A Combinatorial Model for the Macdonald Polynomials

J. Haglund
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
jhaglund@math.upenn.edu

July 8, 2004

Abstract

We introduce a polynomial $\tilde{C}_\mu[Z;q,t]$, depending on a set of variables $Z = z_1, z_2, \ldots$, a partition μ, and two extra parameters q, t. The definition of $\tilde{C}_\mu$ involves a pair of statistics $(\mathrm{maj}(\sigma,\mu), \mathrm{inv}(\sigma,\mu))$ on words σ of positive integers, and the coefficients of the z_i are manifestly in $\mathbb{N}[q,t]$. We conjecture that $\tilde{C}_\mu[Z;q,t]$ is none other than the modified Macdonald polynomial $\tilde{H}_\mu[Z;q,t]$. We further introduce a general family of polynomials $F_T[Z;q,S]$, where T is an arbitrary set of squares in the first quadrant of the xy-plane, and S is an arbitrary subset of T. The coefficients of the $F_T[Z;q,S]$ are in $\mathbb{N}[q]$, and $\tilde{C}_\mu[Z;q,t]$ is a sum of certain $F_T[Z;q,S]$ times nonnegative powers of t. We prove $F_T[Z;q,S]$ is symmetric in the z_i, and satisfies other properties consistent with our conjecture. We also show how the coefficient of a monomial in $F_T[Z;q,S]$ can be expressed recursively. Maple calculations indicate the $F_T[Z;q,S]$ are Schur positive, and we present a combinatorial conjecture for their Schur coefficients when the set T is a partition with at most three columns.

1 Introduction

We refer the reader to Chapter 1 of [Mac95] or Chapter 7 of [Sta99] for basic facts about symmetric functions. Given a sequence $\mu = (\mu_1, \mu_2, \ldots)$ of nonincreasing, nonnegative integers with $\sum_i \mu_i = n$, we say μ is a partition of n, denoted by either $|\mu| = n$ or $\mu \vdash n$. By adding or subtracting parts of size 0 if necessary, we will always assume partitions of n have exactly n parts. We let $\eta(\mu) = \sum_i (i-1)\mu_i$, and if λ is another partition, set $\tilde{K}_{\lambda,\mu}(q,t) = t^{\eta(\mu)} K_{\lambda,\mu}(q, 1/t)$, where $K_{\lambda,\mu}(q,t)$ is Macdonald's q,t-Kostka polynomial [Mac95, p.354]. We call $\tilde{H}_\mu[Z;q,t] = \sum_{\lambda \vdash |\mu|} s_\lambda \tilde{K}_{\lambda,\mu}(q,t)$ the modified Macdonald polynomial, where $s_\lambda = s_\lambda[Z]$ is the Schur function and the sum is over all $\lambda \vdash |\mu|$. The $\tilde{H}_\mu[Z;q,t]$ can be easily transformed by a plethystic substitution into Macdonald's original symmetric functions $P_\mu[Z,q,t]$. Macdonald defined the P_μ in terms of orthogonality with respect to a scalar product, and conjectured $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$ [Mac95, p. 355]. (From their definition, all one can infer is that the $K_{\lambda,\mu}(q,t)$ are rational functions in q,t). He also posed the problem of finding a combinatorial rule to describe these polynomials.

In [GH93] Garsia and Haiman introduced an S_n submodule $V(\mu)$ for each $\mu \vdash n$, and posed the $n!$ Conjecture, which says that $\dim_{\mathbb{Q}} V(\mu)$ equals $n!$, where dim is the dimension as a vector space. This was proved in 2000 by Haiman [Hai01]. It had previously been shown [Hai99] that the $n!$ Conjecture implies the coefficient of $q^i t^j$ in $\tilde{K}_{\lambda,\mu}(q,t)$ equals the multiplicity of the irreducible S_n character χ^λ in the character of a submodule $V(\mu)^{(i,j)}$ of $V(\mu)$. Macdonald's conjecture that $\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$ follows. No purely combinatorial description of the $\tilde{K}_{\lambda,\mu}(q,t)$ is known.

We assign (row,column)-coordinates to squares in the first quadrant, obtained by permuting the (x,y) coordinates of the lower left-hand corner of the square, so the lower-left hand square has coordinates $(0,0)$, the square above it $(1,0)$, etc.. For a square w, we call the first coordinate of w the *row value* of w, denoted $\mathrm{row}(w)$, and the second coordinate of w the *column value* of w, denoted $\mathrm{col}(w)$. Given $\mu \vdash n$, we let μ also stand for the Ferrers diagram of μ (French convention), consisting of the set of n squares with coordinates (i,j), with $0 \le i \le n-1$, $0 \le j \le \mu_i - 1$.

Let T be a finite set of squares in the first quadrant. A subset of squares of T consisting of all those $w \in T$ with a given row value is called a row of T, and a subset of squares of T consisting of all those $w \in T$ with a given column value is called a column of T. Furthermore, we let $T(i)$ denote the ith square of T encountered if we read across rows from left to right, starting with the squares of largest row value and working downwards. Given a square $w \in T$, define the leg of w, denoted $\mathrm{leg}(w)$, to be the number of squares in T which are strictly above and in the same column as w, and the arm of w, denoted $\mathrm{arm}(w)$, to be the number of squares in T strictly to the right and in the same row as w. Also, if w has coordinates (i,j), we let $\mathrm{south}(w)$ denote the square with coordinates $(i-1,j)$.

A word σ of positive integers is a linear sequence $\sigma_1\sigma_2\cdots\sigma_n$, with $\sigma_i \ge 1$. If the letter i occurs α_i times in σ, for each $i \ge 1$, we say σ has content α, denoted $\mathrm{content}(\sigma) = \alpha$. We call a pair (σ, T), where σ is a word of positive integers and T is a set of squares in the first quadrant, a *filling*. We represent (σ,T) geometrically by placing σ_i in square $T(i)$, for $1 \le i \le n$. For $w \in T$, we let $w(\sigma)$ denote the element of σ placed in square w. A *descent* of (σ,T) is a square $w \in T$, with $\mathrm{south}(w) \in T$ and $w(\sigma) > \mathrm{south}(w)(\sigma)$.

Let $\mathrm{Des}(\sigma,T)$ denote the set of all descents of (σ,T). For partitions μ, define a generalized major index statistic $\mathrm{maj}(\sigma,\mu)$ via

$$\mathrm{maj}(\sigma,\mu) = \sum_{w\in \mathrm{Des}(\sigma,\mu)} 1 + \mathrm{leg}(w). \tag{1}$$

An inversion of (σ, T) is a pair of squares (a, b) with $a, b \in T$, $a(\sigma) > b(\sigma)$, and either

$$\begin{cases} \mathrm{row}(a) = \mathrm{row}(b) \text{ and } \mathrm{col}(a) < \mathrm{col}(b), & \text{or} \\ \mathrm{row}(a) = \mathrm{row}(b) + 1 \text{ and } \mathrm{col}(a) > \mathrm{col}(b) \ . \end{cases} \tag{2}$$

Let $\mathrm{Inv}(\sigma, T)$ denote the set of all inversions of (σ, T), and define the inversion statistic $\mathrm{inv}(\sigma, T)$ via

$$\mathrm{inv}(\sigma,T) = |\mathrm{Inv}(\sigma,T)| - \sum_{w\in \mathrm{Des}(\sigma,T)} \mathrm{arm}(w), \tag{3}$$

where $|T|$ denotes the cardinality of a set T. For example, if (σ, T) is the filling on the left in Figure 1, then representing squares by their coordinates,

$$\mathrm{Des}(\sigma,T) = \{(1,1)\}, \tag{4}$$
$$\mathrm{Inv}(\sigma,T) = \{((1,1),(1,2)),((1,1),(0,0)),((0,0),(0,2)),((0,1),(0,2))\}, \tag{5}$$

so $\mathrm{inv}(\sigma, T) = 4 - 1 = 3$. If (σ, μ) is the filling on the right in Figure 1, then

$$\mathrm{Des}(\sigma,\mu) = \{(1,0),(3,1),(1,1)\}, \tag{6}$$
$$\mathrm{Inv}(\sigma,\mu) = \{((2,0),(2,1)),((1,1),(1,2)),((1,1),(0,0)),((1,2),(0,0)),((1,2),(0,1))\}, \tag{7}$$

so $\mathrm{maj}(\sigma, \mu) = 3 + 1 + 3 = 7$, $\mathrm{inv}(\sigma, \mu) = 5 - (2 + 0 + 1) = 2$.

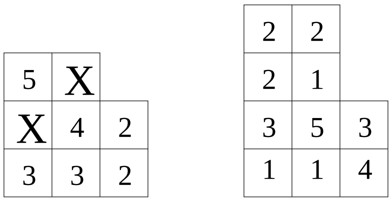

Figure 1: On the left, a set T and a filling of T by the word 542332. The X's indicate squares not in T. On the right, a filling of the partition $(3, 3, 2, 2)$ by the word 2221353114.

For any word σ, as is customary we define the descent set $\mathrm{Des}(\sigma)$ to be $\{i : \sigma_i > \sigma_{i+1}\}$. Note that, if 1^n denotes a column of n cells, then

$$\mathrm{maj}(\sigma, 1^n) = \sum_{i\in \mathrm{Des}(\sigma)} i, \tag{8}$$

the usual major index statistic on the word σ, while

$$\mathrm{inv}(\sigma, (n)) = \sum_{\substack{1\le i<j\le n \\ \sigma_i>\sigma_j}} 1, \tag{9}$$

the usual inversion statistic.

For $\mu \vdash n$, define

$$\tilde{C}_\mu[Z;q,t] = \sum_\sigma t^{\mathrm{maj}(\sigma,\mu)} q^{\mathrm{inv}(\sigma,\mu)} z^\sigma, \tag{10}$$

where $z^\sigma = \prod_{i=1}^n z_{\sigma_i}$ is the "weight" of σ and the sum is over all words σ of n positive integers.

Conjecture 1 *For all partitions μ,*

$$\tilde{C}_\mu[Z;q,t] = \tilde{H}_\mu[Z;q,t]. \tag{11}$$

Conjecture 1 has been verified by the author using Maple for $|\mu| \le 9$. The author would like to thank A. Ulyanov for also verifying it for $|\mu| = 10$ and $|\mu| = 11$. The $n = 11$ run took 67 hours on a Pentium 3 based machine.

Given a set T of squares and a subset $S \subseteq T$, define

$$F_T[Z;q,S] = \sum_{\substack{\sigma \\ \mathrm{Des}(\sigma,T)=S}} q^{\mathrm{inv}(\sigma,T)} z^\sigma. \tag{12}$$

Let $\widehat{T} = \{w \in T : \mathrm{south}(w) \in T\}$. Note that $\mathrm{Des}(\sigma,T) \subseteq \widehat{T}$ for all σ. In Section 2 we prove the following.

Theorem 1 *For all S, T, $F_T[Z;q,S]$ is a symmetric function in the z_i.*

Given $S \subseteq \mu$, let

$$P(S) = \sum_{w \in S} 1 + \mathrm{leg}(w). \tag{13}$$

It follows from Theorem 1 that $\tilde{C}_\mu[Z;q,t]$ is symmetric in the z_i, since by the definition of $\mathrm{maj}(\sigma,\mu)$,

$$\tilde{C}_\mu[Z;q,t] = \sum_{S \subseteq \widehat{\mu}} t^{P(S)} F_\mu[Z;q,S]. \tag{14}$$

A symmetric function $f(z_1, z_2, \ldots)$ of homogeneous degree n is uniquely defined by the coefficients of its monomials in $z_1, \ldots, z_n$ only. Thus one consequence of the symmetry of $\tilde{C}_\mu[Z;q,t]$ is that we can restrict the infinite sum over σ in (10) to only those σ satisfying $1 \le \sigma_i \le n$ for $1 \le i \le n$, and work with the finite set of variables $Z = \{z_1, z_2, \ldots, z_n\}$. For the remainder of the article we will make this assumption.

Remark 1 *The definition of $Inv(\sigma,\mu)$ is motivated by the "dinv" statistic which occurs in a conjectured formula, for the character of the space of diagonal harmonics, occurring in [HHL$^+$, Conjecture 3.1.2]. In fact, that formula can be recast as a sum of certain $F_T[Z;q,\widehat{T}]$ times nonnegative powers of t and q.*

Definition 1 *Given a word σ of content $(\gamma_1, \gamma_2, \ldots)$, construct a permutation σ', the standardization of σ, by replacing the γ_1 1's in σ by the numbers $1, \ldots, \gamma_1$, the γ_2 2's in σ by the numbers $\gamma_1 + 1, \ldots, \gamma_1 + \gamma_2$, etc., in such a way that, for $i < j$, $\sigma_i \le \sigma_j$ if and only if $\sigma'_i < \sigma'_j$. For example, if $\sigma = 224123114$ then $\sigma' = 458167239$.*

Remark 2 *At first glance it may seem that* $inv(\sigma, T)$ *may not always be nonnegative, but given a square* $u \in Des(\sigma, T)$*, for each square* v *in the same row as* u *and to the right of* u*, either* $\sigma(u) > \sigma(v)$*, or* $\sigma(v) > \sigma(south(u))$*, or both. Assume for the moment that* σ *has distinct entries. If we adopt the convention that for a square* $w \notin T$*,* $\sigma(w) = \infty$*, it follows that* $inv(\sigma, T)$ *equals the number of triples of squares* u, v, w*, where* $v \in T$ *and in the same row as* u*, with* u *strictly to the left of* v*,* $south(u) = w$*, and if we draw a circle through* u, v, w*, and read in the* σ *values of* u, v, w *in counterclockwise order around the circle, starting at the smallest value, then the three values form a strictly increasing sequence. Note that this requires at least two of* u, v, w *to be in* T*. If* σ *has repeated entries, first standardize, then count triples in* (σ', T) *as above.*

In Section 3 we include some results related to the expansion of $\tilde{C}_\mu[Z; q, t]$ into quasisymmetric functions. Section 4 contains a discussion of the various special cases of our conjecture that we can prove, and in Section 5 we show how the coefficient of a monomial symmetric function in the $F_T[Z; q, S]$'s can be expressed recursively.

Maple calculations indicate the $F_T[Z; q, S]$ have the following interesting property.

Conjecture 2 *For all* S, T*,* $F_T[Z; q, S]$ *is Schur positive.*

In particular Conjecture 2 is true for all S, T with $T \subseteq \mu$ for some partition μ with $|\mu| \leq 8$, and has also been checked for many selected choices of T with $|T| \in \{9, 10\}$. At this time we are unable to present a combinatorial prediction for the Schur coefficients for general S, T, but in Section 6 we introduce an elegant conjecture for the Schur coefficients of $F_\mu[Z; q, S]$, and hence for the $\tilde{K}_{\lambda,\mu}(q, t)$, whenever μ has at most three columns.

2 Symmetry

Lemma 1 below can be easily obtained from Corollary 5.2.4 in [HHL+] (by letting each μ_i there be a single square, and choosing the offsets s_i appropriately). For any two sets A,B, we let $A - B = \{a \in A, a \notin B\}$.

Definition 2 *Let* $\beta_1 < \beta_2 \cdots < \beta_n$ *be real numbers. Define a* β*-inversion of a function* $f : \{1, \ldots, n\} \to \{1, 2\}$ *to be a pair* i, j *such that* $0 < \beta_j - \beta_i < 1$ *and* $f(i) > f(j)$*. Let* $inv_\beta(f)$ *denote the number of* β*-inversions and set*

$$G_\beta(z_1, z_2; q) = \sum_f q^{inv_\beta(f)} \prod_i z_{f(i)}. \tag{15}$$

Lemma 1 *With the definitions above, we have* $G_\beta(z_1, z_2; q) = G_\beta(z_2, z_1; q)$ *for all* β*.*

Proof of Theorem 1: Given S, T let $A_T[Z; q, S] = q^{\sum_{w \in S} \operatorname{arm}(w)} F_T[Z; q, S]$, so A_T is symmetric in the z_i if and only if F_T is, and

$$A_T[Z; q, S] = \sum_{\substack{\sigma \\ \mathrm{Des}(\sigma, T) = S}} z^\sigma q^{|\mathrm{Inv}(\sigma, T)|}. \tag{16}$$

By inclusion-exclusion, it is sufficient to show that the right-hand side of (16) is symmetric if $S \subseteq \mathrm{Des}(\sigma, T)$, (not necessarily equal to S). We indicate this notationally as in $A_T[z_1, z_2; q, \geq S]$. Furthermore, it suffices to prove $A_T[Z; q, \geq S]$ is symmetric in any two consecutive variables

z_i, z_{i+1}. Given (σ, T), let $R_i(\sigma)$ be the set of those $w \in T$ with $w(\sigma) = i$ or $w(\sigma) = i+1$. Note that if we permute the i and $i+1$ entries amongst themselves we do not affect any inversion pairs or descents which involve a square outside $R_i(\sigma)$. It follows that if we consider the contribution to $A_T[Z; q, \geq S]$ from all σ for which $R_i(\sigma)$ equals a fixed set T_0, and the values of σ at the squares of $T - T_0$ equal a fixed filling $(\beta, T - T_0)$, we get a constant monomial in $\{z_1 \ldots, z_{i-1}, z_{i+2}, \ldots, z_n\}$ times a constant power of q, times the terms in $A_{T_0}[Z; q, \geq S \cap T_0]$ involving only z_i and z_{i+1}. Hence it suffices to show that for any S, T, $A_T[z_1, z_2; q, \geq S] = A_T[z_2, z_1; q, \geq S]$.

Assume $w \in \mathrm{Des}(\sigma, T)$, so $w(\sigma) = 2$ and $\mathrm{south}(w)(\sigma) = 1$. It is easy to check that if c is any square not equal to w or $\mathrm{south}(w)$, then the number of inversion pairs involving c and either w or $\mathrm{south}(w)$ is the same whether $c(\sigma) = 1$ or $c(\sigma) = 2$. Thus pairs of squares $(w, \mathrm{south}(w))$, $w \in S$, contribute a constant power of q times a fixed power of $z_1 z_2$ to $A_T[z_1, z_2; \geq S]$. Thus if $E = T - S - \{\mathrm{south}(w), w \in S\}$, it now suffices to show $A_E[z_1, z_2; q, \geq \emptyset] = A_E[z_2, z_1; q, \geq \emptyset]$.

Let $n = |E|$, and let m be the maximal row value of the squares of E. For $1 \leq i \leq n$, if $E(i)$ has coordinates (y, x) then set $\beta_i = m - y + x\epsilon$, where ϵ is a small positive number. It is easy to check that a pair $(E(i), E(j))$ of squares of E are in $\mathrm{Inv}(\sigma, E)$ if and only if (i, j) forms a β-inversion in the sense of Definition 2, with f replaced by σ. The symmetry of $A_E[z_1, z_2; q, \geq \emptyset]$ now follows from Lemma 1. □

3 Shuffles

We say a sequence $\sigma_1, \ldots, \sigma_n$ is a *shuffle* of a sequence $a_1, \ldots, a_k$ if a_i occurs before a_{i+1} in σ for $1 \leq i < k$. Given a word σ of content α, note that the statistics $\mathrm{maj}(\sigma, T)$ and $\mathrm{inv}(\sigma, T)$ equal $\mathrm{maj}(\sigma', T)$ and $\mathrm{inv}(\sigma', T)$, respectively, where σ' is the standardization of σ. Also, σ' will clearly be a shuffle of increasing sequences of lengths $\alpha_1, \alpha_2, \ldots$, or equivalently $(\sigma')^{-1}$ is a concatenation of increasing sequences of lengths $\alpha_1, \alpha_2, \ldots$. We call such a permutation an α-shuffle, and if β is an α-shuffle, we let $\mathrm{word}(\beta, \alpha)$ denote the corresponding word of content α (so the standardization of $\mathrm{word}(\beta, \alpha)$ equals β). More generally, given a pair of partitions η, λ, with $|\eta| + |\lambda| = n$, we say $\sigma \in S_n$ is a λ, η-shuffle if σ^{-1} is the concatenation of alternating increasing and decreasing sequences $\lambda_1, \eta_1, \lambda_2, \eta_2, \ldots$. This definition also applies if η, λ are any compositions, where by a composition of n we mean a finite sequence of nonnegative integers whose sum is n.

Let $\langle , \rangle >$ denote the Hall scalar product, with respect to which the Schur functions are orthonormal. It is well-known that the coefficient of m_λ in a symmetric function f is given by $\langle f, h_\lambda \rangle$, where $h_k = s_k = \sum_{\lambda \vdash k} m_\lambda$ and $h_\lambda = \prod_i h_{\lambda_i}$. Results in [HHL$^+$] involving the "superization" of symmetric functions connected to the space of diagonal harmonics follow through rather easily to the $F_T[Z; q, S]$. We list a few of the consequences below.

Theorem 2 *For any S, T, $\mu \vdash n$, and compositions η, λ,*

$$\langle F_T[Z; q, S], e_\eta h_\lambda \rangle >= \sum_{\substack{\sigma \in S_n,\ \sigma \text{ is a } \lambda, \eta\text{-shuffle} \\ Des(\sigma, T) = S}} q^{inv(\sigma, T)}. \tag{17}$$

Corollary 1 *If Conjecture 1 holds, then*

$$\left\langle \tilde{H}_\mu[Z; q, t], e_\eta h_\lambda \right\rangle = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ is a } \lambda, \eta\text{-shuffle}}} t^{maj(\sigma, \mu)} q^{inv(\sigma, \mu)}. \tag{18}$$

For a subset D of $\{1, 2, \ldots, n-1\}$, let

$$Q_{n,D}(z_1, \ldots, z_n) = \sum_{\substack{1 \le a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \implies i \notin D}} z_{a_1} z_{a_2} \cdots z_{a_n} \tag{19}$$

denote Gessel's quasisymmetric function.

Proposition 1 *For any S, T,*

$$F_T[Z; q, S] = \sum_{\substack{\sigma \in S_n \\ Des(\sigma, T) = S}} q^{inv(\sigma, T)} Q_{n, Des(\sigma^{-1})}(Z). \tag{20}$$

4 Special Values

Proposition 2 *Conjecture 1 is true if $q = 1$.*

Proof. Clearly $\tilde{C}_\mu[Z; 1, t] = \prod_i \tilde{C}_{(\mu_i)}[Z; 1, t]$, and $\tilde{H}_\mu[Z; 1, t]$ is also known to factor similarly. Thus it suffices to consider the case $\mu = (1^n)$, in which case it follows from the well-known Cauchy identity and MacMahon's result on the equidistribution of $t^{\mathrm{maj}(\sigma)}$ and $t^{\mathrm{inv}(\sigma)}$ over words σ of fixed content. □

One of the basic properties of the $\tilde{H}_\mu[Z; q, t]$ is

$$\tilde{H}_\mu[Z; q, t] = \tilde{H}_{\mu'}[Z; t, q], \tag{21}$$

where μ' is the "conjugate" partition obtained by reflecting μ about the line $y = x$. Using this, the $t = 1$ case of Conjecture 1 follows from the following lemma, which was noticed by the author and proven by N. Loehr and G. Warrington [LW04].

Lemma 2 *Let μ be a partition with 2 rows. Let β be a word of length μ_1, and α a composition of μ_2. Then*

$$\sum_\sigma q^{inv(\sigma + \beta, \mu)} = q^{inv(\beta, (\mu_1))} \begin{bmatrix} \mu_2 \\ \alpha_1, \alpha_2, \ldots \end{bmatrix}_q, \tag{22}$$

where the sum is over all words σ of content α, and $\sigma + \beta$ is the word obtained by concatenating σ and β. Here $\left[\begin{smallmatrix} \mu_2 \\ \alpha_1, \alpha_2, \ldots \end{smallmatrix}\right]_q$ is the q-multinomial coefficient [And98].

A *semi-standard Young tableau* of shape μ is a filling (σ, μ) where the entries are weakly increasing across rows and strictly decreasing down columns. The tableau is called *standard* if $\sigma \in S_n$. We let $SSYT(\mu, \lambda)$ denote the set of semi-standard tableau of shape μ and content λ, and $SYT(\mu)$ denote the set of standard tableaux of shape μ. If Tab is a standard tableau, we define the tableau descent set of Tab, denoted descent(Tab), to be the set of all i for which $i + 1$ is in a row of μ above the row containing i. Note that descent(Tab) is different from the descent set Des when Tab is viewed as a filling. For any word σ of length n with $1 \le \sigma_i \le n$, let $\mathrm{rev}(\sigma) = \sigma_n \cdots \sigma_2 \sigma_1$ and $\mathrm{flip}_n(\sigma) = n - \sigma_1 + 1 \cdots n - \sigma_n + 1$.

Theorem 3 *Conjecture 1 is true if μ is a hook, i.e. $\mu = k1^{n-k}$ for some $1 \le k \le n$.*

Proof. Converting [Ste94, Theorem 2.1] into a statement about the $\tilde{K}_{\lambda,\mu}(q,t)$, we get

$$\tilde{K}_{\lambda,k1^{n-k}}(q,t) = \sum_{Tab\in SYT(\lambda)} q^{\alpha_k(Tab)} t^{\tilde{\beta}_k(Tab)}, \tag{23}$$

where

$$\alpha_k(Tab) = \sum_{1\le i<k,\, i\in \mathrm{descent}(Tab)} i, \quad \tilde{\beta}_k(Tab) = \sum_{k\le i<n,\, i\in \mathrm{descent}(Tab)} n-i. \tag{24}$$

Using (21) we can rephrase (23) as

$$\tilde{K}_{\nu,k1^{n-k}}(q,t) = \tilde{K}_{\nu,(n-k+1)1^{k-1}}(t,q) \tag{25}$$
$$= \sum_{Tab\in SYT(\nu)} t^{\alpha_{n-k+1}(Tab)} q^{\tilde{\beta}_{n-k+1}(Tab)}. \tag{26}$$

Applying the well known fact that

$$s_\nu = \sum_\lambda K_{\nu,\lambda} m_\lambda, \tag{27}$$

where $K_{\nu,\lambda} = |SSYT(\nu,\lambda)|$, we now have

$$\langle \tilde{H}_{k1^{n-k}}, h_\lambda\rangle = \sum_\nu K_{\nu,\lambda} \sum_{Tab\in SYT(\nu)} t^{\alpha_{n-k+1}(Tab)} q^{\tilde{\beta}_{n-k+1}(Tab)}. \tag{28}$$

Foata [Foa68] (see also [FS78]) gave a bijective transformation ϕ on words which satisfies $\mathrm{maj}(\sigma) = \mathrm{inv}(\phi(\sigma))$, and furthermore $\mathrm{content}(\phi(\sigma)) = \mathrm{content}(\sigma)$ and $\phi(\sigma)_n = \sigma_n$. Let $\mathrm{comaj}(\sigma) = \sum_{i\in \mathrm{Des}(\sigma)} n-i$. For $\sigma \in S_n$, let $\pi(\sigma) = (\mathrm{flip}_n \circ \mathrm{rev} \circ \phi \circ \mathrm{rev} \circ \mathrm{flip}_n)(\sigma)$, where $\circ$ denotes composition. For σ a word of content λ, define $\pi(\sigma) = \mathrm{word}(\pi(\sigma'),\lambda)$. One checks that $\mathrm{comaj}(\sigma) = \mathrm{inv}(\pi(\sigma))$, $\pi(\sigma)_1 = \sigma_1$, and π is an invertible map from the set of words of content λ to itself.

Given a λ-shuffle $\zeta \in S_n$, let $\sigma = \mathrm{word}(\zeta,\lambda)$, and let $\gamma = \sigma_1\cdots\sigma_{n-k} + \pi^{-1}(\sigma_{n-k+1}\cdots\sigma_n)$ be the word of content λ obtained by applying the map π^{-1} to the last k letters of σ, and fixing the first $n-k$ letters. The standardization γ' is a λ-shuffle, and if we apply the RSK algorithm to γ', we get a pair $(P_{\gamma'}, Q_{\gamma'})$ of SYT of the same shape, with $\mathrm{Des}(\gamma') = \mathrm{descent}(Q_{\gamma'})$ and $\mathrm{Des}((\gamma')^{-1}) = \mathrm{descent}(P_{\gamma'})$ (see, for example [Sta99, Chapter 7]). Furthermore, the values of $\mathrm{maj}(\zeta, k1^{n-k})$ and $\mathrm{inv}(\zeta, k1^{n-k})$ depend only on $Q_{\gamma'}$. Now $\mathrm{descent}(P_{\gamma'}) \subseteq \{\lambda_1, \lambda_1+\lambda_2, \ldots\}$, hence in $P_{\gamma'}$ the numbers 1 through λ_1 form a horizontal strip, as do the numbers λ_1+1 through $\lambda_1+\lambda_2$, etc.. Thus we can associate a $SSYT$ of content λ to $P_{\gamma'}$. It follows that as we vary ζ over all λ-shuffles in S_n, the number of different $P_{\gamma'}$ that will occur with a given $Q_{\gamma'}$ of shape ν

equals $K_{\nu,\lambda}$. Hence

$$\begin{aligned}
\langle \tilde{C}_{k1^{n-k}}[Z;q,t], h_\lambda\rangle &= \sum_{\zeta \in S_n,\ \zeta \text{ is a } \lambda\text{-shuffle}} t^{\mathrm{maj}(\zeta, k1^{n-k})} q^{\mathrm{inv}(\zeta, k1^{n-k})} && (29)\\
&= \sum_{\gamma' \in S_n,\ \gamma' \text{ is a } \lambda\text{-shuffle}} t^{\mathrm{maj}(\gamma'_1,\ldots,\gamma'_{n-k+1})} q^{\mathrm{comaj}(\gamma'_{n-k+1},\ldots,\gamma'_n)} && (30)\\
&= \sum_{\gamma' \in S_n,\ \gamma' \text{ is a } \lambda\text{-shuffle}} t^{\alpha_{n-k+1}(Q_{\gamma'})} q^{\tilde{\beta}_{n-k+1}(Q_{\gamma'})} && (31)\\
&= \sum_{\nu} K_{\nu,\lambda} \sum_{Tab \in SYT(\nu)} t^{\alpha_{n-k+1}(Tab)} q^{\tilde{\beta}_{n-k+1}(Tab)} && (32)\\
&= \langle \tilde{H}_{k1^{n-k}}[Z;q,t], h_\lambda\rangle && (33)
\end{aligned}$$

by (28). $\square$

Remark 3 *An interesting and perhaps important problem is to show*

$$\tilde{C}_\mu[Z;q,t] = \tilde{C}_{\mu'}[Z;t,q], \tag{34}$$

which by (21) must hold if Conjecture 1 is true. The arguments above show only that $\tilde{C}_\mu[Z;1,t] = \tilde{C}_{\mu'}[Z;t,1]$. *We leave it as an interesting exercise for the reader to verify (34) bijectively for hook shapes using only the fact that* $\tilde{C}_\mu[Z;q,t]$ *is a symmetric function, together with properties of the maps* ϕ *and* π.

We can also prove the following special cases of our conjectures.

Proposition 3 *Let* d *satisfy* $0 \le d \le n$. *Then (18) holds when* $\eta = (n-d), \lambda = (d)$. *Also,*

$$\tilde{C}_\mu[Z;q,0] = \tilde{H}_\mu[Z;q,0] \tag{35}$$

$$\tilde{C}_\mu[Z;q,t]|_{t^{\eta(\mu)}} = \tilde{H}_\mu[Z;q,t]|_{t^{\eta(\mu)}}, \tag{36}$$

where for any polynomial $f(x)$, $f|_{x^j}$ *stands for the coefficient of* x^j *in* f. *In addition, Conjecture 2 holds when* $q = 1$.

Remark 4 *By taking the coefficient of* $z_1 z_2 \cdots z_n$ *in* $\tilde{C}_\mu[Z;q,t]$ *we obtain a conjectured formula for the bigraded Hilbert series of the Garsia-Haiman modules* $V(\mu)$. *In [GH95] Garsia and Haiman derive a statistical description for the Hilbert series when* $\mu = k1^{n-k}$, *which is easily shown to be equivalent to ours. They also obtain statistics for the case where* μ *has two rows, but the author does not know how to show their formula is equivalent to that predicted by Conjecture 1 for this case.*

5 A Recursive Formulation

Given $\lambda \vdash n$ and S, T with $S \subseteq T$, to calculate the coefficient of the monomial symmetric function m_λ in $F_T[Z;q,S]$, by symmetry it suffices to calculate the coefficient of $z_n^{\lambda_1} z_{n-1}^{\lambda_2} \cdots z_1^{\lambda_n}$, so we can consider only fillings involving words of content $\mathrm{rev}(\lambda)$.

Recall the description of $\mathrm{inv}(\sigma, T)$ in Remark 2, involving triples of squares. The fact that all the n's are larger than any other entry of σ will allow us to isolate the contribution to $\mathrm{inv}(\sigma, T)$ from triples involving an n. We will view the set of squares of (σ, T) containing n's as $P \cup Q$, where $P \subseteq T - S$, $Q \subseteq S$.

Definition 3 *Given P, Q, S, T as above, let ind(P, Q, S, T) denote the number of triples of squares u, v, w satisfying*

$$\begin{cases} row(u) = row(v) = row(w) + 1 \\ col(u) < col(v),\ v \in T \text{ and } u \notin S, \\ \text{the triple } u, v, w \text{ form one of the patterns of type } A, B, C \text{ or } D \text{ in Figure 2,} \end{cases} \tag{37}$$

where in these patterns a square occupied by

$$\begin{cases} \text{an } n \text{ means the square is in } P \\ \text{an } X \text{ means the square is not in } T \\ \text{nothing means the square is in } T - P - Q. \end{cases} \tag{38}$$

The following result is easily derived from the definition of $F_T[Z; q, S]$ and ind(P, Q, S, T).

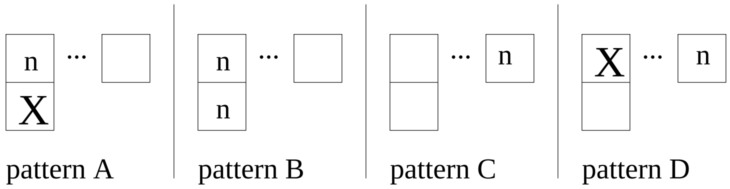

Figure 2: The various possible patterns which contribute to the statistic ind(S, T, P, Q).

Theorem 4 *For $\lambda \vdash n$ and $S \subseteq T$,*

$$\langle F_T[Z; q, S], h_\lambda \rangle = \sum_{\substack{P,Q \\ P \subseteq T-S,\, Q \subseteq S \\ |P|+|Q|=\lambda_1}} q^{ind(S,T,P,Q)} \langle F_{T-P-Q}[Z; q, S-Q], h_{\lambda_2,\lambda_3,\ldots,\lambda_n} \rangle, \tag{39}$$

with the inital condition $F_\emptyset[Z; q, \emptyset] = 1$.

6 Conjectures involving Schur Coefficients

It would be very desirable to have a combinatorial description of the Schur coefficients of the $F_T[Z; q, S]$. Such formulas exist for the $\tilde{K}_{\lambda,\mu}$ when λ or μ is a hook or when μ has two columns or two rows, and for some other shapes obtained by adding a square or two to one of the above shapes. All of the published formulas for the case when μ has two columns are fairly complicated, involving such things as rigged configurations and catabolism [Fis95],[Zab98],[LM03].

We now advance a conjectured combinatorial description for $\langle F_\mu[Z; q, S], s_\lambda \rangle$ whenever μ has at most three columns. Let $F_1 = (12 \cdots n, \mu)$ be the filling of μ by the identity permutation and $F_2 = (n \cdots 21, \mu)$ the filling by the reverse of the identity. For any pair of integers (a, b) with $1 \le a < b \le n$, say a is in square A in F_1 and b is in square B in F_1, i.e. $A = \mu(a)$, $B = \mu(b)$. Define the (multi-t variate) μ-weight of (a, b), denoted wt(μ, a, b), as

$$\mathrm{wt}(\mu, a, b) = \begin{cases} q \text{ if } (A, B) \in \mathrm{Inv}(F_2) \\ q^{-\mathrm{arm}(A)} t_A \text{ if } A \in \mathrm{Des}(F_2) \text{ and } B = \mathrm{south}(A) \\ 1 \text{ otherwise.} \end{cases} \tag{40}$$

For example, for F_1 as on the left in Figure 3,

$$\mathrm{wt}(3221,2,4) = q^{-1}t_{(2,0)}, \quad \mathrm{wt}(3221,3,4) = q, \quad \mathrm{wt}(3221,3,6) = 1, \ldots.$$

1		
2	3	
4	5	6
7	8	9

3	9		
2	6	7	
1	4	5	8

Figure 3: On the left, the filling $F_1 = (123456789, 3321)$, and on the right a tableau in $SYT(432)$.

Given a SYT Tab of partition shape with $|\mu|$ squares, our strategy will be to identify pairs (a,b) as "inversion pairs" of Tab, then weight them as in (40). We begin by partially defining what constitutes an inversion by the following.

1) The pair (a,b) forms an inversion in Tab if $1 \le a < b \le |\mu|$ and b is weakly northwest of a in Tab, i.e. b is not in a column to the right of a.

2) If $a < b$ and b is weakly southeast of a, i.e. is not in a row above a, then (a,b) do not form an inversion pair.

3) If $a+3$ is neither weakly northwest or weakly southeast of a, then $(a,a+3)$ forms an inversion pair if Tab contains the pattern on the left in Figure 4, and does not if Tab contains the pattern on the right.

a+2	a+3
a	a+1

a+1	a+3
a	a+2

Figure 4: The pair $(a,a+3)$ form an inversion if the pattern on the left occurs.

Define

$$\tilde{C}_\mu[Z;q,\vec{t}\,] = \sum_{S\subseteq\widehat{\mu}} F_\mu[Z;q,S] \prod_{w\in S} t_w, \tag{41}$$

which is a "multi-t variate" version of $\tilde{C}_\mu$. By (14), if we replace t_w by $t^{\mathrm{leg}(w)+1}$ for all w, this multi-t version will reduce to $\tilde{C}_\mu[Z;q,t]$.

Let inversion(Tab) denote the set of inversion pairs of Tab.

Conjecture 3 *If μ has at most three columns,*

$$\langle \tilde{C}_\mu[Z;q,\vec{t}\,], s_\lambda\rangle = \sum_{Tab\in SYT(\lambda)} \prod_{(a,b)\in inversion(Tab)} wt(\mu,a,b). \tag{42}$$

Example 1 *If Tab is the tableau on the right in Figure 3, then*

$$(1,2),(2,3),(4,6),(4,7),(5,6),(5,7),(6,9),(7,9),(8,9) \tag{43}$$

form inversions with nontrivial 3321*-weights, so the contribution of Tab to $\langle \tilde{C}_{3321}[Z;q,\vec{t}\,], s_{432}\rangle$ is $t_{(3,0)} * q * q * t_{(1,0)} q^{-2} * q * q * t_{(1,2)} * q * q = t_{(3,0)} t_{(1,0)} t_{(1,2)} q^4$.*

Conjecture 3 has been checked in Maple for all λ, μ with $|\lambda| \leq 12$ and $|\mu| \leq 12$. The calculation made use of tables of the $K_{\lambda,\mu}(q,t)$ supplied by G. Tesler, as well as J. Stembridge's Maple package for symmetric functions SF [Ste], which was also used in testing our other conjectures. Note that if (42) holds, by taking the coefficient of $\prod_{w \in S} t_w$ in the right-hand side of (42) we obtain a formula for $\langle F_\mu[Z;q,S], s_\lambda\rangle$.

7 Acknowledgements

The author's research is partially supported by grant MSPF-02G-193 from the National Security Agency.

References

[And98] G. E. Andrews. *The Theory of Partitions.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[Fis95] S. Fishel. Statistics for special q,t-Kostka polynomials. *Proc. Amer. Math. Soc.*, 123(10):2961–2969, 1995.

[Foa68] D. Foata. On the Netto inversion number of a sequence. *Proc. Amer. Math. Soc.*, 19:236–240, 1968.

[FS78] D. Foata and M.-P. Schützenberger. Major index and inversion number of permutations. *Math. Nachr.*, 83:143–159, 1978.

[GH93] A. M. Garsia and M. Haiman. A graded representation model for Macdonald polynomials. *Proc. Nat. Acad. Sci. U.S.A.*, 90(8):3607–3610, 1993.

[GH95] A. M. Garsia and M. Haiman. Factorizations of Pieri rules for Macdonald polynomials. *Discrete Math.*, 139(1-3):219–256, 1995. Formal power series and algebraic combinatorics (Montreal, PQ, 1992).

[Hai99] M. Haiman. Macdonald polynomials and geometry. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 207–254. Cambridge Univ. Press, Cambridge, 1999.

[Hai01] M. Haiman. Hilbert schemes, polygraphs, and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14:941–1006, 2001.

[HHL$^+$] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke J. Math., to appear.* arXiv:math.CO/0310424.

[LM03] L. Lapointe and J. Morse. Tableaux statistics for two part Macdonald polynomials. In *Algebraic combinatorics and quantum groups*, pages 61–84. World Sci. Publishing, River Edge, NJ, 2003.

[LW04] N. Loehr and G. Warrington. Private communication, 2004.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs, second ed., Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1995.

[SSW03] A. Schilling, M. Shimozono, and D. E. White. Branching formula for q-Littlewood-Richardson coefficients. *Adv. in Appl. Math.*, 30(1-2):258–272, 2003. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).

[Sta99] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, United Kingdom, 1999.

[Ste] J. R. Stembridge. A Maple package for symmetric functions.

[Ste94] J. R. Stembridge. Some particular entries of the two-parameter Kostka matrix. *Proc. Amer. Math. Soc.*, 121:469–490, 1994.

[Zab98] M. Zabrocki. A Macdonald vertex operator and standard tableaux statistics for the two-column (q,t)-Kostka coefficients. *Electron. J. Combin.*, 5:Research Paper 45, 46 pp. (electronic), 1998.