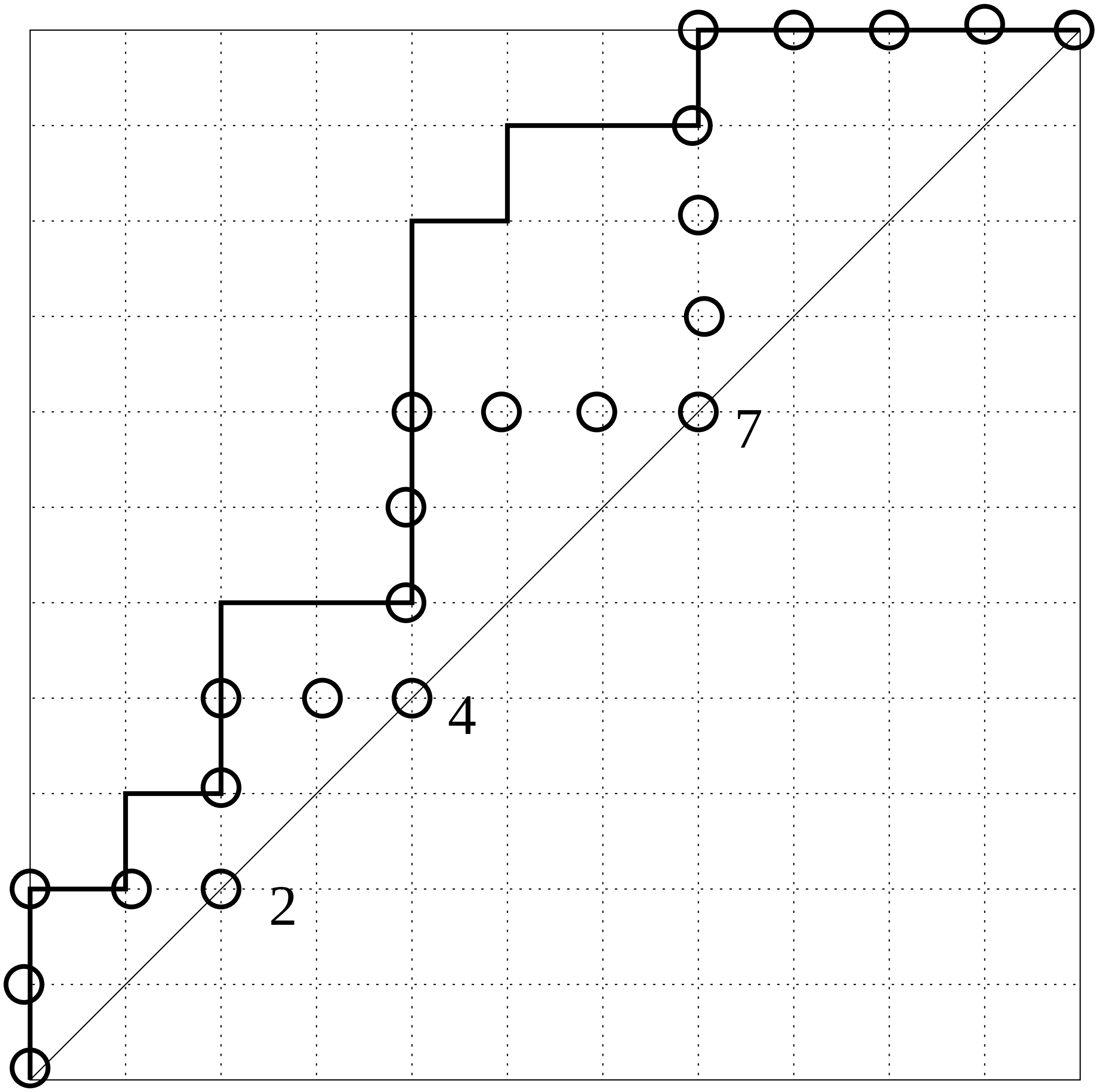
$K_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$ *and* $dim(V(\mu)) = n!$.

So far no pair of statistics for the $K_{\lambda,\mu}(q, t)$ have been proposed.

Theorem. *(Garsia, H.; PNAS 2001)*

$$C_n(q, t) \in \mathbb{N}[q, t].$$

Pf: Intricate application of plethystic identities involving ∇ after an empirical discovery of a recurrence.



The circles form the bounce path.
The bounce statistic is $2 + 4 + 7 = 13$.

Definition.

$$F_n(q,t) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{area}(\sigma)} t^{\mathrm{bounce}(\sigma)}.$$

Conjecture. *(H.; To appear in Adv. in Math.)* *For all* $n \in \mathbb{N}$,

$$F_n(q,t) = C_n(q,t).$$

(Verified in Maple for $n \leq 14$).

Definition. *Say* σ *ends in* $end(\sigma)$ *E steps. For* $n, s \in \mathbb{N}$, *set*

$$F_{n,s}(q,t) = \sum_{\substack{\sigma \in \mathcal{D}_n \\ \mathrm{end}(\sigma)=s}} q^{\mathrm{area}(\sigma)} t^{\mathrm{bounce}(\sigma)}.$$

Theorem.

$$F_{n,s}(q,t) = \sum_{r=0}^{n-s} q^{\binom{s}{2}} t^{n-s} \times F_{n-s,r}(q,t) \binom{r+s-1}{r}_q.$$

Corollary.

$$q^{\binom{n}{2}} F_n(q,q^{-1}) = \frac{1}{(n+1)_q} \binom{2n}{n}_q.$$

Theorem. *(Garsia, H.; PNAS 2001)*
For all $n, s \in \mathbb{N}$,

$$t^{n-s} q^{\binom{s}{2}} \nabla e_{n-s}[X \frac{1-q^s}{1-q}]|_{s_{1^{n-s}}(X)} = F_{n,s}(q,t).$$

Corollary.

$$C_n(q,t) = F_n(q,t).$$

Corollary.

$$F_{n,s} = (1-q^s) \sum_{\mu \vdash n} t^{\eta(\mu)} q^{\eta(\mu')}$$

$$\times \frac{\prod'(1-q^{a'}t^{l'}) h_s[(1-t)\sum q^{a'}t^{l'}]}{\prod(q^a - t^{l+1})(t^l - q^{a+1})}.$$

Corollary. $F_n(q,t) = F_n(t,q)$.

Haiman discovered another pair of statistics for the q,t-Catalan.

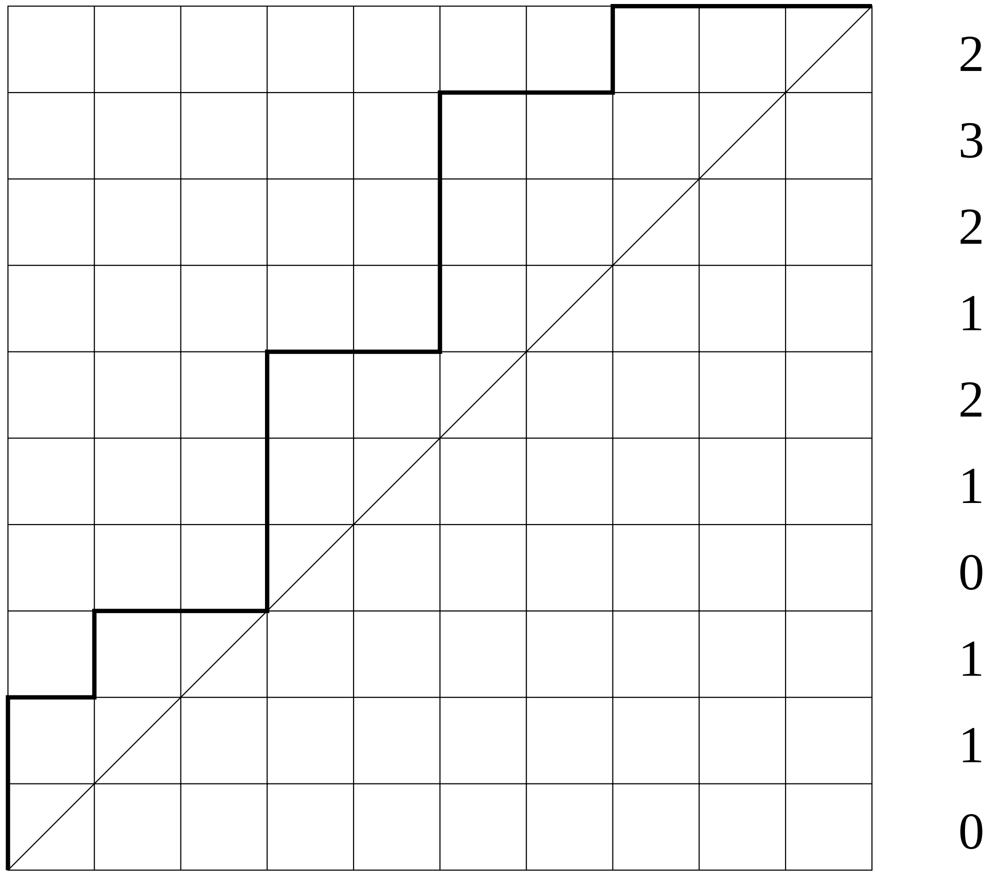
Conjecture. *(Haiman)*

$$C_n(q,t) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{area}(\sigma)} t^{\mathrm{dinv}(\sigma)}.$$

Proposition.

$$\sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{area}(\sigma)} t^{\mathrm{dinv}(\sigma)} = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{bounce}(\sigma)} t^{\mathrm{area}(\sigma)}.$$

Corollary. *Haiman's conjecture above is true.*



$t^{14} \quad q^{13}$

The statistic dinv is the # of pairs $(i, j), i < j$ with the lengths r_i and r_j of rows i, j satisfying $r_j - r_i \in \{0, 1\}$.

Corollary. $F_n(q,1) = F_n(1,q)$.

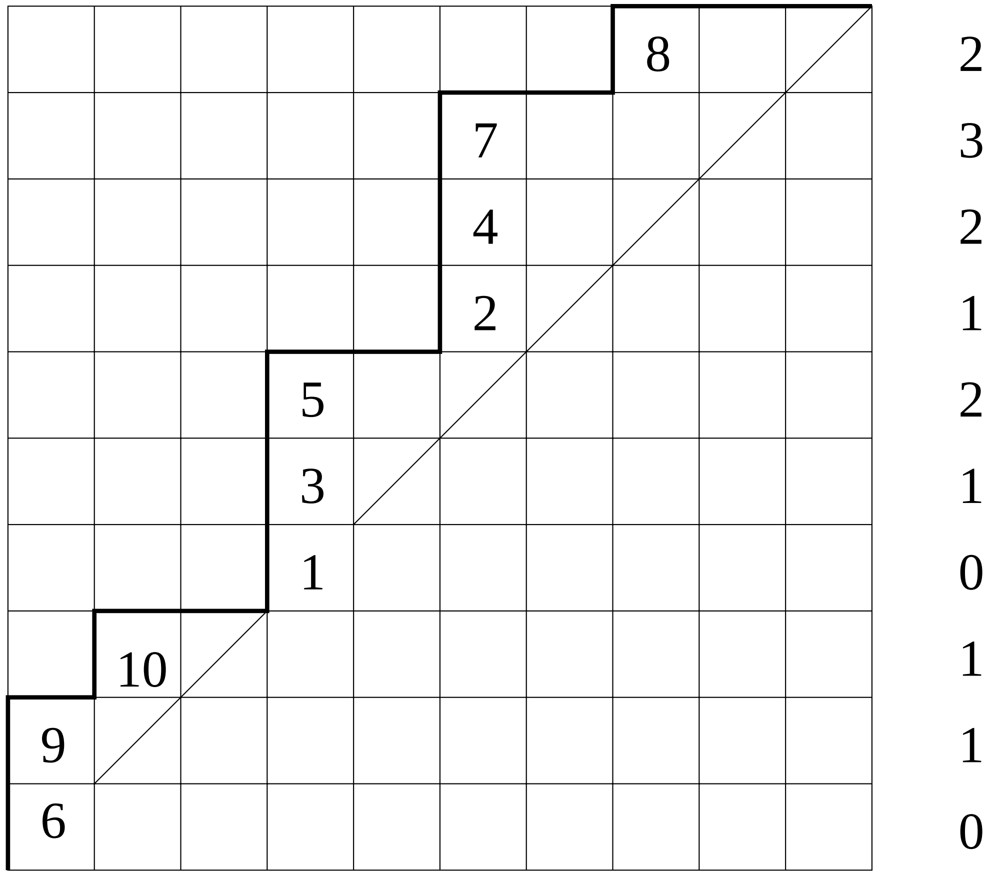
Open Question. *Find a bijective proof that* $F_n(q,t) = F_n(t,q)$.

Theorem. *(Haiman; Invent. Math. 2002)* $\nabla e_n(X)$ *is the Frobenius Series of* R_n.

Corollary. *The* (q,t)*-Catalan* $C_n(q,t)$ *is the Hilbert Series of the space of alternates* R_n^{ϵ}.

Corollary. $dim(R_n) = (n+1)^{n-1}$.

The number $(n+1)^{n-1}$ is the number of rooted, labeled trees on $n+1$ vertices, with root node labeled 0, and also the number of *parking functions* on n cars.



$t^{6} q^{13}$

$$\text{dinv} = \#(i,j), i < j : r_i = r_j \text{ and } car_i > car_j \text{ or } r_i = r_j - 1 \text{ and } car_i < car_j .$$

Conjecture. *(H., Loehr) The Hilbert Series of R_n is given by*

$$W_n(q,t) = \sum_{\sigma} q^{\mathrm{area}(\sigma)} t^{\mathrm{dinv}(\sigma)},$$

where the sum is over all parking functions on n cars.

Using Maple, we have verified our conjecture for $n \leq 7$. We can't prove, by any method, that $W_n(q,t) = W_n(t,q)$, nor can we prove that

$$q^{\binom{n}{2}} W_n(q,1/q) = (1+q+\ldots+q^n)^{n-1},$$

which is the value for the Hilbert Series at $t = 1/q$ conjectured by Stanley and now proven by Haiman. Loehr has a proof that $W_n(q,1) = W_n(1,q)$.

Garsia and Haiman define

$$C_n^m(q,t) = \nabla^m e_n(X)|_{s_{1^n}(X)}, \quad m \in \mathbb{N}.$$

Note $C_n^1(q,t) = C_n(q,t)$. These are connected to lattice paths from $(0,0)$ to (nm,n) which never go below the diagonal, and also have an algebraic description.

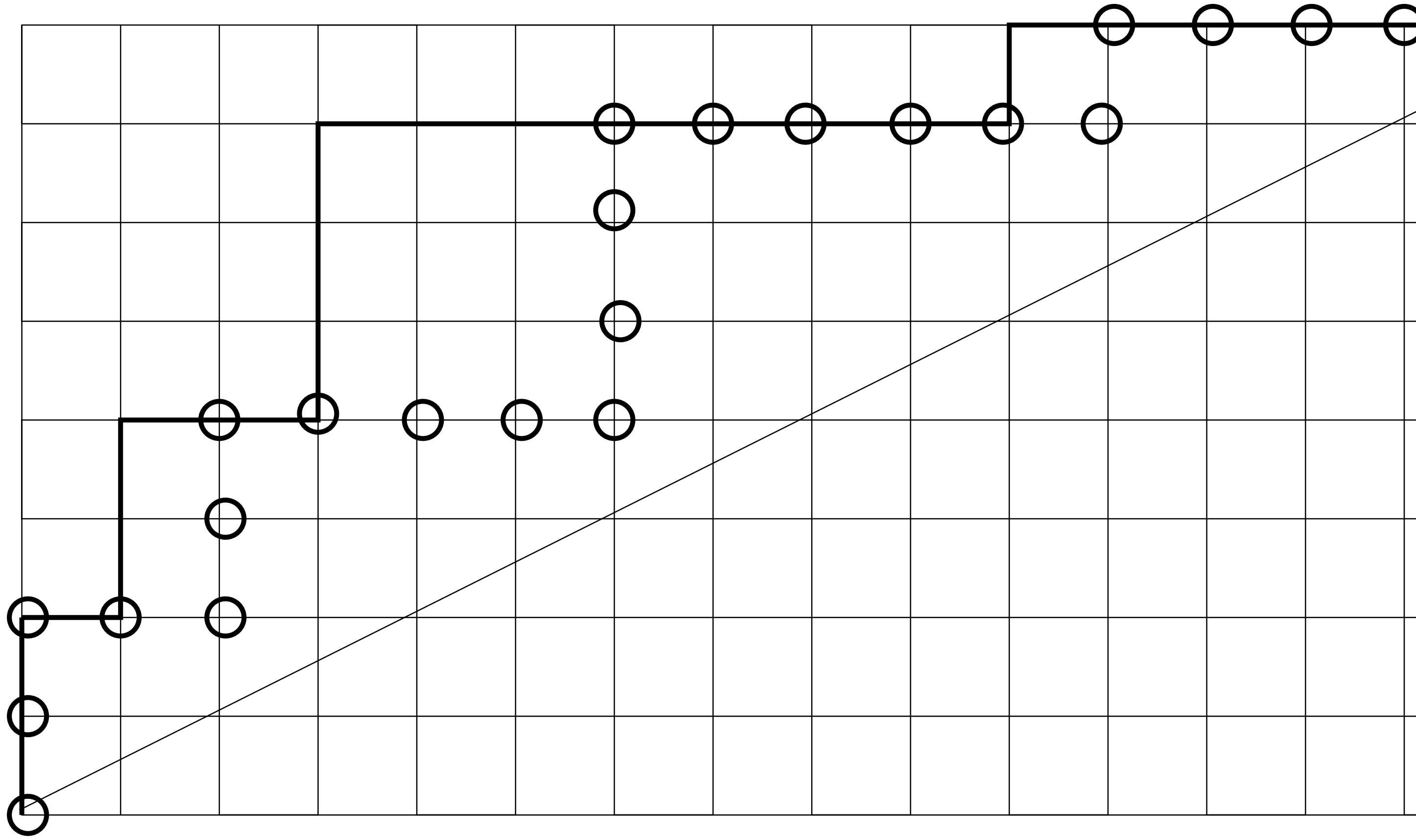
Conjecture. *(Haiman, Loehr)*

$$\sum_{\sigma \in \mathcal{D}_n^m} q^{\mathrm{area}(\sigma)} t^{m\text{-}\mathrm{dinv}(\sigma)} = C_n^m(q,t)$$

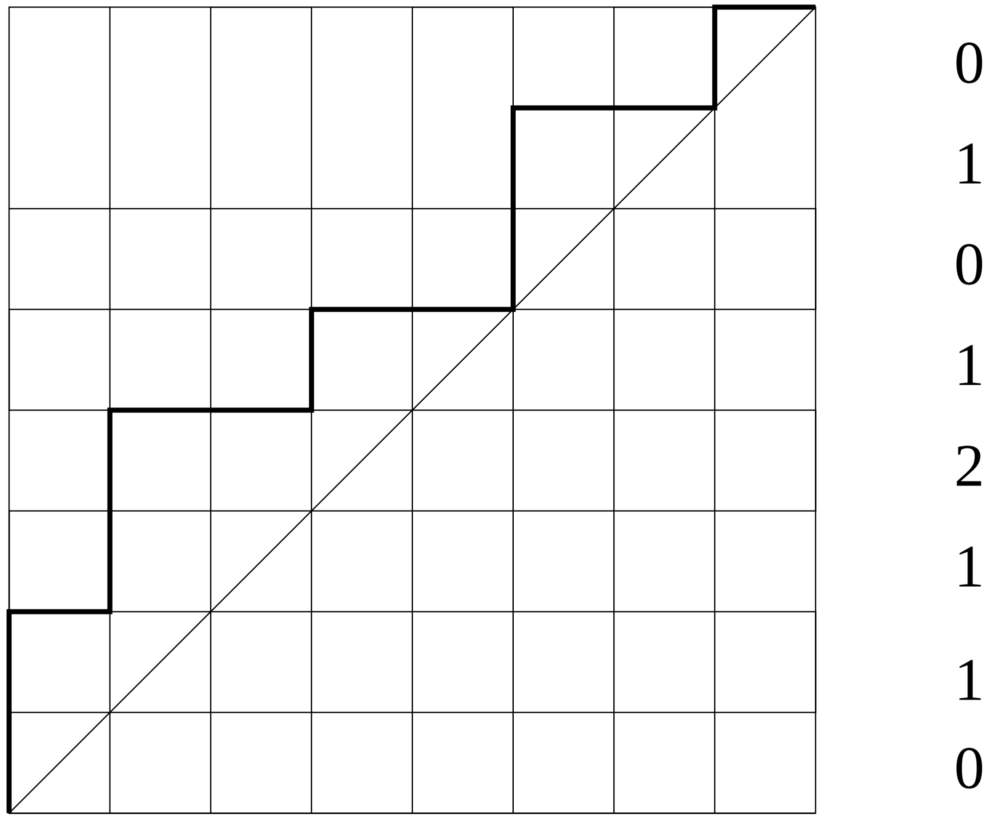
$$= \sum_{\sigma \in \mathcal{D}_n^m} q^{\mathrm{area}(\sigma)} t^{m\text{-}\mathrm{bounce}(\sigma)}.$$

Loehr obtains recurrences involving the parameter m which extend the recurrence for $F_{n,s}(q,t)$.

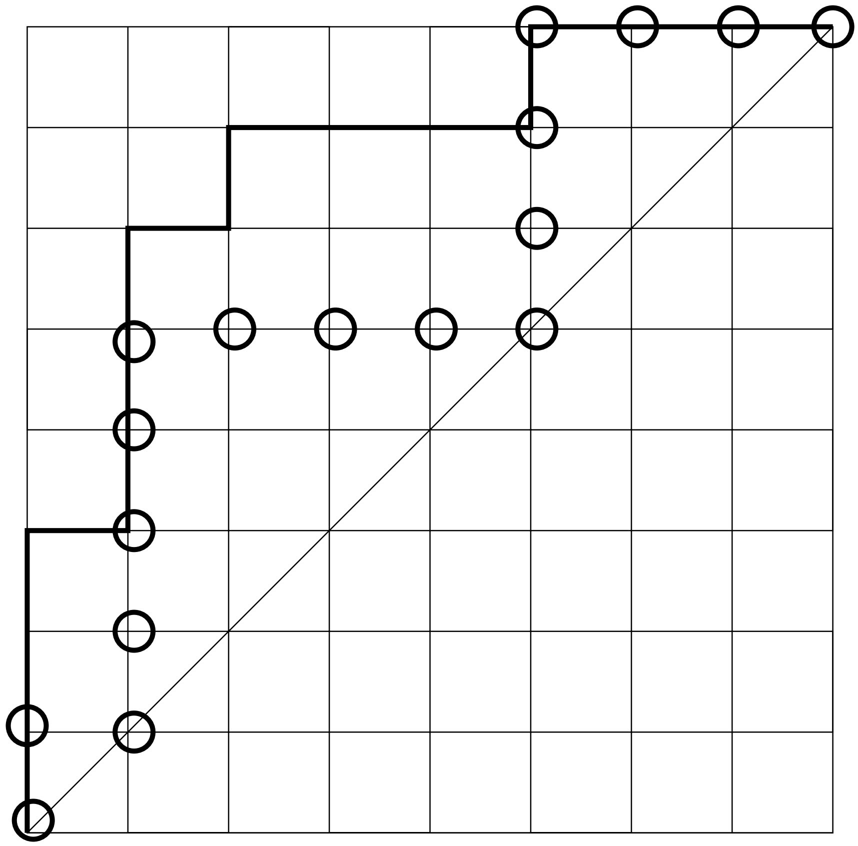
Lapointe, Lascoux and Morse have introduced a generalization of Schur functions they call “Atoms”, which depend on X, t, a positive integer k, and a partition λ satisfying $\lambda_1 \leq k$. The coefficients in the expansion of the Atoms in terms of Schur functions are in $\mathbb{N}[t]$, and they conjecture that if $\mu_1 \leq k$, the coefficients in the expansion of the $\tilde{H}_\mu(X; q, t)$ in terms of the Atoms are in $\mathbb{N}[q, t]$. This conjecture thus implies $K_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$. Hear more about this in the **special session** on **Algebraic and Enumerative Combinatorics**.



The bounce path for the case $m = 2$. Go up distance a_1 to the path, then over a_1, then up distance a_2, then over $a_1 + a_2$, then up a_3, then over $a_2 + a_3$, etc.



Start with the path above. Form the bounce path (circles, next page) whose top step is the # of rows length zero, etc. Then start at corner of top step, and look at subword of 0's and 1's on previ-



(area, dinv) ⟶ (bounce, area)

ous page, starting at bottom. For each 0 go down, for each 1 go left. Then iterate with subword of 1's and 2's.