Recent Combinatorial Results Involving Macdonald Polynomials and Diagonal Harmonics

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q-Dyson Conjecture (Andrews 1975)

$$\mathrm{CT} \prod_{1 \le i < j \le n} (x_i/x_j;q)_{a_i}(x_jq/x_i;q)_{a_j}$$

$$= \frac{(q;q)_{a_1+a_2+\ldots+a_n}}{(q;q)_{a_1}\cdots(q;q)_{a_n}}, \qquad \text{where}$$

$$(w;q)_k = (1-w)(1-wq)\cdots(1-wq^{k-1}), \quad a_i \in \mathbb{N}$$

and CT denotes "the constant term in".

- case $q=1$ Dyson Conjecture 1962: proved by Wilson 1962 and Gunson 1962
- Proof of q-Dyson: Bressoud and Zeilberger 1985

Macdonald's Constant Term Conjecture (1982)

$$\mathrm{CT} \prod_{\alpha \in R^+} (x^{\alpha};q)_k(x^{-\alpha}q;q)_k \quad = \quad \prod_i \begin{bmatrix} kd_i \\ k \end{bmatrix},$$

where R is a reduced root system and

$$k \in \mathbb{N}, \quad \sum_{w \in W} q^{\ell(w)} = \prod_i \frac{1-q^{d_i}}{1-q},$$

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q;q)_m}{(q;q)_k(q;q)_{m-k}}.$$

- Type A_{n-1} is $a_i \equiv k$ case of q-Dyson
- Kadell proved cases B_n and D_n

Recall that a polynomial f is a *symmetric function* if $f(x_1, \ldots, x_n) = f(x_{\sigma_1}, \ldots, x_{\sigma_n})$ for all $\sigma \in S_n$.

Example: The Schur function S_λ

$$S_\lambda = \sum_{T \in SSYT(\lambda)} x^T$$

$$s_{2,1} = x_1^2 x_2 + x_1 x_2^2 + 2 x_1 x_2 x_3 + \ldots = m_{2,1} + 2 m_{1,1,1}$$

corresponding to the weighted sum over the SSYT(2,1) below. Here $m_{2,1}$ and $m_{1,1,1}$ are *monomial* symmetric functions.

$$\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array}$$

Selberg's Integral For relevant $k, a, b \in \mathbb{C}$,

$$\int_{(0,1)^n} \left| \prod_{1 \le i < j \le n} (x_i - x_j) \right|^{2k} \prod_{i=1}^{n} x_i^{a-1} (1-x_i)^{b-1} dx_1 \cdots dx_n$$
$$= \prod_{i=1}^{n} \frac{\Gamma(a+(i-1)k)\Gamma(b+(i-1)k)}{\Gamma(a+b+(n+i-2)k)} \times \frac{\Gamma(ik+1)}{\Gamma(k+1)}.$$

• In the 1980's researchers realized additional symmetric functions of the x_i could be inserted in the integrand of Selberg's Integral and still get a nice product evaluation. In particular, Kadell found a formula involving polynomials that were later realized to be symmetric functions introduced earlier by statistician H. Jack. In 1987 Kadell conjectured that there existed a family of q-analogoues of these, i.e. q-Jack symmetric functions, which satisfied a q-analog of Selberg's Integral, and he showed how to define them for $n = 1, 2$.

Macdonald (1988) solved Kadell's problem, introducing the $P_\lambda(X;q,t)$, λ a partition. They satisfy

• q-analog of Selberg's Integral

• orthogonal w.r.t. the scalar product

$$\langle f,g\rangle_0 =$$
$$\frac{1}{n!}\mathrm{CT}\left[f(x_1,\dots,x_n)g(1/x_1,\dots,1/x_n)\Delta(X;q,q^k)\right],$$

where

$$\begin{aligned}\Delta(X;q,t) &= \prod_{i\neq j}\frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}\\ &= \prod_{i\neq j}(x_i/x_j;q)_k \quad \text{when } t=q^k.\end{aligned}$$

• contain Schur, Hall-Littlewood, Jack polynomials as special cases

• there are versions of P_λ for other root systems.

Macdonald formed "norm conjecture"

$$\langle P_\lambda,P_\lambda\rangle_0 = \frac{1}{|W|}\prod_{\alpha\in R^+}\prod_{i=1}^{k-1}\frac{(1-q^{<\lambda+k\rho,\alpha^\vee+i>})}{(1-q^{<\lambda+k\rho,\alpha^\vee-i>})},$$

where ρ is 1/2 the sum of positive roots, $<,>$ is the standard dot product on $\mathbb{R}^n$, and $\alpha^\vee = 2\alpha/<\alpha,\alpha>$ is the coroot of α.

• 1993 – 1995: Cherednik proves CT and its generalization, the norm conjecture, in full generality. Macdonald shows $\exists!$ family $E_\beta(X;q,t)$ of (non-symmetric) polynomials satisfying

$$\begin{aligned} &(a) \qquad \langle E_\lambda, E_\mu \rangle_0 = 0 \quad \text{for} \quad \lambda \neq \mu \\ &(b) \qquad E_\lambda = x^\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu}(q,t) x^\mu \end{aligned}$$

for a certain partial order on elements μ, λ of the weight lattice of the underlying root system. He obtains a norm formula for the E_β. Cherednik in turn obtains recurrence relations (known as the intertwiner relations) for E_β and outlines a proof of the "duality theorem".

Current theory involves "affine root systems". An affine root is a pair (r, m), where $r \in R$, $m \in \mathbb{Z}$. Let

$$\langle (r,m), (\alpha,p) \rangle = \langle r, \alpha \rangle ,$$

$$(r,m)^\vee = \frac{2(r,m)}{< r, r >}$$

$$s_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha^\vee .$$

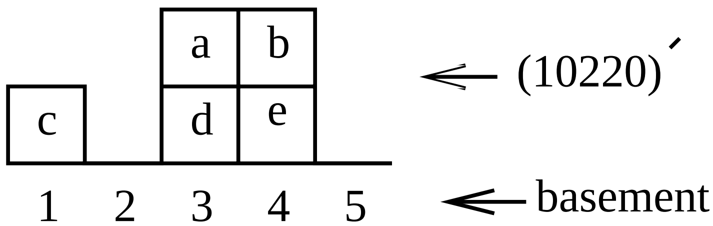
Affine root systems are sets of affine roots satisfying a set of axioms, analogous to those for ordinary root systems. In his book "Affine Hecke Algebras and Orthogonal Polynomials" Macdonald classifies all affine root systems. There are ones corresponding to ordinary root systems, such as

$$A_n, E_6, E_7, E_8, F_4, F_4^\vee, G_2, G_2^\vee.$$

There are systems of type $(C_n^\vee, C_n)$, which are the same as polynomials introduced by Koornwinder, and contain 5 t-variables. When $n = 1$ they become the Askey Wilson polynomials. Specializations of the t-variables yield most of the other infinite families of affine root systems.

- There are P_λ and E_β for each affine root system.
- Haiman has a new preprint on his website "Cherednik Algebras, Macdonald Polynomials, and Combinatorics" which develops the theory in somewhat greater generality, uses simpler notation, and simplifies some proofs. A. Kirillov also has a nice exposition of Cherednik's work in a paper in the Bulletin of the AMS.

We now describe a new combinatorial formula for the type A_{n-1} nonsymmetric Macdonald polynomials, due to M. Haiman, N. Loehr and the speaker. In type A_{n-1} we can assume our weight lattice elements β are compositions, i.e. $\beta \in \mathbb{N}^n$.



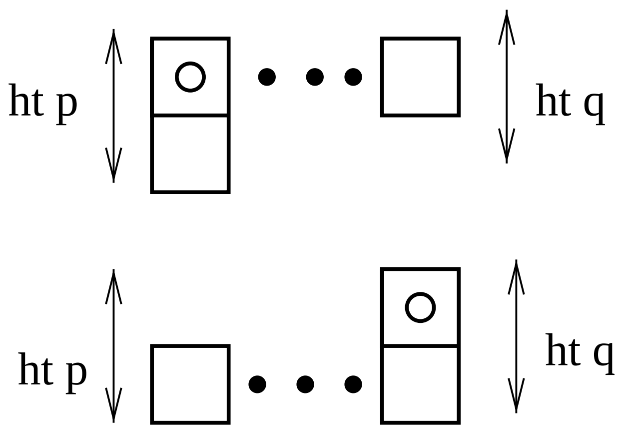
reading order: abcde12345

type A triples:

p>= q

type B triples:

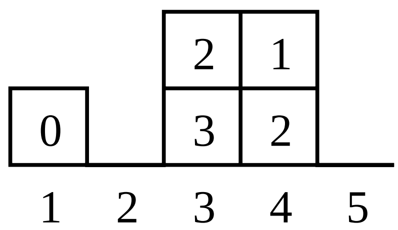
p<q



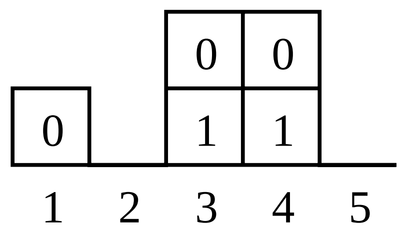
○ = base square

$\operatorname{arm}(w) = \#$ triples with w as base square,
$\operatorname{leg}(w) = \#$ squares above w in column

Example:



arms

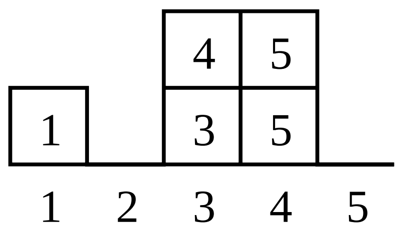


legs

Each triple comes with a clockwise or counterclockwise *orientation*, determined by starting at the smallest number of the triple, and go in a circular direction towards the next smallest then to the largest. (If the triple has two equal numbers, regard the one which occurrs first in the reading order as smaller.)

Let $\operatorname{coinv}(\sigma)$ be the #of clockwise ype A triples plus the # of counterclockwise type B triples, and let $\operatorname{maj}(\sigma)$ be the sum, over all descents, of 1 plus the leg of the topmost square of the descent. We say two equal numbers *attack* each other if they are in the same row, or in successive rows, with the square in the row below strictly to the left.

A filling σ of $(10220)'$ $\qquad$ $x^{\sigma} = x_1x_3x_4x_5^2$



type B triples:

$\begin{matrix} & 4 \\ 1 & 3 \end{matrix}$

$\begin{matrix} & 5 \\ 1 & 5 \end{matrix}$

$\begin{matrix} & 3 \\ 1 & 3 \end{matrix}$

$\begin{matrix} & 3 \\ 2 & 3 \end{matrix}$

$\begin{matrix} & 5 \\ 1 & 4 \end{matrix}$

$\begin{matrix} & 5 \\ 2 & 4 \end{matrix}$

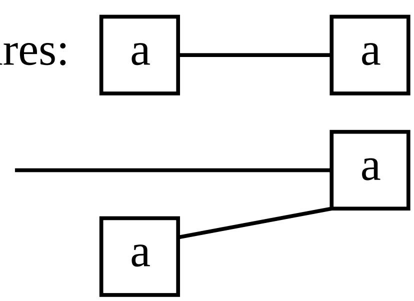
type A triples:

$\begin{matrix} 4 & 5 \\ 3 & \end{matrix}$

$\begin{matrix} 3 & 5 \\ 3 & \end{matrix}$

Descents: $\begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline \end{array}$, $\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array}$

Attacking Squares:



or

Theorem. *(Haiman, Loehr, H. 2006). For any* $\beta \in \mathbb{N}^n$,

$$\mathcal{E}_\beta(X;q,t) = \sum_{\substack{\sigma:\beta' \to \{1,\dots,n\} \\ \text{non-atttacking}}} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)}$$

$$\prod_{\substack{w\in\beta' \\ \sigma(w)=\sigma(\mathrm{South}(w))}} (1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1}) \prod_{\substack{w\in\beta' \\ \sigma(w)\neq\sigma(\mathrm{South}(w))}} (1-t),$$

where

$$\mathcal{E}_\beta = E_{(\beta_n,\dots,\beta_1)}(x_n,\dots,x_1;1/q,1/t)\prod_w (1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1}).$$

Pf: In type A_{n-1}, Knop found a simplification in Cherednik's recurrence. Let

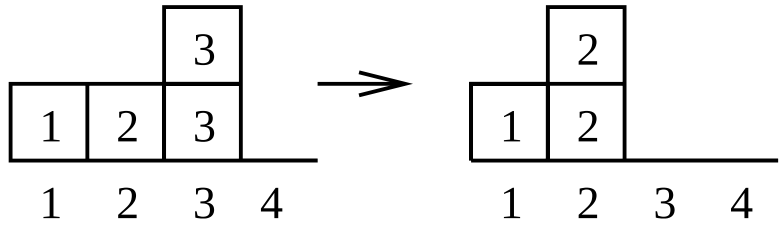
$$\tilde{T}_i = ts_i - \frac{1-t}{1-x_i/x_{i+1}}(1-s_i),$$

where s_i is the transposition $(i, i+1)$. Then

(*a*) $E_{(0,0,\dots,0)} = 1$

(*b*) $E_{(\lambda_n+1,\lambda_1,\dots,\lambda_{n-1})} = q^{\lambda_n} x_1 E_\lambda(x_2,\dots,x_n,x_1/q)$

(*c*) $E_{s_i(\lambda)} = \left(\tilde{T}_i + \dfrac{1-t}{1-q^{\lambda_i-\lambda_{i+1}}t^{c_i}}\right)E_\lambda, \quad \lambda_i > \lambda_{i+1}$



where $c_i \in \mathbb{N}$. Part (a) is trivial, and (b) holds on a filling-by-filling basis (see figure above). Part (c) is the hard part to prove; we were only able to show it held when $\lambda_{i+1} = 0$, but this is enough to recursively generate all the E_β.

Let $J_\mu(X; q, t) = \prod(1 - q^a t^{l+1}) P_\mu$ be Macdonald's integral form, and let β^+ be the partition formed by rearranging β into nonincreasing order.

Corollary. *For any β with $\beta^+ = \mu$, if in the formula for $\mathcal{E}_\beta$ we change the basement to $n+1, \ldots, n+1$, we get a formula for J_μ.*

Corollary. *Letting $q = t^\alpha$ in $\mathcal{E}_\beta$, dividing by $(1-t)^{|\beta|}$ and letting $t \to 1$, we get Sahi and Knop's combinatorial formula for the nonsymmetric Jack polynomial $\mathcal{E}_\beta^{(\alpha)}(X)$.*

Remark. *It was comparing the Sahi-Knop formula with our earlier J_μ formula which led to the $\mathcal{E}_\beta$ formula.*

Example. *Taking the coef of $x_1^3 x_2^2 x_1$ in $J_{3,2,1}$ results in a sum of 8 terms, while taking the same coef in $J_{(1,2,3)}$ gives an equivalent formula with only 1 term.*

n=6 coefficient of $x_1^3 x_2^2 x_3^1$ in $J_{3\,2\,1}$

$$
\begin{array}{ccc} 1 & & \\ 1 & 2 & \\ 1 & 2 & 3 \\ \hline 7 & 7 & 7 \end{array}
\qquad
\begin{array}{ccc} 1 & & \\ 1 & 2 & \\ 1 & 3 & 2 \\ \hline 7 & 7 & 7 \end{array}
\qquad
\begin{array}{ccc} 1 & & \\ 1 & 2 & \\ 3 & 1 & 2 \\ \hline 7 & 7 & 7 \end{array}
\qquad
\begin{array}{ccc} 1 & & \\ 1 & 2 & \\ 3 & 2 & 1 \\ \hline 7 & 7 & 7 \end{array}
$$

$$
\begin{array}{ccc} 1 & & \\ 2 & 1 & \\ 2 & 1 & 3 \\ \hline 7 & 7 & 7 \end{array}
\qquad
\begin{array}{ccc} 1 & & \\ 2 & 1 & \\ 3 & 1 & 2 \\ \hline 7 & 7 & 7 \end{array}
\qquad
\begin{array}{ccc} 1 & & \\ 2 & 1 & \\ 2 & 3 & 1 \\ \hline 7 & 7 & 7 \end{array}
\qquad
\begin{array}{ccc} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \\ \hline 7 & 7 & 7 \end{array}
$$

coefficient of $x_1^3 x_2^2 x_3^1$ in $J_{1\,2\,3}$

$$
\begin{array}{ccc} & & 1 \\ & 2 & 1 \\ 3 & 2 & 1 \\ \hline 7 & 7 & 7 \end{array}
$$

It is known that

$$J_\mu = \sum_{\beta^+=\mu} \prod \frac{1-q^*t^*}{1-q^*t^*} \mathcal{E}_\beta,$$

for simple expressions * involving arms and legs. Letting $q = t = 0$ above we get

$$s_\mu = \sum_{\beta^+=\mu} \mathcal{E}_\beta(X;0,0) = \sum_{\beta^+=\mu} NS_\beta$$

say. S. Mason has a bijective proof of this. The NS_β are "standard bases" introduced by Lascoux and Schützenberger in the study of Schubert varieties.

Let $p_k(X) = \sum_i x_i^k$ and define the plethystic substitution of $X(1-t)$ into p_k via

$$p_k\left[X(1-t)\right] = \sum_i x_i^k(1-t^k).$$

Define $f[X(1-t)]$ by expanding $f(X)$ into the p_k's and using the above formula. Set

$$J_\mu(X;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t) S_\lambda\left[X(1-t)\right],$$

$$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,1/t), \quad n(\mu) = \sum_i (i-1)\mu_i.$$

$$\tilde{H}_\mu(X;q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t) S_\lambda(X).$$

Corollary. *For any β with $\beta^+ = \mu'$,*

$$\tilde{H}_\mu = \sum_{\sigma:\beta' \to \{1,2,\ldots,n\}} x^\sigma t^{\mathrm{maj}} q^{\mathrm{inv}},$$

where inv is the number of inversion triples (counterclockwise type A or clockwise type B triples). Here we use $n+1$'s in the basement as in the formula for J_μ.

Diagonal Harmonics

Let

$$DH_n = \{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] : \\ \sum_{i=1}^{n} \partial_{x_i}^j \partial_{y_i}^k f = 0, \quad \forall j+k > 0\}.$$

Diagonal Action:

$$\sigma f = f(x_{\sigma_1}, \ldots, x_{\sigma_n}, y_{\sigma_1}, \ldots, y_{\sigma_n}).$$

Alternates: $\sigma f = \mathrm{sign}(\sigma) f \quad \forall \sigma \in S_n$.

Example. $n = 2$: *basis* $\{1, x_2 - x_1, y_2 - y_1\}$.

$$\begin{aligned} Hilb(DH_2) &= 1 + q + t \\ Hilb(DH_2^\epsilon) &= q + t \quad (\textit{subspace of alternates}) \\ Frob(DH_2) &= S_2 + S_{1,1}(q + t). \end{aligned}$$

Here Hilb is the Hilbert series, bigraded by x and y degree, and Frob is the "Frobenius series", i.e. the bigraded character, with each occurrence of the irreducible S_n-character χ^λ weighted by the Schur function S_λ.

Let ∇ be the linear operator

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu.$$

F. Bergeron first noticed that many identities involving Macdonald polynomials can be elegantly phrased in terms of the ∇ operator.

Theorem. *(Haiman* 2000*)*

$$Frob(DH_n) = \nabla S_{1^n}.$$

Pf. Based on the geometry of $\mathcal{H}_n$, the Hilbert scheme of n points in the plane

$$\mathcal{H}_n = \{I \subset \mathbb{C}[x, y] : \dim(\mathbb{C}[x, y]/I) = n\}.$$

Corollary.

$$dim(DH_n) = (n+1)^{n-1} \qquad dim(DH_n^\epsilon) = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem. *(Garsia, H. 2000)*

$$Hilb(DH_n^\epsilon) = \sum_{\pi \in \mathcal{D}_n} q^{\text{area}} t^{\text{bounce}}.$$

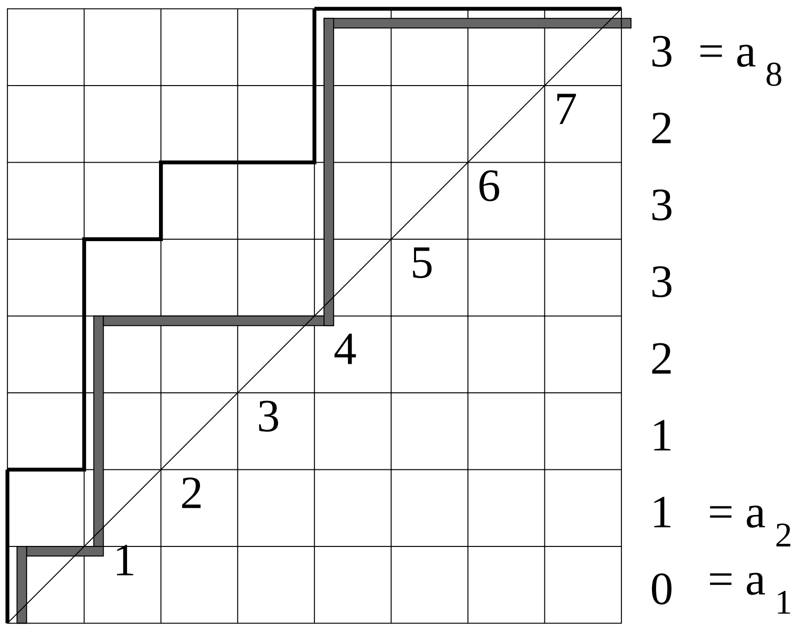
Pf. Uses ∇ results of F. Bergeron, Garsia, Tesler and others.

Example.

$$
\begin{aligned}
< \nabla S_{1^n}, S_{1^n} > &= \text{ \textit{the ``(q,t)-Catalan sequence''} } C_n(q,t) \\
&= Hilb(DH_n^{\epsilon}).
\end{aligned}
$$

$$
\begin{aligned}
C_1(q,t) &= \frac{t-q}{t-q} = 1 \\
C_2(q,t) &= \frac{t^2}{t-q} + \frac{q^2}{q-t} = \frac{t^2-q^2}{t-q} = t+q \\
C_3(q,t) &= \frac{t^6}{(t^2-q)(t-q)} + \frac{q^6}{(q^2-t)(q-t)} + \\
&\quad \frac{q^2t^2(1+q+t)}{(q-t^2)(t-q^2)} = t^3 + t^2q + qt + tq^2 + q^3.
\end{aligned}
$$

Here the sum is over all Dyck paths π (lattice paths from $(0,0)$ to (n,n) consisting of unit N and E steps which never go below the main diagonal $x=y$) and area is the number of complete squares below π and above the diagonal. We let $a_i(\pi)$ denote the number of area squares in the ith row (from the bottom) of π. To compute the bounce statistic, first form the bounce path (shaded line in figure on next page) by starting at (n,n) and going left until you hit a N step of π, then bounce



area = 0+1+1+2+3+3+2+3 = 15

bounce = 1+4 = 5

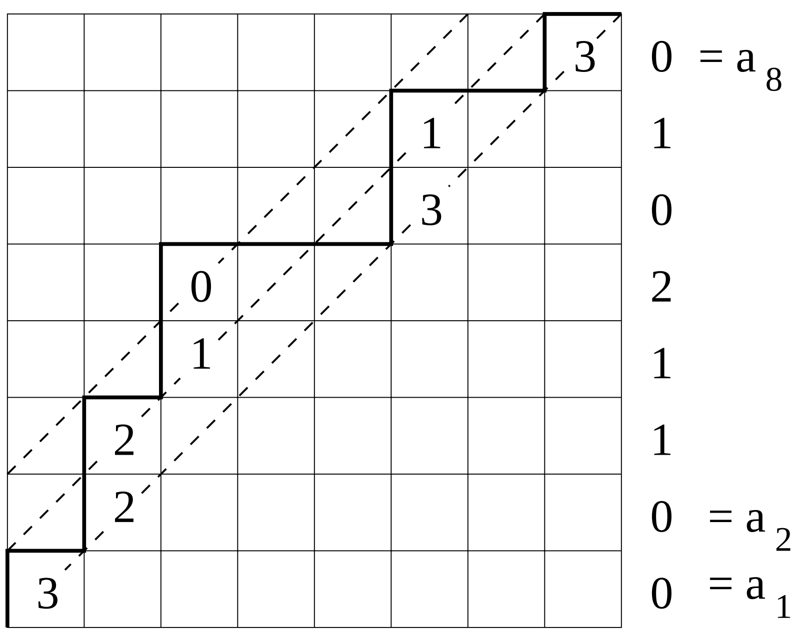
down to the diagonal and iterate. The statistic bounce(π) is the sum of the coordinates where the bounce path intersects the diagonal.

The fact that (area, bounce) generates $C_n(q,t)$ was conjectured by Haglund. Haiman, independently, conjectured that

$$C_n(q,t) = \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{dinv}} t^{\mathrm{area}},$$

where

$$\mathrm{dinv} = \#(i,j): \quad 1 \le i < j \le n \text{ and } a_i = a_j \text{ or } a_i = a_j{+}1.$$



area = 0+0+1+1+2+0+1+0 = 5

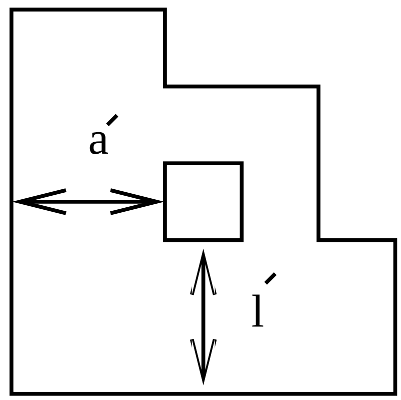
dinv = 0+1+1+2+3+3+2+3 = 15

$b_1, b_2, \ldots$ = 0,1,1,2,3,3,2,3

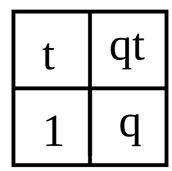
It is easy to see that (dinv, area) and (area, bounce) have the same distribution. Start with a typical π as on the previous page, and create a new path $\phi(\pi)$ by first counting the number of $a_i = 0$ in π: this is the length of your first bounce step in ϕ. The number of $a_i = 1$ is the length of your 2nd bounce step, etc. We get the bounce path on the page two previous to this. Now consider the subsequence of the a_i from π consisting of $a_i = 0$ or $a_i = 1$ (all a_i except a_5 in our example). Start at the topmost NE corner of the bounce path and for each $a_i = 0$ go S one step, and for each $a_i = 1$ go W one step. We end up going $SSWWSWS$, drawing the corresponding portion of ϕ. Next consider the subsequence of a_i equaling 1 or 2, and draw the next section of ϕ, etc.

If we read along 45° diagonals, top to bottom, outside to in, then the order in which we encounter the squares whose left border is a N step of π is called the *reading order*. For the ith square w in this order, let b_i be the number of such squares before w in the reading order whose row forms a contribution to dinv with the row containing w, so $b_1 = 0$ and dinv $= b_1 + \ldots + b_n$. Note that $b_i(\pi) = a_i(\phi(\pi))$.

For a partition μ, let $B_\mu = \{q^{a'}t^{l'}\}$, i.e. the set of all coarms and colegs, as in the figure below.



a´ = coarm, l´ = coleg



$B_{2,2} = \{1, q, t, qt\}$

Let Δ_f be a linear operator defined via

$$\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu,$$

where $f[B_\mu]$ is f evaluated at the alphabet B_μ. Below $<,>$ is the Hall scalar product, with respect to which the Schur functions are orthonormal, and $|_{z^k}$ means "the coefficient of z^k in".

Conjecture. *(Can, H.) For any $k \in \mathbb{N}$, the following four expressions are all equal*

$$
\begin{aligned}
&(a) \quad \langle \Delta_{S_k} \nabla S_{1^{n-k}}, S_{1^{n-k}} \rangle \\
&(b) \quad \left\langle \nabla S_{1^n}, S_{k+1,1^{n-k-1}} \right\rangle \\
&(c) \quad \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{dinv}} t^{\mathrm{area}} \prod_{a_i > a_{i-1}} (1 + z/t^{a_i})|_{z^k} \\
&(d) \quad \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{dinv}} \prod_{b_i > b_{i-1}} (1 + z/q^{b_i}) t^{\mathrm{area}}|_{z^k}.
\end{aligned}
$$

Remarks. *Can has recently extended some of Haiman's results on the Hilbert scheme to the nested Hilbert scheme $\mathcal{H}_{n,n-1}$, defined as*

$$
\begin{aligned}
\mathcal{H}_{n,n-1} = \{(I_1, I_2) : dim(\mathbb{C}[x,y]/I_j) = n, \\
dim(\mathbb{C}[x,y]/I_2) = n-1, I_1 \subset I_2\}.
\end{aligned}
$$

Can obtains an associated (q,t)-Catalan sequence $C_{n,n-1}(q,t)$, which is (a) above. The statement that (b) = (d) is a theorem of H., first conjectured by Egge, Killpatrick, Kremer and H. It gives a formula for the hook shapes in $Frob(DH_n)$. The proof involves extensions of the ∇ identities occurring in the work of Garsia and H. on $C_n(q,t)$.

- *case $k = 1$ of (a) is Can's $\mathcal{H}_{n-1,n-2}$.*

Let

$$F_n(q,t,z,w) = \sum_{\pi \in \mathcal{D}_n} q^{\mathrm{dinv}} \prod_{b_i > b_{i-1}} (1 + z/q^{b_i})$$
$$t^{\mathrm{area}} \prod_{a_i > a_{i-1}} (1 + w/t^{a_i}).$$

Conjecture. *(Can, H.)*

$$F_n(q,t,z,w) = F_n(q,t,w,z) = F_n(t,q,z,w).$$

Remarks. *It is an easy exercise using the map ϕ to show that $F_n(1,1,z,w) = F_n(1,1,w,z)$. The fact that $F_n(q,t,0,0) = F_n(t,q,0,0)$ is equivalent to symmetry of $C_n(q,t)$, which follows from the result of Garsia and H., but for which there is (still) no known combinatorial proof. Perhaps the study of this four parameter function will shed some light on this problem.*

- $F_n(q,t,z,w)$ satisfies a recurrence, which generalizes the known recurrence for $C_n(q,t)$.