Notes on Rook Polynomials

F. Butler

J. Haglund

J. B. Remmel

Department of Physical Sciences, York College of Pennsylvania, York, PA 17403

Current address: Department of Physical Sciences, York College of Pennsylvania, York, PA 17403

E-mail address: `fbutler@ycp.edu`

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395

Current address: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395

E-mail address: `jhaglund@math.upenn.edu`

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112

Current address: Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112

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Abstract. This is a great Book on Rook Theory

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CHAPTER 1

Rook Theory and Simon Newcomb's Problem

Rook Placements and Permutations

Throughout we abbreviate left-hand-side and right-hand-side by LHS and RHS, respectively. The theory of rook polynomials was introduced by Kaplansky and Riordan [**KR46**], and developed further by Riordan [**Rio02**]. We refer the reader to Stanley [**Sta12**, Chap. 2] for a nice exposition of some of the basics of rook polynomials and permutations with forbidden positions. A *board* is a subset of an $n \times n$ grid of squares. We label the squares of the grid by the same $(\text{row}, \text{column})$ coordinates as the squares of an $n \times n$ matrix, i.e. the lower-left-hand square has label $(n, 1)$, etc. We let $r_k(B)$ denote the number of ways of placing k rooks on the squares of B, no two attacking, i.e. no two in the same row or column. By convention we set $r_0(B) = 1$. We define the kth hit number of B, denoted $t_k(B)$, to be the number of ways of placing n nonattacking rooks on the $n \times n$ grid, with exactly k rooks on B. Note that $t_n(B) = r_n(B)$. We have the fundamental identity

$$\sum_{k=0}^{n} r_k(B)(x-1)^k(n-k)! = \sum_{j=0}^{n} x^j t_j(B). \tag{1.1}$$

Permutations $\pi = (\pi_1\pi_2\cdots\pi_n) \in S_n$ in one-line notation can be identified with placements $P(\pi)$ of n rooks on the $n \times n$ grid by letting a rook on (j, i) correspond to $\pi_i = j$. Hence $t_k(B)$ can be viewed as the number of π which violate k of the "forbidden positions" represented by the squares of B. For example, if B is the "derangement board" consisting of squares $(i, i), 1 \le i \le n$, then $t_0(B)$ counts the number of permutations with no derangements. Clearly $r_k(B) = \binom{n}{k}$ here, and applying (1.1) we get the well-known formula

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k(n-k)! \tag{1.2}$$

for the number of derangements in S_n.

Another way to identify rook placements with permutations is to start with a permutation π in one-line notation, then create another permutation $\beta(\pi)$ by viewing each left-to-right minima of π as the last element in a cycle of $\beta(\pi)$. For example, if $\pi = 361295784$, then $\beta(\pi) = (361)(2)(95784)$. In this example $P(\beta(\pi))$ consists of rooks on

$$\{(6,3),(1,6),(3,1),(2,2),(5,9),(7,5),(8,7),(4,8),(9,4)\}. \tag{1.3}$$

Note that the number of permutations with k cycles is hence equal to the number of permutations with k left-to-right minima. Now for the rook placement $P(\beta(\pi))$, a rook on (i, j) can be interpreted as meaning i immediately follows j in some cycle of β, and if $j > i$, this will happen iff π contains the descent $\cdots ji \cdots$. If we let

B_n denote the "triangular board" consisting of squares $(i, j), 1 \leq i < j \leq n$, then rooks on B_n in $P(\beta(\pi))$ correspond to descents in π. Hence we have

$$t_k(B_n) = A_{k+1}(n), \tag{1.4}$$

where $A_j(n)$ is the jth "Eulerian number", i.e. the number of permutations in S_n with $j-1$ descents.

A Ferrers board is a board with the property that if $(i, j) \in B$, then $(k, l) \in B$, for all $1 \leq k \leq i, j \leq l \leq n$. We can identify a Ferrers board with the numbers of squares c_i in the ith column of B, so $c_1 \leq c_2 \leq \cdots \leq c_n$. We will often refer to the Ferrers board with these columns heights by $B(c_1, \ldots, c_n)$. Note in this convention $B_n = B(0, 1, \ldots, n-1)$.

Identity (1.4) has a nice generalization to multiset permutations. Given $v \in \mathbb{N}^p$, define a map ζ from S_n to the set of multiset permutations $M(v_1, v_2, \ldots, v_p)$ of $\{1^{v_1} \cdots p^{v_p}\}$ by starting with $\pi \in S_n$ and replacing the smallest v_1 numbers by 1's, the next v_2 smallest numbers by 2's, etc. For example, if $\pi = 361295784$ and $v = (3, 5, 1)$, then $\zeta(\pi) = 121132222$. Note that with this v, rooks on squares $(1, 2), (1, 3), (2, 3)$ no longer correspond to descents, and neither do rooks on $(4, 5), (5, 6), (4, 6), \ldots, (7, 8)$.

Let $N_k(v)$ denote the number of multiset permutations of elements of $M(v)$ with $k-1$ descents. The $N_k(v)$ are named after British astronomer Simon Newcomb who, while playing a card game called patience, posed the following problem: if we deal the cards of a 52 card-deck out one at a time, starting a new pile whenever the face value of the card is less than that of the previous card, in how many ways can we end up with exactly k piles? MacMahon noted this is equivalent to asking for a formula for $N_k(13, 13, 13, 13)$, and studied the more general question of finding a formula for $N_k(v)$ [**Mac60**]. Riordan [**Rio02**] noted that since ζ is a 1 to $\prod_i v_i!$ map, if we let G_v denote the Ferrers board whose first v_1 columns are of height 0, next v_2 of height v_1, next v_3 of height $v_1 + v_2$, ..., and last v_p of height $v_1 + \ldots v_{p-1}$, it follows that

$$t_k(G_v) = \prod_i v_i! N_{k+1}(v). \tag{1.5}$$

Algebraic Identities for Ferrers Boards

If $B = B(c_1, \ldots, c_n)$ is a Ferrers board, let

$$PR(x, B) = \prod_{i=1}^{n} (x + c_i - i + 1). \tag{1.6}$$

Goldman, Joichi and White [**GJW75**] proved that

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1) r_{n-k}(B) = PR(x, B). \tag{1.7}$$

We recall the well-known proof, which we will generalize later. First note that, since both sides of (1.7) are polynomials of degree n in x, it suffices to prove (1.7) for $x \in N$. For such an x, let B_x denote the board obtained by adjoining an $x \times n$ rectangle of squares above B. We count the number of ways to place n nonattacting rooks on B_x in two ways. First of all, we can place a rook in column 1 of B_x in $x + c_1$ ways, then a rook in column two in any of $x + c_2 - 1$ ways, etc., thus generating the RHS of (1.7). Alternatively, we can begin by placing say $n - k$

rooks on B in $r_{n-k}(B)$ ways. Each such placement eliminates $n-k$ columns of B_x, leaving k rooks to place on an $n-k$ by x rectangle, which can clearly be done in $x(x-1)\cdots(x-k+1)$ ways.

COROLLARY 1.0.1. *Let $B = B(c_1, \ldots, c_n)$ be a Ferrers board. Then*

$$k!r_{n-k}(B) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} PR(j, B) \tag{1.8}$$

$$t_{n-k}(B) = \sum_{j=0}^{k} \binom{n+1}{k-j} (-1)^{k-j} PR(j, B). \tag{1.9}$$

PROOF. By (1.7), the RHS of (1.8) equals

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{s} \binom{j}{s} s!r_{n-s} \tag{1.10}$$

$$= \sum_{s\geq 0} s!r_{n-s} \sum_{j\geq s} \binom{k}{j} (-1)^{k-j} \binom{j}{s}$$

$$= \sum_{s\geq 0} s!r_{n-s} \sum_{j\geq s} (1-z)^k|_{z^{k-j}} (1-z)^{-s-1}|_{z^{j-s}} \tag{1.11}$$

$$= \sum_{s\geq 0} s!r_{n-s} \frac{(1-z)^k}{(1-z)^{s+1}}|_{z^{k-s}}$$

$$= \sum_{s\geq 0} s!r_{n-s}\delta_{s,k},$$

where

$$\delta_{k,j} = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ else.} \end{cases}$$

Also using (1.7), the RHS of (1.9) equals

$$\sum_{j=0}^{k} \binom{n+1}{k-j} (-1)^{k-j} \sum_{s\leq j} \binom{j}{s} s!r_{n-s} \tag{1.12}$$

$$= \sum_{s} s!r_{n-s} \sum_{j\geq s} \binom{n+1}{k-j} (-1)^{k-j} \binom{j}{s}$$

$$= \sum_{s} s!r_{n-s} \sum_{j\geq s} (1-x)^{n+1}|_{z^{k-j}} (1-z)^{-s-1}|_{z^{j-s}}$$

$$= \sum_{s} s!r_{n-s} \frac{(1-x)^{n+1}}{(1-x)^{s+1}}|_{x^{k-s}}$$

by the binomial theorem. Now use (1.1). □

A unitary vector is a nonzero vector all of whose coordinates are 0 or 1. For a vector v of nonnegative integers, let $g_k(v)$ denote the number of ways of writing v as a sum of k unitary vectors. By convention we set $g_0(0) = 0$. For example,

$g_2(2,1) = 2$ and $g_3(2,1) = 3$ since

$$\begin{aligned}(2,1) &= (1,1)+(1,0)\\ &= (1,0)+(1,1)\\ &= (1,0)+(1,0)+(0,1)\\ &= (1,0)+(0,1)+(1,0)\\ &= (0,1)+(1,0)+(1,0).\end{aligned} \tag{1.13}$$

Letting 1^n stand for the vector with n ones, it is easy to see that $g_k(1^n) = k!S(n,k)$, where $S(n,k)$ is the Stirling number of the second kind. MacMahon derived a number of identities for $g_k(v)$ in connection with his work on Simon Newcomb's problem. Here we show how these numbers can be connected with rook theory. We let $n = v_1 + \dots v_p$.

THEOREM 1.1. *For any v,*

$$g_k(v) = \frac{k!r_{n-k}(G_v)}{\prod_i v_i!}. \tag{1.14}$$

PROOF. By definition we have

$$\sum_v \prod_i x_i^{v_i} g_k(v) = (\prod_i (1+x_i) - 1)^k. \tag{1.15}$$

Hence

$$\begin{aligned}\prod_i (1+x_i)^z &= \sum_{k\geq 0} \binom{z}{k} (\prod_i (1+x_i) - 1)^k\\ &= \sum_{k\geq 0} \binom{z}{k} \sum_w \prod_i x_i^{w_i} g_k(w).\end{aligned} \tag{1.16}$$

Taking the coefficient of $\prod_i x_i^{v_i}$ on both sides above yields

$$\prod_i \binom{z}{v_i} = \sum_{k\geq 0} \binom{z}{k} g_k(v). \tag{1.17}$$

Next note that $PR(z, G_v) = \prod_i v_i! \binom{z}{v_i}$. Comparing (1.17) with the $B = G_v$ case of (1.7) we obtain (1.14). □

COROLLARY 1.1.1.

$$\sum_k g_k(v) x^{n-k} = \sum_j N_{j+1}(v)(x+1)^j. \tag{1.18}$$

PROOF. This follows from (1.14), (1.5), and (1.1). We also provide a direct combinatorial proof, which is based on a argument in [**And98**, p. 61] proving a closely related identity. By comparing coefficients of x^{n-k} on both sides of (1.18) we get

$$g_k(v) = \sum_j N_{j+1}(v) \binom{j}{n-k}. \tag{1.19}$$

To prove (1.19), start with a unitary composition C into k parts, say

$$w_1 + w_2 + \dots + w_k = v. \tag{1.20}$$

For each vector w_i in C, associate a subset $S(w_i)$ by letting $p \in S(w_i)$ iff $w_{i,p} = 1$. For example, if $z = (1,0,0,1,1,0,1)$, $S(z) = \{1,4,5,7\}$. Next form a multiset permutation $M(C)$ with bars between some elements by listing the elements of $S(w_1)$ in decreasing order, followed by a bar and then the elements of $S(w_2)$, in decreasing order, followed by a bar, ..., followed by the elements of $S(w_k)$, in decreasing order. If C is the composition

$$\begin{aligned}(1,1,0,0,0,0)+(1,0,0,0,0,0)+(0,0,1,0,0,0)+(0,0,0,1,0,0)+(0,0,0,0,0,1)\\ +(1,1,0,0,1,0)+(0,0,0,1,0,0)+(0,0,0,1,0,0)+(0,1,0,0,0,0)\end{aligned}$$

then $M(C) = 21|1|3|4|6|521|4|4|2$. Note that if $M(C)$ has j descents, then we have bars at each of the $n-1-j$ non-descents, together with an additional $j-n+k$ bars at descents for a total of $n-1-j+j-n+k = k-1$ bars. Thus we have a map from unitary compositions with k parts to multiset permutations with say j descents, with an additional $j-n+k$ bars chosen from the descents, which is counted by the RHS of (1.19). It is easy to see the map is invertible. □

THEOREM 1.2. *For any Ferrers board B,*

$$\sum_{j=0}^{\infty} PR(j,B)z^j = \frac{\sum_{k=0}^{n} z^k t_{n-k}(B)}{(1-z)^{n+1}}. \tag{1.21}$$

PROOF.

$$\begin{aligned}(1-z)^{n+1}\sum_{j=0}^{\infty} PR(j,B)z^j|_{z^k} &= \sum_{j=0}^{k}\binom{n+1}{k-j}(-1)^{k-j}PR(j,B)\\ &= t_{n-k}(B)\end{aligned} \tag{1.22}$$

by (1.9). □

Letting $v = 1^n$ in (1.21) we get

COROLLARY 1.2.1.

$$\frac{\sum_{k=0}^{n} z^k N_{k+1}(1^n)}{(1-z)^{n+1}} = \sum_{j=0}^{\infty} z^j j^n. \tag{1.23}$$

THEOREM 1.3. *For any Ferrers board B,*

$$\sum_{k=0}^{n}\binom{x+k}{n}t_k(B) = PR(x,B). \tag{1.24}$$

PROOF. It suffices to prove (1.24) under the assumption that $x \in \mathbb{N}$. Then the RHS of (1.24) equals

$$\left(\sum_{k=0}^{\infty} y^k PR(k,B)\right)|_{y^x} = \left(\frac{\sum_j t_{n-j}(B)y^j}{(1-y)^{n+1}}\right)|_{y^x} \tag{1.25}$$
$$= \left(\sum_j t_{n-j}(B)y^j\right)\left(\sum_{m=0}^{\infty} y^m \binom{n+m}{m}\right)|_{y^x}$$
$$= \sum_{j=0}^{x} t_{n-j}(B)\binom{n+x-j}{x-j}$$
$$= \sum_{k\geq 0} t_k(B)\binom{x+k}{n}.$$

□

By letting $B = G_v$ in (1.24) and using (1.5) we get

COROLLARY 1.3.1. *For any $v \in \mathbb{N}^p$,*

$$\sum_{k=0}^{n} \binom{x+k}{n} N_{k+1}(v) = \prod_{i=1}^{p} \binom{x}{v_i}. \tag{1.26}$$

REMARK 1.4. When $v = 1^n$, (1.26) is known as Worpitsky's identity.

Vector Compositions

For $v \in \mathbb{N}^p$, let $f_k(v)$ denote the number of ways of writing

$$v = w_1 + \ldots + w_k, \tag{1.27}$$

where $w_i \in \mathbb{N}^p$ with $|w_i| = \sum_j w_{ij} > 0$. For example if $v = (2,1)$, in addition to the ways of decomposing v into unitary vectors as in (1.13), we have

$$(2,1) = (2,1) \tag{1.28}$$
$$= (2,0) + (0,1) = (0,1) + (2,0), \tag{1.29}$$

so $f_1(2,1) = 1$, $f_2(2,1) = 4$, and $f_3(2,1) = 3$. MacMahon first defined and studied $f_k(v)$, deriving of (1.30) and (1.34) below.

PROPOSITION 1.4.1. *For any $v \in \mathbb{N}^p$,*

$$\prod_i \binom{z+v_i-1}{v_i} = \sum_{k\geq 0} \binom{z}{k} f_k(v), \tag{1.30}$$

where we define $f_0(v) = \delta_{n,0}$.

PROOF. By definition we have

$$\sum_v \prod_i x_i^{v_i} f_k(v) = (\prod_i \frac{1}{(1-x_i)} - 1)^k. \tag{1.31}$$

Hence

$$(\prod_i \frac{1}{(1-x_i)})^z = \sum_{k\geq 0}\binom{z}{k}(\prod_i \frac{1}{(1-x_i)}-1)^k \tag{1.32}$$
$$= \sum_{k\geq 0}\binom{z}{k}\sum_w \prod_i x_i^{w_i} f_k(w).$$

Taking the coefficient of $\prod_i x_i^{v_i}$ on both sides above yields (1.30). $\square$

Corollary 1.4.1. *Let F_v be the Ferrers board whose first v_1 columns are of height $v_1 - 1$, whose next v_2 columns are of height $v_1 + v_2 - 1$, ..., and whose last v_p columns are of height $v_1 + \ldots + v_p - 1$, so $PR(z, F_v) = \prod_i v_i!\binom{z+v_i-1}{v_i}$. Then*

$$f_k(v) = \frac{k! r_{n-k}(F_v)}{\prod_i v_i!}. \tag{1.33}$$

Theorem 1.5. *(MacMahon [**Mac60**,]).*

$$\sum_k f_k(v)x^{n-k} = \sum_j N_j(v)(x+1)^{n-j}. \tag{1.34}$$

Exercise 1.6. Prove (1.34) combinatorially using an argument similar to the one above proving (1.18).

By combining (1.1), (1.33) and (1.34) we obtain

Corollary 1.6.1.

$$N_k(v) = \frac{1}{\prod_i v_i!} t_{n-k}(F_v). \tag{1.35}$$

q-Rook Polynomials

For $x \in \mathbb{R}$, let $[x] = (1-q^x)/(1-q)$. By L'Hopital's rule, $[x] \to x$ as $q \to 1$. For $k \in \mathbb{N}$ set $[k]! = [1][2]\cdots[k]$, and define the generalized q-binomial coefficient via

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x][x-1]\cdots[x-k+1]}{[k]!}. \tag{1.36}$$

For B a Ferrers board, Garsia and Remmel [GaRe] introduced the following q-analogue of the rook number.

$$R_k(B) := \sum_C q^{\mathrm{inv}(C,B)}, \tag{1.37}$$

where the sum is over all placements C of k non-attacking rooks on the squares of B. To calculate the statistic $\mathrm{inv}(C, B)$, cross out all squares which either contain a rook, or are above or to the right of any rook. The number of squares of B not crossed out is $\mathrm{inv}(C, B)$. See Figure 1.

Garsia and Remmel showed that the R_k enjoy many of the same properties as the r_k. For example, they proved

Theorem 1.7. *For any Ferrers board $B(c_1, \ldots, c_n)$,*

$$\sum_{k=0}^{n} [x][x-1]\cdots[x-k+1]R_{n-k}(B) = \prod_{i=1}^{n}[x + c_i - i + 1]. \tag{1.38}$$

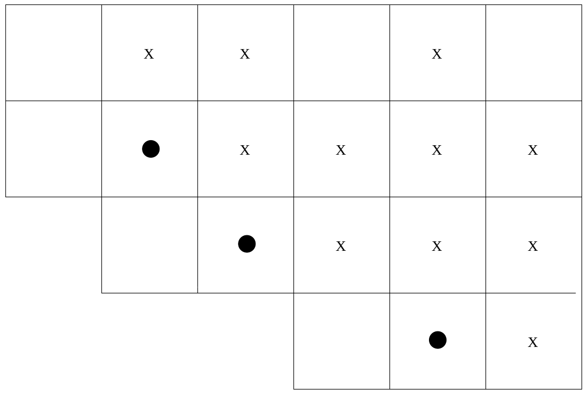


FIGURE 1. A placement of 3 rooks with inv = 6.

PROOF. Since both sides of (1.38) can be viewed as polynomials in the variable q^x, it suffices to prove (1.38) for $x \in \mathbb{N}$. For such an x, consider the Ferrers board $B_x = B(c_1 + x, \ldots, c_n + x)$ obtained by adjoining an x by n rectangle above B. We add up $q^{\mathrm{inv}(C,B_x)}$ over all placements C of n nonattacking rooks on B_x. If we place a rook in column 1, then the inversions in column 1 generate a $[x+c_1]$ factor. Then in column 2, one square is eliminated by the rook in column 1, so we generate a $[x + c_2 - 1]$ factor, and by iterating this argument we get the RHS of (1.38). Alternatively, we could begin by placing $n-k$ rooks on B in $R_{n-k}(B)$ ways (taking into account the contribution to inv from squares on B only). Each placement of $n - k$ rooks eliminates $n - k$ columns of the x by n rectangle, and placing the remaining k rooks on the k open columns, taking into account contributions to inv from squares on the x by n rectangle only, gives the $[x][x - 1] \cdots [x - k + 1]$ factor in the LHS of (1.38). □

Unless otherwise stated, we assume $c_n \leq n$ (such boards are called *admissible* in the literature). As noted by Garsia and Remmel, an interesting consequence of (1.7) is that two Ferrers boards have the same rook numbers if and only if they have the same q-rook numbers, since both of these are determined by the multiset whose elements are the shifted column heights $c_i - i + 1$.

For B a Ferrers board, define the q-hit numbers $T_k(B)$ via

$$\sum_{k=0}^{n}[k]!R_{n-k}(B) \prod_{i=k+1}^{n} (x - q^i) = \sum_{j=0}^{n} T_j x^j. \tag{1.39}$$

Garsia and Remmel proved that

$$T_k(B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\mathrm{stat}(C,\,B)}, \tag{1.40}$$

for some statistic $\mathrm{stat}(C, B) \in \mathbb{N}$ which they defined recursively. They left it as an open problem to find a more explicit description of the $T_k(B)$. This problem was solved independently by Dworkin [**Dwo98**] and Haglund [**Hag98**], who found slightly different ways of generating $T_k(B)$. Given a placement C of n rooks on the $n \times n$ grid, we define the Dworkin statistic $\xi(C, B)$ by means of the following procedure.

First place a bullet under each rook, and an x to the right of any rook. Next, for each rook on B, place a circle in the empty cells of B that are below it in the column. Then for each rook off B, place a circle in the empty cells below it in the column, and also in the empty cells of B above it in the column. Then $\xi(C, B)$ is the number of circles. See Figure 2.

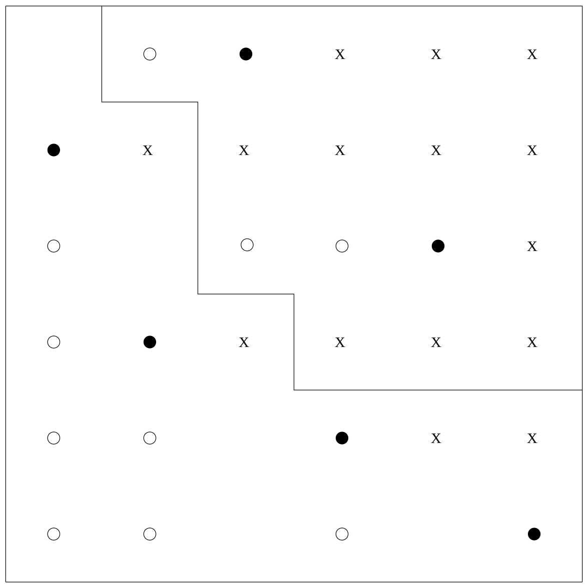


FIGURE 2. A placement of 6 rooks with 2 rooks on B. Here $\xi = 10$.

Haglund's statistic $\beta(C, B)$ is defined by the same procedure used to calculate ξ, except that for the rooks on B, instead of placing circles in the empty cells of B which are below and in the column, place circles in the empty cells of B which are above and in the column.

THEOREM 1.8. *For any Ferrers board B,*

$$T_k(B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\xi(C,B)} \tag{1.41}$$

$$= \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\beta(C,B)}. \tag{1.42}$$

EXERCISE 1.9. Prove that

$$\sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\xi(C,B)} = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\beta(C,B)}. \tag{1.43}$$

Thus if we know that $\xi(C, B)$ generates the $T_k(B)$, then so does $\beta(C, B)$.

Dworkin proves (1.41) by showing both sides satisfy the same (somewhat complicated) recurrences. Haglund's proof of the equivalent identity (1.42) uses a connection between rook placements and Gaussian elimination in matrices over finite fields, an idea occurring in work of Solomon [**Sol90**]. K. Ding has used this connection to answer topological questions involving algebraic varieties associated to matrices over the complex numbers in the shape of a Ferrers board [**Din97a**],[**Din97b**],[**Din01**].

The lemma below generalizes a result of Solomon to Ferrers boards. The proof is a straightforward extension of his.

DEFINITION 1.10. For B a Ferrers board with n columns (some of which may be empty), let $P_k(B)$ be the number of $n \times n$ matrices A with entries in $\mathbb{F}_q$, of rank k, and with the restriction that all the entries of A in those squares of A outside of B are zero. For example, if $B(0,1,2)$ is the triangular board B_3, then $P_0 = 1, P_1 = 2q^2 - q - 1, P_2 = q(q-1)^2$, and $P_3 = 0$.

LEMMA 1.11. *For any Ferrers board B,*

$$P_k(B) = (q-1)^k q^{Area(B)-k} R_k(B, q^{-1}), \tag{1.44}$$

where $Area(B)$ is the number of squares of B.

PROOF. Let A be a matrix of rank k, with entries in $\mathbb{F}_q$, and zero outside of B. We perform an operation on A which we call the *elimination procedure*. Starting at the bottom of column 1 of A, travel up until you arrive at a nonzero square β (if the whole first column is zero go to column 2 and iterate). Call this nonzero square a *pivot spot*. Next add multiples of the column containing β to the columns to the right of it to produce zeros in the row containing β to the right of β. Also add multiples of the row containing β to the rows above it to produce zeros in the column containing β above β. Now go to the bottom of the next column and iterate; find the lowest nonzero square, call it a pivot spot, then zero-out entries above and to the right as before.

If we place rooks on the square β and the other pivot spots we end up with k non-attacking rooks. The number of matrices which generate a specific rook placement C is

$$\begin{aligned}(q-1)^k q^{\#\text{ of squares to the right of or above a rook}} \\ = (q-1)^k q^{\text{Area}(B)-k-\text{inv}(C,B)}.\end{aligned} \tag{1.45}$$

□

COROLLARY 1.11.1. *Let P_k be the number of $n \times n$ upper triangular matrices of rank k with entries in $\mathbb{F}_q$. Then*

$$P_k = (q-1)^k q^{\binom{n+1}{2}-k} S_{n+1,n+1-k}(q^{-1}), \tag{1.46}$$

where $S_{n,k}(q)$ is the q-Stirling number of the second kind defined by the recurrences

$$S_{n+1,k}(q) := q^{k-1} S_{n,k-1}(q) + [k] S_{n,k}(q) \qquad (0 \le k \le n+1), \tag{1.47}$$

with the initial conditions $S_{0,0}(q) = 1$ and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.

PROOF. It is known [**GR86**, p.248] that

$$R_k(B_{n+1}) = S_{n+1,n+1-k}(q). \tag{1.48}$$

Now apply (1.44). □

Using (1.44) and (1.38) you can easily derive

COROLLARY 1.11.2. *For any Ferrers board B,*

$$\sum_{k=0}^{n}(1-x)(1-xq)\cdots(1-xq^{k-1})P_{n-k}(B)=\prod_{i=1}^{n}(q^{c_i}-xq^{i-1}). \tag{1.49}$$

REMARK 1.12. In [**Hag96**] the following identity was derived as a limiting case of a hypergeometric result.

$$\sum_{k} R_k(B)(1-q)^k = 1, \tag{1.50}$$

which can also be obtained by letting $x \to \infty$ in (1.38). Using (1.44), this is equivalent to the trivial statement

$$\sum_{k} P_k(B) = q^{\mathrm{Area}(B)}. \tag{1.51}$$

Algebraic Identities for q-Rook and q-Hit Numbers

For any Ferrers board $B(c_1,\ldots,c_n)$, let

$$PR[x,B]=\prod_{i=1}^{n}[x+c_i-i+1]. \tag{1.52}$$

THEOREM 1.13. *For any Ferrers board B,*

$$[k]!R_{n-k}(B)=\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}PR[j,B] \tag{1.53}$$

$$T_{n-k}(B)=\sum_{j=0}^{k}\begin{bmatrix}n+1\\k-j\end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}PR[j,B] \tag{1.54}$$

$$[k]!R_{n-k}(B)=\sum_{j=0}^{k}T_{n-j}(B)\begin{bmatrix}n-k\\k-j\end{bmatrix}q^{k(k-j)}. \tag{1.55}$$

PROOF. Applying (1.38) to the RHS of (1.53) we get

(1.56)
$$\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}\sum_{s=0}^{j}\begin{bmatrix}j\\s\end{bmatrix}[s]!R_{n-s}(B)=\sum_{s=0}^{k}[s]!R_{n-s}(B)\sum_{j=s}^{k}\begin{bmatrix}k\\j\end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}j\\s\end{bmatrix}$$

$$=\sum_{s=0}^{k}[s]!R_{n-s}(B)\frac{(z;q)_k}{(z;q)_{s+1}}|_{z^{k-s}} \tag{1.57}$$

$$=\sum_{s=0}^{k}[s]!R_{n-s}(B)\delta_{k,s}=[k]!R_{n-k}(B)$$

where we have used the q-binomial theorem to evaluate (1.57). Similarly, applying (1.38) to the RHS of (1.54),

$$
\begin{aligned}
(1.58)\qquad & \sum_{j=0}^{k} \begin{bmatrix} n+1 \\ k-j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \sum_{s=0}^{j} \begin{bmatrix} j \\ s \end{bmatrix} [s]! R_{n-s}(B) \\
&= \sum_{s=0}^{n} [s]! R_{n-s}(B) \sum_{j=s}^{k} \begin{bmatrix} n+1 \\ k-j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} j \\ s \end{bmatrix} \\
&= \sum_{s=0}^{n} [s]! R_{n-s}(B) \frac{(z;q)_{n+1}}{(z;q)_{s+1}} |_{z^{k-s}} \\
&= \sum_{s=0}^{n} [s]! R_{n-s}(B) (zq^{s+1};q)_{n-s} |_{z^{k-s}} \\
&= \sum_{s=0}^{n} [s]! R_{n-s}(B) \begin{bmatrix} n-s \\ k-s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}}
\end{aligned}
$$

by the q-binomial theorem. But this is exactly equal to the coefficient of x^{n-k} in the LHS of (1.39), again by the q-binomial theorem. □

THEOREM 1.14. *For any Ferrers board B,*

$$
(1.59)\qquad \sum_{j=0}^{\infty} PR[j,B] z^j = \frac{\sum_{k=0}^{n} z^k T_{n-k}(B)}{(z;q)_{n+1}}.
$$

PROOF.

$$
\begin{aligned}
(1.60)\qquad \left((z;q)_{n+1} \sum_{j=0}^{\infty} PR[j,B] z^j \right) |_{z^k} &= \sum_{j=0}^{k} \begin{bmatrix} n+1 \\ k-j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j,B] \\
&= T_{n-k}(B)
\end{aligned}
$$

by (1.54). □

THEOREM 1.15. *For any Ferrers board B,*

$$
(1.61)\qquad \sum_{k=0}^{n} \begin{bmatrix} x+k \\ n \end{bmatrix} T_k(B) = PR[x,B].
$$

PROOF. Again, it suffices to prove (1.61) for $x \in \mathbb{N}$. Then by (1.23), the RHS of (1.61) equals

$$\left(\sum_{j=0}^{\infty} PR[j,B] z^j\right)|_{z^x} = \left(\frac{\sum_{k=0}^{n} z^k T_{n-k}(B)}{(z;q)_{n+1}}\right)|_{z^x} \tag{1.62}$$
$$= \left(\sum_j T_{n-j}(B) z^j\right)\left(\sum_{m=0}^{\infty} z^m \begin{bmatrix} n+m \\ m \end{bmatrix}\right)|_{z^x}$$
$$= \sum_{j=0}^{x} T_{n-j}(B) \begin{bmatrix} n+x-j \\ x-j \end{bmatrix}$$
$$= \sum_{k=0}^{n} T_k(B) \begin{bmatrix} x+k \\ n \end{bmatrix}.$$

□

THEOREM 1.16. *(Lemma 4.3.5 from* [**Hag93**]*) For any Ferrers board B,*

$$[k]! R_{n-k}(B) = \sum_{j=0}^{k} T_{n-j}(B) \begin{bmatrix} n-k \\ k-j \end{bmatrix} q^{k(k-j)}. \tag{1.63}$$

PROOF. Using (**??**), the RHS of (1.63) equals

$$\sum_{j=0}^{k} \begin{bmatrix} n-k \\ k-j \end{bmatrix} q^{k(k-j)} \times \sum_{s=0}^{j} R_{n-s}(B)[s]! \begin{bmatrix} n-s \\ j-s \end{bmatrix} q^{\binom{j-s}{2}} (-1)^{j-s} q^{(j-s)(s+1)} \tag{1.64}$$

$$\times \sum_{s=0}^{k} R_{n-s}(B)[s]! \sum_{j=s}^{k} \begin{bmatrix} n-j \\ k-j \end{bmatrix} q^{k(k-j)} \begin{bmatrix} n-s \\ j-s \end{bmatrix} q^{\binom{j-s}{2}} (-1)^{j-s} q^{(j-s)(s+1)} \tag{1.65}$$

(1.66)

$$= R_{n-k}[k]! + \sum s = 0^{k-1} R_{n-s}(B)[s]! \times \frac{(1-xq^{s+1})(1-xq^{s+2})\cdots(1-xq^n)}{(1-xq^k)(1-xq^{k+1})\cdots(1-xq^n)}|_{\text{coeff of } x^{k-s}}$$

$$= R_{n-k}[k]!, \tag{1.67}$$

since the fraction above simplifies to $(1-xq^{s+1})\cdots(1-xq^{k-1})$, which has degree $k-s-1$ in x. □

The q-Simon Newcomb problem

MacMahon also studied a q-anologue of the Simon Newcomb Problem [**Mac60**, Vol. 2, p. 211]

DEFINITION 1.17. For any vector v, let

$$N_k[v] = \sum_{\substack{\pi \in M(v) \\ k-1 \text{ descents}}} q^{\mathrm{maj}(\pi)}, \tag{1.68}$$

where the sum is over all multiset permutations π of $M(v)$ with $k-1$ descents.

THEOREM 1.18.

$$\sum_{k=0}^{n} \begin{bmatrix} x+k \\ n \end{bmatrix} N_{k+1}[v] = \prod_i \begin{bmatrix} x \\ v_i \end{bmatrix}. \tag{1.69}$$

Theorem 1.18 and (1.61) together with (1.54) imply

COROLLARY 1.18.1.

$$T_k(G_v) = \prod_i [v_i]! N_{k+1}(v) \tag{1.70}$$

$$N_{n-k-1}(v) = \sum_{j=0}^{k} \begin{bmatrix} n+1 \\ k-j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_i \begin{bmatrix} j \\ v_i \end{bmatrix}. \tag{1.71}$$

REMARK 1.19. There is also a q-analogue of unitary compositions introduced in [**Hag93**], which features in a q-analogue of (1.14) and (1.19), but it is a bit complicated to describe.

Compositions and (P, ω)-partitions

Many of the identities involving unitary and vector compositions have an interpretation in terms of Stanley's (P, ω)-partitions. We include a brief discussion of this here, which is based on material in [**Sta12**, Section 4.5] and [**Bre89**]. For P a partially ordered set, we identify P with its Hasse diagram, and throughout this section we let n be the number of vertices of P. We let $a <_P b$ denote the statement that vertex a is less than vertex b in P. Let ω be a *labelling*, that is a bijective assignment of the numbers $1, 2, \ldots, n$ to the vertices of P. The labelling is called natural if $a <_P b \implies \omega(a) < \omega(b)$ for all $a, b \in P$. A (P, ω)-partition σ with largest part $\leq s$ is an assignment of the numbers $1, 2, \ldots, s$ to the elements of P where

$$a <_P b \implies \sigma(a) \geq \sigma(b) \tag{1.72}$$

$$a <_P b \text{ and } \omega(a) > \omega(b) \implies \sigma(a) > \sigma(b). \tag{1.73}$$

The (P, ω) partition is called surjective if all the numbers $1, 2, \ldots, s$ appear as $\sigma(i)$ for some $1 \leq i \leq n$. Let $\Omega(P, \omega; x)$ (the "order polynomial") denote the number of (P, ω) partitions with largest part $\leq x$, and let $e_s(P, \omega)$ denote the number of surjective (P, ω) partitions with largest part s. For example, if (P, ω) is the labelled poset in Figure 3 then

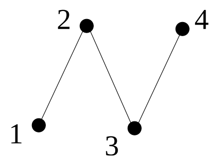


FIGURE 3. The labelled poset (P, ω).

we need

$$\begin{aligned} \sigma(1) &\geq \sigma(2) \\ \sigma(3) &> \sigma(2) \\ \sigma(3) &\geq \sigma(4). \end{aligned} \tag{1.74}$$

The set of surjective (P,ω) partitions $\sigma = (\sigma(1),\sigma(2),\sigma(3),\sigma(4))$ with largest part 3 for this poset is

$$\begin{aligned}\{\sigma\} = \{&(2,1,3,3),(2,1,3,2),(2,1,3,1),(3,1,3,2),\\ &(3,1,2,1),(3,1,2,2),(3,2,3,1)\}\end{aligned} \tag{1.75}$$

so $e_3(P,\omega) = 7$.

Theorem 1.20. *For any labelled poset (P,ω) and $x \in \mathbb{N}$,*

$$\sum_s \binom{x}{s} e_s(P,\omega) = \Omega(P,\omega;x). \tag{1.76}$$

Proof. Say we have a $P,\omega)$ partition σ with s different values from the set $\{1,\ldots,x\}$. Then without changing any of the relative inequalities between elements, we can replace these s different values by the numbers $\{1,\ldots,s\}$ in an order-preserving way. Eq. (1.76) is now transparent. □

Corollary 1.20.1. *$\Omega(P,\omega;x)$ is a polynomial in x.*

Proposition 1.20.1. *If (P,ω) is a naturally labelled disjoint union of chains of lengths $v_1,v_2,\ldots,v_p$ then $e_k(P,\omega) = f_k(v)$. If each chain has decreasing labels going up its portion of the Hasse diagram ("unnaturally labelled" so to speak) then $e_k(P,\omega) = g_k(v)$.*

Proof. Asumme without loss of generality that the first chain has labels $1,2,\ldots,v_1$ going up the Hasse diagram, the second chain labels $v_1+1,\ldots,v_1+v_2$, etc. Given a surjective (P,ω)-partition σ, the constraints on σ are

$$\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(v_1) \tag{1.77}$$
$$\sigma(v_1+1) \geq \sigma(v_1+2) \geq \cdots \geq \sigma(v_1+v_2) \tag{1.78}$$
$$\vdots \tag{1.79}$$
$$\sigma(n-v_1+1) \geq \sigma(n-v_1+2) \geq \cdots \geq \sigma(n). \tag{1.80}$$

Given such a σ let w_i denote the vector in $\mathbb{N}^p$ whose jth coordinate is the number of times σ takes on the value j in the ith chain. Then since σ is surjective, $|w_i| > 0$ and moreover

$$w_1 + \ldots + w_k = v, \tag{1.81}$$

so $e_k(P,\omega) = f_k(v)$. If (P,ω) is "unnaturally labelled" then there are strict inequalities in (1.77), which means $w_{i,j} \leq 1$ and the sum in (1.81) involves unitary compositions. □

The Jordan-Hölder set $\mathcal{L}(P,\omega)$ of (P,ω) is the set of all permutations π in S_n which satisfy

$$a <_P b \implies \omega(a) \text{ occurs before } \omega(b) \text{ in the sequence } \pi_1\pi_2\cdots\pi_n. \tag{1.82}$$

The set of (P,ω)-partitions decomposes fundamentally into classes corresponding to the elements of $\mathcal{L}(P,\omega)$. For example, if (P,ω) is the poset from Figure 3, then

$$\mathcal{L}(P,\omega) = \{1324, 3124, 1342, 3142, 3412\}, \tag{1.83}$$

and every (P,ω)-partition σ satisfies exactly one of the corresponding conditions

$$
\begin{aligned}
&\sigma(1) \geq \sigma(3) > \sigma(2) \geq \sigma(4)\\
&\sigma(3) > \sigma(1) \geq \sigma(2) \geq \sigma(4)\\
&\sigma(1) \geq \sigma(3) \geq \sigma(4) > \sigma(2)\\
&\sigma(3) > \sigma(1) \geq \sigma(4) > \sigma(2)\\
&\sigma(3) \geq \sigma(4) > \sigma(1) \geq \sigma(2).
\end{aligned} \tag{1.84}
$$

(Here the strict inequalities correspond to descents in the associated permutation π.) It is easy to see that each (P,ω) partition falls into one of the classes corresponding to an element of $\mathcal{L}(P,\omega)$. It remains to show they are mutally exclusive. If π,β are two elements of $\mathcal{L}(P,\omega)$, then there are two elements $c<d$ with c occurring before d in π and d occurring before c in β (or vice-versa). In β, there must be a descent between d and c. Thus, (P,ω) partitions corresponding to β will satisfy $\sigma(d)>\sigma(c)$, while paritions falling into the π-class will satisfy $\sigma(c)\geq\sigma(d)$. □

The decomposition described above implies

THEOREM 1.21.

$$
\Omega(P,\omega;x) = \sum_{i=1}^{n} \binom{x+n-i}{n} \sum_{\substack{\pi in \mathcal{L}(P,\omega)\\ des(\pi)=i-1}} 1, \tag{1.85}
$$

where the inner sum is over all permutations in the Jordan-Hölder set with $i-1$ descents.

PROOF. Consider the number of σ which satisfy a given set of inequalities corresponding to a given $\pi\in\mathcal{L}(P,\omega)$ (as in one of the sets from (1.84)). If there are $i-1$ descents in π, then by subtracting $i-1$ from the elements of σ to the left of the first descent, $i-2$ from the elements between the first and second descents, etc., we get a sequence σ' with

$$
\begin{aligned}
\sigma'(\pi_j) &\geq \sigma'(\pi_{j+1}), && 1\leq j\leq n-1,\\
x-(i-1) \geq \sigma'(\pi_j) &\geq 1, && 1\leq j\leq n.
\end{aligned} \tag{1.86}
$$

The number of solutions to (1.86) is the number of partitions fitting inside a n by $x-i$ rectangle, which is $\binom{x-i+n}{n}$. □

Theorem 1.21 has a natural q-analog. Define the q-order polynomial $\Omega[P,\omega;x]$ via

$$
\Omega[P,\omega;x] = \sum_{\substack{\sigma\\ \sigma(j)\leq x}} q^{\sigma(1)+\ldots+\sigma(n)-n}, \tag{1.87}
$$

where the sum is over all (P,ω)-partitions with largest part $\leq x$. If we consider the portion of (1.87) corresponding to a given π as in the proof of Theorem 1.21, the difference between the q-weight of σ and that of σ' is clearly $\mathrm{maj}(\pi)$. Then summing q-weights over all σ' satisfying (1.86) we get a factor of $\begin{bmatrix} x-i+n\\ n\end{bmatrix}$. The values of σ' are between 1 and $x-i+1$, but in (1.87) we also subtract n from the sum of the σ', which places them between 0 and $x-i$, and we get the standard sum of partitions in a rectangle weighted by q to the area. Hence we have

Theorem 1.22.

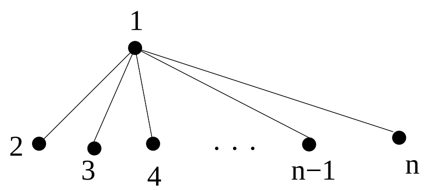
$$\Omega[P,\omega;x] = \sum_{i=1}^{n} \begin{bmatrix} x-i+n \\ n \end{bmatrix} \sum_{\substack{\pi\in\mathcal{L}(P,\omega) \\ des(\pi)=i-1}} q^{maj(\pi)}. \tag{1.88}$$

By similar reasoning one can prove [?]

Theorem 1.23.

$$\sum_{k=0}^{\infty} \Omega[P,\omega;k]x^k = \frac{\sum_{\pi\in\mathcal{L}(P,\omega)} q^{maj(\pi)} x^{des(\pi)+1}}{(1-x)(1-xq)\cdots(1-xq^n)}. \tag{1.89}$$

Exercise 1.24. Let (P,ω) be the labelled poset in Figure 1. For this labelled poset, compute the numerator in the right-hand-side of Theorem 1.23, and also express $\lim_{x\to\infty}\Omega[P,\omega;x]$ as a rational function in q.



CHAPTER 2

Zeros of Rook Polynomials

Rook Polynomials and the Heilmann-Lieb Theorem

Let $f(z) = \sum_{k=0}^{n} b_k z^k$ be a polynomial of degree n. If all the zeros of f happen to be real, there are a number of interesting relations that the coefficients must satisfy. For example, Newton stated (see [**HLP52**, p. 52]) that all real zeros implies

$$b_k^2 > b_{k-1}b_{k+1}(1+1/k)(1+1/(n-k)), \qquad 1 \le k < n. \tag{2.1}$$

If in addition all the b_k are nonnegative, (2.1) implies that the b_k are *unimodal*, i.e. there is a value of k for which

$$b_0 \le b_1 \le \cdots \le b_k \ge b_{k+1} \ge \cdots \ge b_n, \tag{2.2}$$

and also implies that the b_k are *log-concave*, i.e. that $b_k^2 \ge b_{k+1}b_{k-1}$ for $2 \le k \le n-1$.

Definition 2.1. A sequence of real numbers $\{b_k\}_{k=0,1,2,\dots}$ is called a Polya frequency sequence of order r, or a PF_r sequence, if for all $1 \le m \le r$, the determinants of all the minors of order r of the infinite matrix $(b_{j-i})_{i,j=0,1,2,\dots}$ are nonnegative. Here we let $b_k = 0$ for $k < 0$ or $k > n$, so if $n = 3$ we have the matrix

$$\begin{bmatrix} b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 & \cdots & \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 & \cdots & \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 & \cdots & \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.3}$$

In fact, the polynomial $\sum_{k=0}^{n} b_k z^k$, $b_k \ge 0$ for all k, has only real zeros iff all the determinants of all the minors of (2.3) are nonnegative. A detailed study of Polya frequency sequences and their connections to polynomials arising in combinatorial theory was undertaken by Brenti [**Bre89**]. One of Brenti's results from this paper, which we will use later, is the following.

Theorem 2.2. *Let $f(z) = \sum_{k=0}^{n} \binom{z+k}{n} b_k$ be a polynomial with all real zeros, with smallest root $\lambda(f)$ and largest root $\Lambda(f)$. If all the integers in the intervals $[\lambda, -1]$ and $[0, \Lambda]$ are also roots of f, then all the roots of $\sum_{k=0}^{n} b_k z^k$ are real.*

If f and g are polynomials with only real zeros of degrees n and $n-1$, and the roots $f_1 \le f_2 \le \cdots \le f_n$ of f and $g_1 \le g_2 \le \cdots \le g_{n-1}$ of g satisfy

$$f_1 \le g_1 \le f_2 \le g_2 \le \cdots \le f_{n-1} \le g_{n-1} \le f_n \tag{2.4}$$

we say that g *interlaces* f. If g is also of degree n, then we say that g interlaces f if (2.4) holds and in addition $f_n \le g_n$. We say g and f interlace if either g interlaces

f or f interlaces g, and that f and g strictly interlace if all the weak inequalities in (2.4) are strict. One of the most useful techniques for proving a sequence of polynomials has only real zeros is to prove by induction that the roots interlace.

EXAMPLE 2.3. Let

$$A_n(z) = \sum_{\pi \in S_n} z^{\mathrm{des}(\pi)+1} = \sum_k z^k A_{n,k} \tag{2.5}$$

be the Eulerian polynomial. In this example we show that $A_n(z)$ has only distinct, real zeros, which are interlaced by those of $A_{n-1}(z)$.

PROOF. First, by starting with a permutation $\beta \in S_{n-1}$ and considering what happens to descents when we insert n into β at the various n possible places we get the recurrence

$$A_{n,k} = kA_{n-1,k} + (n-k+1)A_{n-1,k-1}.$$

Thus

$$\begin{aligned} \sum_k z^k A_{n,k} &= \sum_k z^k (kA_{n-1,k} + (n+1-k)A_{n-1,k-1}) \\ &= z\frac{d}{dz}A_{n-1}(z) + nzA_{n-1}(z) - z^2\frac{d}{dz}A_{n-1}(z) \\ &= nzA_{n-1}(z) + (z-z^2)\frac{d}{dz}A_{n-1}(z). \end{aligned} \tag{2.6}$$

Now assume by induction that all the zeros of $A_{n-1}(z)$ are real and distinct, which are neccessarily nonpositive since $A_{n,k} > 0$ for $1 \le k \le n$. Let $\alpha < \beta$ be two consecutive, negative zeros of $A_{n-1}(z)$ Then isince $\frac{d}{dz}A_{n-1}(z)$ switches sign when going from α to β, we see by (2.6) that $A_n(z)$ also switches sign, and so has a zero between α and β. In addition we have $A_n(0) = 0$, and also that $A_n(z)$ has degree n, while $A_{n-1}(z)$ has degree $n-1$. Hence $A_n(z)$ has another zero ζ smaller than all the zeros of $A_{n-1}(z)$, which completes the induction. □

Let E_n denote the upper triangular array of real numbers $E = \{e_{ij},\ 1 \le i < j \le n\}$. One of the most important examples of a family of polynomials with only real zeros is given by the Heilmann-Lieb theorem [**HL72**]. Their result deals with a complete, weighted graph on n vertices, where there is a weight $e_{i,j}$ associated to the edge between vertices i and j. We weight a matching in this graph by the product, over the edges in the matching, of the edge weights, and define $m_k(E_n)$ to be the sum of these weights, over all k-edge matchings in K_n. For example, for $n = 4$ the weighted matching numbers are

$$m_2(E_4) = e_{12}e_{34} + e_{13}e_{24} + e_{14}e_{23} \tag{2.7}$$

$$m_1(E_4) = e_{12} + e_{13} + e_{14} + e_{23} + e_{24} + e_{34}, \tag{2.8}$$

and $m_0(E_4) = 1$. The weighted matching polynomial $W(E_n, z)$ is defined as $\sum_{0 \le k \le n/2} m_k(E_n)z^k$.

THEOREM 2.4. *Let E_n be an upper triangular array of nonnegative edge weights. Then $W(E_n; z)$ has only real zeros.*

PROOF. (Sketch) Let $E_n - i$ stand for the array obtained by starting with E_n and setting $e_{ij} = 0$ for $j \ne i$. We prove by induction on n that the roots of $W(E_n, z)$ are real and furthermore that $W(E_n - i, z)$ and $W(E_n, z)$ interlace for all $1 \le i \le n$.

By considering matchings which either use vertex i or not, we get the recurrence

$$m_k(E_n) = m_k(E_n - i) + \sum_{j \neq i} e_{ij} m_{k-1}(E_n - i - j). \tag{2.9}$$

Hence

$$W(E_n, z) = 1 + \sum_{k \geq 1} z^k \left(m_k(E_n - i) + \sum_{j \neq i} e_{ij} m_{k-1}(E_n - i - j) \right) \tag{2.10}$$

$$= W(E_n - i, z) + z \sum_{j \neq i} W(E_n - i - j, z). \tag{2.11}$$

We can now use the method of interlacing roots as in Example 2.3. □

Note that as a corollary of the Heilmann-Lieb theorem, for any graph G on n vertices, the matching polynomial $\sum_k m_k(G) z^k$ has only real zeros, since graphs correspond to the special case $e_{ij} \in \{0, 1\}$.

Inspired by a conjecture of Goldman, Joihi and White [**GJW75**], Nijenhuis [**Nij76**] proved that for any board B the rook polynomial $\sum_{k=0}^n r_k(B) z^k$ has only real zeros. Recall that for any $n \times n$ matrix A, the permanent of A, denoted $\mathrm{per}(A)$, is defined as

$$\mathrm{per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^n a_{i, \pi_i}. \tag{2.12}$$

In fact, Nijenhuis proved that if we start with any $n \times n$ matrix A of nonnegative real numbers, than the polynomial $\sum_{k=0}^n r_k(A) z^k$ has only real zeros, where $r_k(A)$ is the sum of the permanents of all $k \times k$ minors of A, which can also be viewed as the sum, over all placements of k nonattacking rooks on the squares of the $n \times n$ grid, of the product of the a_{ij} corresponding to the squares containing rooks. For example, if $n = 3$ we have

$$\begin{aligned} r_3(A) &= a_{11}(a_{22}a_{33} + a_{23}a_{32}) \\ &\quad + a_{12}(a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{22}a_{31} + a_{21}a_{32}) \\ r_2(A) &= a_{11}(a_{22} + a_{23} + a_{32} + a_{33}) + a_{12}(a_{21} + a_{23} + a_{31} + a_{33}) \\ &\quad + a_{13}(a_{21} + a_{22} + a_{31} + a_{32}) \\ r_1(A) &= a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} + a_{31} + a_{32} + a_{33} \\ r_0(A) &= 1. \end{aligned} \tag{2.13}$$

Note that the rook polynomial of a $0,1$-matrix equals the rook polynomial of the board whose squares are the entries which equal 1. As noted shortly after this by Ed Bender, Nijenhuis' result follows from the Heilmann-Lieb theorem. To see how, given a rook placement C, say with rooks on squares $(i_1, j_1), \ldots, (i_k, j_k)$, we can construct a corresponding matching $\alpha(C)$ in the complete bipartitie graph $K_{n,n}$ by letting $\alpha(C)$ consist of edges from vertices i_m above to j_m below, for $1 \leq m \leq k$. No two rooks in the same row or column of C tranlates into no two edges in $\alpha(C)$ incident to a common vertex, i.e. $\alpha(C)$ is a matching. The weights a_{i_m, j_m} on the rooks in C become the edge weights in $\alpha(C)$.

If a sequence $f_0, f_1, f_2, \ldots$ has the property that, for any polynomial $\sum_{k=0}^n b_k z^k$ with only real zeros, the polynomial $\sum_{k=0}^n b_k f_k z^k$ also has only real zeros, then $f_0, f_1, f_2, \ldots$ is called a *factor sequence*.

THEOREM 2.5. *(Laguerre) The sequence $1/k!, k = 0, 1, 2, \ldots$ is a factor sequence.*

We now consider the question of when the hit polynomial $\sum_{k=0}^{n} t_k(B)z^k$ of a board B with n columns has only real zeros. By (1.1) and the fact that the transformation $z \to z+1$ sends the real line to itself, the hit polynomial has only real zeros iff $\sum_{k=0}^{n} r_k(B)(n-k)!z^k$ does. By Theorem 2.5, the hit polynomial having only real zeros is a stronger condition than the rook polynomial having only real zeros.

THEOREM 2.6. *(Haglund, Ono, Wagner* [**HOW99**]*). If B is a Ferrers board, the hit polynomial has only real zeros.*

REMARK 2.7. Not all boards have hit polynomial with only real zeros. For example, if B is the derangement board for $n = 2$ then the hit polynomial is $1 + z^2$.

EXERCISE 2.8. Show that Theorem 2.6 follows from Theorem 2.2.

By combining Theorem 2.6 and (1.5) we get a result of Simion [**Sim84**].

COROLLARY 2.8.1. *For any vector $v \in \mathbb{N}^p$, the polynomial $\sum_k z^k N_{k+1}(v)$ has only real zeros.*

Since the rook polynomial of an arbitrary $n \times n$ matrix of real numbers has only real zeros, one may suspect that Theorem 2.6 has a similar generalization. The following result, first conjectured in [**HOW99**] and proved in [**BHVW11**], gives an elegant answer to this question. We use the fact that $\sum_{k=0}^{n} z^{n-k} r_k(A)(n-k)! = \text{per}(A + zJ)$, where J is the $n \times n$ matrix of all ones.

THEOREM 2.9. *(The Monotone Column Permanent (MCP) Theorem). Let A be an $n \times n$ matrix, weakly increasing down columns, i.e. $a_{ij} \le a_{i+1,j}$ for $1 \le i < n, 1 \le j \le n$. Then as a polynomial in z, per$(A + zJ)$ has only real zeros.*

Grace's Apolarity Theorem

Let $f(z) = \sum_{k=0}^{n} b_k z^k$ and $g(z) = \sum_{k=0}^{n} d_k z^k$ be two polynomials of degree n. We say that f and g are *apolar* if

$$\sum_{k=0}^{n} (-1)^k k!(n-k)! b_k d_{n-k} = 0. \tag{2.14}$$

A *circular domain* in the complex plane is the closed interior or closed exterior of a disk, or a closed half plane. One of the classic results on the zeros of polynomials is the following theorem of Grace [**Gra02**].

THEOREM 2.10. *Let f and g be two polynomial of degree n, and assume they are apolar. Then any circular domain which contains all the zeros of f contains at least one of the zeros of g.*

The book [**PS98**, Part Five, Chap. 2] contains a lot of useful results involving Grace's theorem.

EXERCISE 2.11. Show that Grace's apolarity theorem can be expressed in the following way: Let $w_1, \ldots, w_n, z_1, \ldots, z_n$ be $2n$ complex numbers. Assume that $\text{per}(w_i - z_j) = 0$. Then if C is the closed interior of a disk, or a closed half-plane, containing all of the w_i, then C contains at least one of the z_j.

Szegö [**Sze22**] gave a new proof of Grace's theorem, and also derived the following interesting Corollaries.

COROLLARY 2.11.1. *(Szego's composition theorem). Let*

$$f(z) = \sum_{k=0}^{n} b_k z^k \tag{2.15}$$

be a polynomial of degree n*, all of whose zeros lie in a circular domain* C*. Let*

$$g(z) = \sum_{k=0}^{n} d_k z^k \tag{2.16}$$

have zeros $\beta_1, \beta_2, \ldots, \beta_n$*. Then all the zeros of*

$$h(z) = \sum_{k=0}^{n} k!(n-k)! b_k d_k z^k \tag{2.17}$$

are of the form $\gamma = -\beta_j \kappa$ *for some* $\kappa \in C$*.*

PROOF. Let γ be a zero of h. Then replacing z by γ in (2.17) we see that f and $z^n g(-\gamma/z)$ are apolar. Hence by Grace's theorem C contains one of the zeros of $z^n g(-\gamma/z)$, i.e. one of the numbers $-\gamma/\beta_1, \ldots, -\gamma/\beta_n$. In other words $-\gamma = \kappa\beta_j$ for some $\kappa \in C$ and some $1 \leq j \leq n$. □

COROLLARY 2.11.2. *Let* a, b *be nonnegative real numbers. Assume* $f(z) = \sum_{k=0}^{n} b_k z^k$ *is a polynomial with only real zeros, all in the interval* $[-a, a]$*, and let* $g(z) = \sum_{k=0}^{n} d_k z^k$ *be a polynomial with only real zeros, either all in the interval* $[-b, 0]$*, or all in the interval* $[0, b]$*. Then all the zeros of the polynomial*

$$h(z) = \sum_{k=0}^{n} k!(n-k)! b_k d_k z^k \tag{2.18}$$

are real and lie in the interval $[-ab, ab]$*.*

PROOF. Let C be the upper-half-plane $\Im(z) \geq 0$. Then by Theorem 2.11.1, all the zeros of $h(z)$ are of the form $-\beta_j\kappa$, where $\Im(\kappa) \geq 0$. Hence if all the zeros of g are in $[-b, 0]$ (resp. $[0, b]$) then all the zeros of h have nonnegative (resp. nonpositive) imaginary part. Letting C be the lower-half-plane we can similarly conclude that all the zeros of h have nonpositive (resp. nonnegative) imaginary part, and hence are real. Now letting C be the closed circle of radius a centered at the origin, we see all the zeros of h must be in $[-ab, ab]$. □

EXERCISE 2.12. Let $p_1, \ldots, p_n$ be arbitrary real numbers, and $q_1, \ldots, q_n$ nonnegative real numbers. Show that the special case $a_{ij} = p_i q_j$ of the MCP theorem is equivalent to Corollary 2.11.2.

Here is a conjecture which contains Grace's apolarity theorem and the MCP theorem as special cases. Can you find a counterexample, or better yet, prove the conjecture? (I dont know how to prove it even in the case $m = 2$.)

CONJECTURE 2.13. *Assume* $n \geq m$*. Let* $w_1, \ldots, w_n$ *and* $z_1, \ldots, z_m$ *be complex numbers. Let* A *be an* $n \times m$ *matrix of nonnegative reals, weakly increasing down columns. Assume* $per(w_i a_{ij} - z_j) = 0$*, where the permanent of an* $n \times m$ *matrix is defined as the sum of the permanents of all* $m \times m$ *minors. Then if* C *is the closed* *interior of a disk, or a closed half-plane, which contains all of the nm numbers $\{w_i a_{ij}\}$, then C contains at least one of the z_i.*

Another beautiful result closely related to Grace's apolarity theorem is the Grace-Szegö-Walsh Coincidence theorem:

THEOREM 2.14. *Let $F(z_1, z_2, \ldots, z_n)$ be a multivariate polynomial with complex coefficients, which is linear in the z_i. Further assume that F is symmetric in the z_i (i.e. invariant under any permutation fo the variables), and that $F(\zeta_1, \ldots, \zeta_n) = 0$, $\zeta_i \in \mathbb{C}$ for $1 \leq i \leq n$. Then any closed disk or closed half-plane which contains all of the ζ_i contains at least one of the zeros of $F(z, z, \ldots, z)$.*

CHAPTER 3

α-Rook Polynomials

The α Parameter

In this model, first introduced in [**GH00**], rook placements can have at most one rook in any column but more than one rook in a given row. For a Ferrers board B, a row containing u rooks will have weight

$$\begin{cases} 1 & \text{if } 0 \leq u \leq 1 \\ \alpha(2\alpha-1)(3\alpha-2)\cdots((u-1)\alpha-(u-2)) & \text{if } u \geq 2. \end{cases}$$

The weight $\mathrm{wt}(C)$ of a placement C on B is just the product of the weights of each row of B. We can then define with *kth α-rook number* as

$$r_k^{(\alpha)}(B) = \sum_{\substack{C \\ k \text{ rooks on } B}} \mathrm{wt}(C). \tag{3.1}$$

For B a subset of the $n \times n$ grid, we can also define the *kth α-hit number* via

$$h_k^{(\alpha)}(B) = \sum_{\substack{C\ n \text{ rooks on } n \times n \text{ grid} \\ k \text{ rooks on } B}} \mathrm{wt}(C). \tag{3.2}$$

Note that when $\alpha = 0$, the alpha rook and hit numbers reduce to the ordinary rook and hit numbers, respectively. Recall we use the notation $B(c_1, \ldots, c_n)$ to denote the Ferrers board with column heights $b_1 \leq \cdots \leq b_n$. We also introduce the notation $x^{(a,b)} = x(x+b)(x+2b)\cdots(x+(a-1)b)$ for $a \in \mathbb{N}$ and $b \in \mathbb{C}$. The first important theorem we prove for this model is a version of the factorization theorem (1.7) for Ferrers boards, like we he seen for every other rook theory model. First, we need a simple lemma.

Lemma 3.1. *Suppose $B = B(c_1, \ldots, c_n)$ is a Ferrers board, and let C' be a fixed placement of k rooks in columns 1 through $i-1$ of B, for $i < n$. Then*

$$\sum_{\substack{C \supset C' \\ k+1 \text{ rooks on } B}} wt(C) = (k(\alpha-1)+c_i)wt(C'), \tag{3.3}$$

where the sum is taken over all placements C which extend C' by placing an additional rook in column i.

Proof. Suppose for the placement C' that there are l_1 rooks in row $j_1, \ldots, l_m$ rooks in row j_m (where each $l_p > 0$). Then $l_1 + \cdots + l_m = k$, and there are $c_i - m$ rows in column i of B with no rooks. Extending the placement C' to a placement C by placing a rook in column i in one of these $c_i - m$ unoccupied rows will add a factor 1 to the weight C', while placing a rook in occupied row j_p containing l_p rooks will add a factor of $l_p\alpha - (l_p - 1)$ to the weight of C'. Thus

$$\sum_{\substack{C \supset C' \\ k+1 \text{ rooks on } B}} \mathrm{wt}(C) \tag{3.4}$$

$$= \{(c_i - m) + (l_1\alpha - (l_1 - 1)) + \cdots + (l_m\alpha - (l_m - 1))\}\mathrm{wt}(C') \tag{3.5}$$

$$= \{c_i - m + (l_1 + \cdots + l_m)\alpha - (l_1 + \cdots + l_m) + m\}\mathrm{wt}(C') \tag{3.6}$$

$$= \{k(\alpha - 1) + c_i\}\mathrm{wt}(C'). \tag{3.7}$$

□

Theorem 3.2. *For the Ferrers board $B = B(c_1, \ldots, c_n)$,*

$$\sum_{k=0}^{n} r_k^{(\alpha)}(B) x^{(n-k,\alpha-1)} = \prod_{j=1}^{n}(x + c_j + (j-1)(\alpha - 1)). \tag{3.8}$$

Proof. We mimic the proofs of all of the other versions of the factorization theorem. As before is suffices to prove the identity for the case when $x \in \mathbb{N}$. Let B_x denote the board obtained from B by affixing an $x \times n$ rectangle below B, as in the proof of 1.7. We count the weighted sum

$$\sum_{\substack{C \\ n \text{ rooks on } B_x}} \mathrm{wt}(C) \tag{3.9}$$

in two different ways, as follows.

First we count the weighted sum obtained by placing a rook in the first column of B_x, then the second column of B_x, etc. By Lemma 3.1, placing a rook in all possible rows of column 1 contributes a factor of $x + c_1$ to (3.9), placing a rook in all possible rows of column 2 contributes a factor of $x + c_2 - 1 + \alpha$, ..., placing a rook in all possible rows of column n contributes a factor of $x + c_n - n + 1 + (n-1)\alpha$. Multiplying the contribution from from each column yields the RHS of (3.8).

The second way to count is to first place k rooks on the B part of B_x, for a fixed value of k between 0 and n. This contributes $r_k^{(\alpha)}(B)$ to (3.9). Each of these placements uses k of the columns of the $x \times n$ rectangle of B_x below the B part of the board. Placing the remaining rooks successively in these $n - k$ columns of the $x \times n$ rectangle contributes a factor $x(x + \alpha - 1)\cdots(x + (n - k - 1)(\alpha - 1)$ by arguments like in Lemma 3.1. Summing over all k gives the LHS of (3.8).

□

When $B = B(c_1, \ldots, c_n)$ is a Ferrers board, the α-rook numbers satisfy the recurrence

$$r_k^{(\alpha)}(B) = r_k^{(\alpha)}(B') + (c_n + (k-1)(\alpha - 1))r_{k-1}^{(\alpha)}(B'), \tag{3.10}$$

where $B' = B(c_1, \ldots, c_{n-1})$ is the Ferrers board obtained from B by removing the nth column. The proof of this recurrence uses ideas similar to those used in the proof of Theorem 3.2, and is left as an exercise.

Exercise 3.3. Give a combinatorial proof of the recurrence in (3.10).

The α-rook and α-hit numbers are also related by the following generalization of (1.1).

THEOREM 3.4. *For any board B,*

$$\sum_{k=0}^{n} h_k^{(\alpha)}(B)x^k \tag{3.11}$$

$$= \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B)((n-k)\alpha+k)((n-k+1)\alpha+k-1)\cdots((n-1)\alpha+1)(x-1)^{n-k}.$$

PROOF. After replacing x by $x+1$, the coefficient of x^k on the LHS of (3.11) is

$$\sum_{j=k}^{n} \binom{j}{k} h_j^{(\alpha)}(B), \tag{3.12}$$

and on the RHS it is

$$r_{n-k}^{(\alpha)}(B)((n-k)\alpha+k)((n-k+1)\alpha+k-1)\cdots((n-1)\alpha+1). \tag{3.13}$$

Both of these represent different ways of organizing the terms in the weighted count

$$\sum_{(P,\pi)} \mathrm{wt}(P), \tag{3.14}$$

where the sum is taken over all pairs (P,π) with P is a placement of n rooks on the $n\times n$ grid, and π is a subset of P of k rooks which all lie on the board B. $\square$

Special Values of α

In this section, we will see that the α-rook numbers specialize to well known combinatorial sequences for certain boards and values of α. We have already noted that when $\alpha=0$, the $r_k^{(\alpha)}(B)$ are just the ordinary rook numbers of B. When α is a negative integer, $r_k^{(\alpha)}(B)$ equals the number of rook placements where each rook deletes $1-\alpha$ rows to the right of the rook as in the theory of Remmel and Wachs REFERENCE NEEDED.

For a Ferrers board $B=B(c_1,\ldots,c_n)$ when α is a positive integer i, $r_k^{(\alpha)}(B)$ is equal to the *kth i-creation rook number* $r_k^{(i)}(B)$ discussed extensively in [**GH00**]. In this theory, each rook placed from left to right in turn creates i new rows to the right and immediately above where the rook was placed on B. A brief sketch of the proof that the $r_k^{(\alpha)}(B)$ equal the i-creation rook numbers when $\alpha=i$ follows. First note that the $r_k^{(i)}(B)$ are shown in [**GH00**] to satisfy the factorization theorem

$$\sum_{k=0}^{n} r_k^{(i)}(B)x^{(n-k,i-1)} = \prod_{j=1}^{n}(x+c_j+(j-1)(i-1)). \tag{3.15}$$

The proof of (3.15) uses the similar arguments to those in the proofs of other versions of the factorization theorem. When $\alpha=i$, the RHS of (3.15) and (3.8) are equal. Since the $n+1$ polynomials

$$1, x, x(x+i-1),\ldots, x(x+i-1)\cdots(x+(n-1)(i-1)) \tag{3.16}$$

form a basis for the vector space of degree n polynomials with coefficients in $\mathbb{R}$, the degree n polynomial $\prod_{j=1}^{n}(x+c_j+(j-1)(i-1))$ has a unique expansion in terms of this basis. Another way to see that the α and i-creation rook number are equal in this case is to use induction, and the fact that the $r_k^{(i)}(B)$ satisfy a version of the recurrence (3.10).

We will now examine $r_k^{(\alpha)}(B)$ for some specific Ferrers boards and positive integer values of α. For appropriate B and α, we obtain some familiar combinatorial sequences.

Absolute Stirling Numbers of the First Kind. For the board $B_n = B(0,1,\dots,n-1)$ and $\alpha = 1$, we can show that $r_k^{(1)}(B_n) = c(n,k)$. Here $c(n,k)$ denotes the *absolute Stirling number of the first kind* which counts the number of permutations of $\{1,2,\dots,n\}$ with k cycles. In this case, the polynomial

$$\prod_{j=1}^{n}(x+c_j+(j-1)(\alpha-1)) \tag{3.17}$$

reduces to $x(x+1)\cdots(x+n-1)$. By the factorization theorem, we know that

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} r_k^{(1)}(B_n)x^{(n-k,0)} = \sum_{k=0}^{n} r_k^{(1)}(B_n)x^{n-k}. \tag{3.18}$$

However, it is also well known that

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} c(n,n-k)x^{n-k}, \tag{3.19}$$

so by the uniqueness of the expansion in the basis $1, x, \dots, x^n$ we get that $r_k^{(1)}(B_n) = c(n,n-k)$. We now give a bijective proof of this fact, which is a slight modification of a proof given in [**GH00**].

THEOREM 3.5.

$$r_k^{(1)}(B_n) = c(n,n-k). \tag{3.20}$$

PROOF. The number $r_k^{(1)}(B_n)$ counts the number of placements of k rooks on B_n with any number of rooks in each row (that is, each placement receives a weight of 1). Let I_n denote the identity permutation on $\{1,2,\dots,n\}$. That is, in cycle notation $I_n = (1)(2)\cdots(n)$. Suppose C is a placement of k rooks on B_n, with rooks on the squares $(i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)$, where $i_1 < i_2 < \cdots < i_k$. Note that since each of these squares are on B_n, we'll always have $j_r < i_r$ for each r. Under the bijection we map the placement C to the permutation

$$\pi_C = I_n(i_1j_1)(i_2j_2)\cdots(i_kj_k). \tag{3.21}$$

With the multiplication of each subsequent two-cycle, we are merging a one-cycle with another cycle. Thus since there are a total of k two-cycles in (3.21), the resulting permutation will consist of $n-k$ cycles.

The inverse map for the bijection is clear. Suppose $\sigma \in S_n$ is a permutation with k cycles. We associate to σ a placement C_σ of k rooks on B_n as follows. If n is in a one-cycle of σ, erase the cycle, obtaining a permutation in S_{n-1}. If n is immediately followed by j (in cyclic order) in σ, then erase n from this cycle, obtaining a permutation in S_{n-1}. For the placement C_σ, place a rook on square

(n, j). In either of these cases, we now repeat this procedure on the permutation from S_{n-1}. $\square$

Finally, we see that in this case (3.10) reduces to the well-known recurrence

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k). \tag{3.22}$$

Matching Numbers of the Complete Graph. For a graph G, recall that a *matching* for G is a set of edges of G, none of which shares a common vertex. The number of k-edge matchings for G will be denoted $m_k(G)$. Also recall that a graph on n vertices containing every possible edge is called a *complete graph*, denoted K_n.

We can now give the following combinatorial proof relating α-rook numbers when $\alpha = 2$ to the matching numbers of the complete graph. Note that in the proof we will make use of the fact that $r_k^{(2)}(B_n)$ is equal to the kth 2-creation rook number for B_n. We will say that a rook from an i-creation rook placement on a Ferrers board B has *coordinates* (s, t) if the rook is in column s of B, and was placed in the tth available space from the bottom as the rooks are placed from left to right.

THEOREM 3.6. *For the board $B_n = B(0, 1, \ldots, n-1)$,*

$$r_k^{(2)}(B_n) = m_k(K_{n+k-1}). \tag{3.23}$$

PROOF. We begin with a 2-creation placement of k rooks on B_n. If there is no rook in the last column of B_n, use induction to obtain a k edge matching for the complete graph on $n-1+k-1$ vertices. This can also be considered a k edge matching on $n+k-1$ vertices, but one not containing the vertex $n+k-1$.

Now suppose there is a rook in the last column of B_n, with coordinates (n, j). Since each of the $k-1$ rooks in columns 1 through $n-1$ creates two new rows, we have that $1 \leq j \leq n+k-2$. For the matching associated to this rook placement, first choose the edge between vertices j and $n+k-1$. This leaves $n+k-3$ vertices unmatched in K_{n+k-1}. Now by induction, the $k-1$ rooks in columns 1 through $n-1$ determine a $(k-1)$-edge matching on the remaining $n-1+k-1-1 = n+k-3$ vertices. We can use the edges from this matching, with vertices $j, j+1, \ldots, n+k-3$ relabeled as $j+1, j+2, \ldots, n+k-2$.

For the inverse of this correspondence, suppose we have a matching M. If M does not contain the vertex $n+k-1$, then by induction we can associate to M a 2-creation placement of k rooks on B_n. We can consider this to be a 2-creation placement of k rooks on $B(0, 1, \ldots, n-1)$ without a rook in column n. If M contains an edge between vertices $n+k-1$ and j, the again by induction we can associate a 2-creation placement of $k-1$ rooks on B_{n-1} to $M - \{(j, n+k-1)\}$. If we add to the board a column of height $n+k-1$, we can then add to this placement a rook in column n with coordinates (n, j). This gives us a 2-creation placement of k rooks on B_n as desired. $\square$

By a simple combinatorial argument

$$m_k(K_n) = \binom{n}{2k}\frac{(2k)!}{k!2^k}. \tag{3.24}$$

Combining this with Theorem 3.6 gives

$$r_k^{(2)}(B_n) = \binom{n+k-1}{2k}\frac{(2k)!}{k!2^k}, \tag{3.25}$$

and substituting this into the factorization theorem gives the identity

$$\sum_{k=0}^{n}\binom{n+k-1}{2k}\frac{(2k)!}{k!2^k}x^{(n-k,1)} = x^{(n,2)}. \tag{3.26}$$

Number of Labeled Forests. In this section we consider the *Abel* board $B(0, n, \ldots, n)$ contained in the $n \times n$ board. By the factorization theorem for the 1-rook numbers of this board, we obtain

$$\sum_{k=0}^{n} r_k^{(1)}(A_n)x^{n-k} = x(x+n)^{n-1}. \tag{3.27}$$

It is known [] REFERENCE NEEDED that the coefficient of x^k in the polynomial $x(x+n)^{n-1}$ counts the number of labeled forests on n vertices composed of k rooted trees. If we denote this number by $t_{n,k}$, then we obtain the following.

THEOREM 3.7.

$$r_k^{(1)}(A_n) = t_{n,n-k}. \tag{3.28}$$

A combinatorial proof of this fact is given in [**GH00**] using partial endofunctions.

A q-Analog of $r_k^{(\alpha)}(B)$

Recall the notation $[x] = (1-q^x)/(1-q)$ for the standard q-analog of the real number x. As with other rook theory q-analogs, we assume that B is a Ferrers board. We still allow more than one rook in each row, but only one rook in each column of B. Suppose C is a fixed placement of rooks on B satisfying these properties. If γ is a square of B, let $v(\gamma)$ denote the number of rooks from C that are strictly to the left of, and in the same row as, γ. The weight of the square γ, denoted $\mathrm{wt}_q(\gamma)$, is defined by

$$\mathrm{wt}_q(\gamma) = \begin{cases} 1 & \text{if there is a rook above} \\ & \quad \text{and in the same column as } \gamma \\ [(\alpha-1)v(\gamma)+1] & \text{if } \gamma \text{ contains a rook} \\ q^{(\alpha-1)v(\gamma)+1} & \text{otherwise.} \end{cases}$$

Then define the q-weight of the placement C by $\mathrm{wt}_q(C) = \prod_{\beta \in B} \mathrm{wt}_q(\beta)$, and the *$q$-analog of the kth α-rook number* via

$$R_k^{(\alpha)}(B) = \sum_{\substack{C \\ k \text{ rooks on } B}} \mathrm{wt}_q(C). \tag{3.29}$$

LEMMA 3.8. *Suppose $B = B(c_1, \ldots, c_n)$ is a Ferrers board, and let C' be a fixed placement of k rooks in columns 1 through $i-1$ of B, for $i < n$. Then*

$$\sum_{\substack{C \supset C' \\ k+1 \text{ rooks on } B}} wt_q(C) = [k(\alpha-1)+c_i]\,wt_q(C'), \tag{3.30}$$

where the sum is taken over all placements C which extend C' by placing an additional rook in column i.

PROOF. Suppose the top row in column i of B contains v_1 rooks from C', the next row down contains v_2 rooks, ..., the bottom row contains v_{c_i} rooks (where each v_j may or may not be 0). Then placing a rook in the top position of column i contributes a factor of $[(\alpha-1)v_1+1]$ to the LHS of (3.30), placing a rook in the next position down contributes a factor of $q^{(\alpha-1)v_1+1}[(\alpha-1)v_2+1]$, ..., placing a rook in the bottom position contributes a factor of

$$q^{(\alpha-1)v_1+1+(\alpha-1)v_2+1+\cdots+(\alpha-1)v_{c_i-1}+1}[(\alpha-1)v_{c_i}+1]. \tag{3.31}$$

Note that $v_1+\cdots+v_{c_i}=k$. Then the sum

$$[(\alpha-1)v_1+1]+q^{(\alpha-1)v_1+1}[(\alpha-1)v_2+1]+\cdots$$

$$+q^{(\alpha-1)v_1+1+(\alpha-1)v_2+1+\cdots+(\alpha-1)v_{c_i-1}+1}[(\alpha-1)v_{c_i}+1] \tag{3.32}$$

simplifies to

$$[(\alpha-1)v_1+1+(\alpha-1)v_2+1+\cdots+(\alpha-1)v_{c_i}+1], \tag{3.33}$$

which is equal to $[k(\alpha-1)+c_i]$. Thus (3.30) holds as desired. □

As in all the other cases, we obtain a version of the factorization theorem.

THEOREM 3.9. *For the Ferrers board $B=B(c_1,\ldots,c_n)$,*

$$\sum_{k=0}^{n} R_k^{(\alpha)}(B)[x][x+\alpha-1]\cdots[x+(n-k-1)(\alpha-1)]$$

$$=\prod_{j=1}^{n}[x+c_j+(j-1)(\alpha-1)]. \tag{3.34}$$

EXERCISE 3.10. Using Lemma 3.8, give a combinatorial proof of Theorem 3.9.

The $R_k^{(\alpha)}(B)$ also satisfy a version of recurrence (3.10). Namely, if $B=B(c_1,\ldots,c_n)$ and $B'=B(c_1,\ldots,c_{n-1})$, then

$$R_k^{(\alpha)}(B)=R_k^{(\alpha)}(B')+[(k-1)(\alpha-1)+c_n]R_{k-1}^{(\alpha)}(B'). \tag{3.35}$$

This is proven similarly to (3.10), by breaking placements on B into those with a rook in column n and those without a rook in column n, and using Lemma 3.8.

$R_k^{(\alpha)}(B)$ **for Special Values of** α**.** Since $R_k^{(\alpha)}(B)$ equals $r_k^{(\alpha)}(B)$ when $q=1$, there are several interesting q-analogs for the special values of B and α from the previous section.

For example, for the board $B_n=B(0,1,\ldots,n-1)$ and $\alpha=1$, we obtain a q-analog of the absolute Stirling numbers of the first kind that we denote $C(n,k)$. Namely, we have the relationship

$$R_k^{(1)}(B_n)=C(n,n-k). \tag{3.36}$$

This $C(n,k)$ is the well-known q-analog of these Stirling numbers, introduced by Gould and studied by several others REFERENCES NEEDED. This can easily be shown using the recurrence

$$C(n,k)=C(n-1,k-1)+[n-1]C(n-1,k) \tag{3.37}$$

derived from (3.35), and is left as an exercise.

We can get a q-analog of the matching numbers for the complete graph on n vertices, denoted $M_k(K_n)$, via the definition $M_k(K_{n+k-1}) := R_k^{(2)}(B_n)$ These q-matching numbers satisfy the identity

$$M_k(K_n) = q^{\binom{n-k}{2}} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix} \prod_{j=1}^{k} [2j-1] \tag{3.38}$$

(which reduces to equation (3.24) when $q = 1$, after some algebraic simplification). This can be easily proven using recurrence (3.35) and induction, and is left as an exercise.

CHAPTER 4

Rook Theory and Cycle Counting

In this chapter, we will look at some rook theory models which incorporate the cycle structure of rook placements. Throughout this chapter (unless otherwise noted), n is a positive integer, B is a Ferrers board contained in the $n \times n$ grid, and k is an integer between 0 and n.

Rook Placements and Directed Graphs

The models in this chapter begin with an observation of Gessel [**?**]. He noted that any placement C of k non-attacking rooks on a board B can be associated to a directed graph G_C on n vertices as follows. The placement C has a rook on square (i, j) iff the graph G_C has a directed edge from i to j. See Figure 1 for an example.

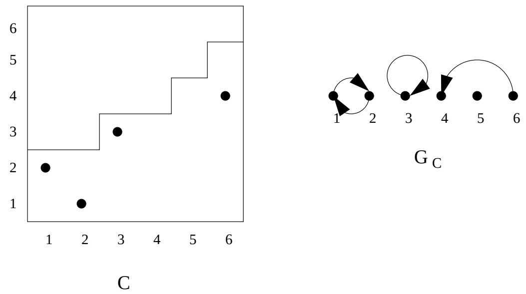


FIGURE 1. A placement C and its corresponding digraph G_C.

We denote by $\mathrm{cyc}(C)$ the number of cycles in the corresponding digraph G_C (directed paths in G_C that are not cycles are not counted in $\mathrm{cyc}(C)$). So for the placement C in Figure 1, $\mathrm{cyc}(C) = 2$. Note that if π is a permutation in S_n and $P(\pi)$ is a the associated n-rook placement as on page **??** REFERENCE NEEDED FROM CH 1, then $\mathrm{cyc}(P(\pi))$ is equal to the number of cycles in the disjoint cycle decomposition of π.

We can then define the *cycle-counting rook numbers* $r_k(y, B)$ via

$$r_k(y, B) := \sum_{\substack{C \\ k \text{ rooks on } B}} y^{\mathrm{cyc}(C)}, \tag{4.1}$$

and the *cycle-counting hit numbers* $t_k(y, B)$ as

$$t_k(y, B) := \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} y^{\mathrm{cyc}(C)}. \tag{4.2}$$

Recall we use the notation $B(c_1, \ldots, c_n)$ to denote the Ferrers board with column heights $c_1 \leq \cdots \leq c_n$. The following fact will be used many times in this chapter.

LEMMA 4.1. *Let $B = B(c_1, \ldots, c_n)$, $1 \le i \le n$, $1 \le j \le i-1$, and consider a placement of j non-attacking rooks on columns 1 through $i-1$ of B. If the height c_i of the ith column of B satisfies $c_i \ge i$, then there is exactly one square in this column of B on which a rook can be placed to create a new cycle in the corresponding digraph. If $c_i < i$, then there is no such square.*

PROOF. In the case when $c_i \ge i$, there are two possibilities. The first is that there is no directed edge in the digraph going into i, in which case (i, i) is the unique square where placing a rook will create a new cycle. The second is that there is a directed path starting at some vertex v and ending at i. Since $c_i \ge i$ we have that $v < i \le c_i$, so the square (i, v) is on B, and it is the unique square that creates a new cycle in the digraph.

If $c_i < i$, then we cannot place a rook on (i, i) to complete a cycle, and since B is a Ferrers board (hence for any $k < i$ we have $c_k \le c_i < i$), there can be no directed path ending at i. Thus there is no square in column i on which a rook can be placed to create a new cycle in the digraph. $\square$

EXERCISE 4.2. Consider the following generalization of the Stirling numbers of the second kind,

$$S_2(y, n+1, k) := \sum_{\substack{\lambda \text{ partitions of } n+1 \\ \text{elements into } k \text{ blocks}}} y^{\text{num}(\lambda)}, \tag{4.3}$$

where $\text{num}(\lambda)$ denotes the number of values $i, 1 \le i \le n$ such that the ith and $(i{+}1)$st elements are in the same block of λ. Recall that B_{n+1} denotes the triangular board $B(0, 1, 2, \ldots, n)$. Prove combinatorially that

$$S_2(y, n+1, k) = r_{n+1-k}(y, B_{n+1}). \tag{4.4}$$

Gessel also showed that the $r_k(y, B)$ and $t_k(y, B)$ are related by the following generalization of (1.1)

$$\sum_{k=0}^{n} r_{n-k}(y, B)(y)_k (z-1)^{n-k} = \sum_{k=0}^{n} z^k t_k(y, B), \tag{4.5}$$

where $(y)_k := y(y{+}1)\cdots(y{+}k{-}1)$. We give the following proof of (4.5), mimicking that of (1.1) in [**Sta12**, p. 72]. First let $z = z+1$ in (4.5), obtaining

$$\sum_{k=0}^{n} r_{n-k}(y, B)(y)_k z^{n-k} = \sum_{k=0}^{n} (z+1)^k t_k(y, B). \tag{4.6}$$

The coefficient of z^{n-k} on the LHS of (4.6) is $r_{n-k}(y, B)(y)_k$, and the coefficient of z^{n-k} on the RHS is $\sum_{j \ge n-k} \binom{j}{n-k} t_j(y, B)$. We now show that these are different ways of organizing the terms in the weighted count

$$\sum_{\substack{(C,\pi),\ \pi \subseteq C \cap B, \\ |\pi| = n-k}} y^{\text{cyc}(C)}, \tag{4.7}$$

where C is a placement of n rooks on the $n \times n$ grid, and π is a subset of $n-k$ rooks from C, all of which are on B. Note that since the height of each column in the $n \times n$ grid is n, each column will have exactly one square which creates a new cycle when considering only those rooks to its left, as in the proof of Lemma 4.1. The first way to organize the terms is to place $n-k$ rooks on B (thus ensuring that there are at least $n-k$ rooks on B), then extend this $(n-k)$-rook placement to to an n-rook placement by placing a rook in each of the k unoccupied columns of the $n \times n$ grid. Summing over all $(n-k)$-rook placements on B gives $r_{n-k}(y, B)$, and summing over all placements of rooks in the k unoccupied columns yields $(y)_k$, giving the LHS. The second way to organize the terms is to place n rooks on the $n \times n$ grid with j rooks on B (where $j \geq n-k$), then choose a subset of the rooks on B of size $n-k$. For a fixed j this contributes $\binom{j}{n-k} t_j(y, B)$, and summing over all $j \geq n-k$ yields the RHS.

In [**Hag96**], Haglund generalizes the $t_k(y, B)$, by algebraically defining $t_k(x, y, B)$ as

$$\sum_{k=0}^{n} r_{n-k}(y, B)(x)_k (z-1)^{n-k} = \sum_{k=0}^{n} z^k t_k(x, y, B) \tag{4.8}$$

(and does similarly for a q-version). It is clear from (4.5) that $t_k(y, y, B) = t_k(y, B)$ as defined above. A number of algebraic identities for the $t_k(x, y, B)$ and their q-version are proved in [**Hag96**], using results from the theory of hypergeometric series. One such hypergeometric result which Haglund uses repeatedly is the well-known *Vandermonde convolution*,

$$\sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(c)_k} = \frac{(c-b)_n}{(c)_n}, \tag{4.9}$$

where the arguments a, b and c can be any complex numbers with the real part of $c-a-b$ greater than 0, and n is a positive integer. In the next section and the one that follows, we will give versions of Haglund's proofs concerning the $t_k(x, y, B)$ for the $x = y$ case.

Algebraic Identities for Ferrers Boards

For a Ferrers board $B = B(c_1, \ldots, c_n)$, define

$$PR(z, y, B) = \prod_{c_i \geq i} (z + c_i - i + y) \prod_{c_i < i} (z + c_i - i + 1). \tag{4.10}$$

Then we have the following analog of (1.7)

$$\sum_{k=0}^{n} r_{n-k}(y, B) z(z-1) \cdots (z-k+1) = PR(z, y, B). \tag{4.11}$$

Exercise 4.3. Prove (4.11) combinatorially, mimicking the proof of (1.7) in Chapter **??** and making use of Lemma 4.1.

We also obtain an analogue of Corollary **??**, giving algebraic expansions of the $r_k(y, B)$ and $t_n(y, k)B$ in terms of binomial coefficients.

Corollary 4.3.1. *Let $B = B(c_1, \ldots, c_n)$ be a Ferrers board. Then*

$$k! r_k(y, B) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} PR(j, y, B) \tag{4.12}$$

$$t_{n-k}(y, B) = \sum_{j=0}^{k} \binom{n+y}{k-j} (-1)^{k-j} \binom{y+j-1}{j} PR(j, y, B). \tag{4.13}$$

PROOF. Using (4.11), the RHS of (4.12) equals

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{s=0}^{n}r_{n-s}(y,B)j(j-1)\cdots(j-s+1) \tag{4.14}$$
$$=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{s=0}^{n}r_{n-s}(y,B)\binom{j}{s}s!.$$

Reversing the order of summation gives

$$\sum_{s\geq 0}s!r_{n-s}(y,B)\sum_{j\geq s}\binom{k}{j}(-1)^{k-j}\binom{j}{s}, \tag{4.15}$$

which equals

$$\sum_{s\geq 0}s!r_{n-s}(y,B)\delta_{s,k} \tag{4.16}$$

as in the proof of Corollary **??**.

Similarly by using (4.11), reversing the order of summation, and rewriting the $\binom{y+j-1}{j}$ term, the RHS of (4.13) equals

$$\sum_{s=0}^{n}r_{n-s}(y,B)\sum_{j\geq s}\binom{n+y}{k-j}(-1)^{k-j}\frac{(y)_j j(j-1)\cdots(j-s+1)}{(1)_j}. \tag{4.17}$$

By defining $u=j-s$, (4.17) becomes

$$\sum_{s=0}^{n}r_{n-s}(y,B)\sum_{u\geq 0}\binom{n+y}{k-u-s}(-1)^{k-u-s}\frac{(y)_{u+s}(u+1)_s}{(1)_{u+s}} \tag{4.18}$$

$$=\sum_{s=0}^{n}r_{n-s}(y,B)\sum_{u\geq 0}\binom{n+y}{k-s}(-1)^{u}\frac{(s-k)_u}{(n+y-k+s+1)_u} \tag{4.19}$$
$$\times(-1)^{k-s-u}\frac{(y)_s(y+s)_u}{(s+1)_u}\binom{u+s}{s}$$

$$=\sum_{s=0}^{n}r_{n-s}(y,B)\binom{n+y}{k-s}(-1)^{k-s}(y)_s\sum_{u\geq 0}\frac{-(k-s)_u(y+s)_u}{u!(n+y-k+s+1)_u}, \tag{4.20}$$

which by the Vandermonde convolution (4.9) equals

$$\sum_{s=0}^{n}r_{n-s}(y,B)\binom{n+y}{k-s}(-1)^{k-s}(y)_s\frac{(n-k+1)_{k-s}}{(n+y-k+s+1)_{k-s}}. \tag{4.21}$$

Then (4.21) equals

$$\sum_{s=0}^{n}(y)_s r_{n-s}(y,B)\binom{n-s}{k-s}(-1)^{k-s}, \tag{4.22}$$

which is exactly $t_{n-k}(y,B)$ by (4.5).

□

Cycle-Counting q-Rook Polynomials

We will now define a q-version of the cycle-counting rook numbers for Ferrers boards, as in [**Hag96**]. Throughout this section let $B = B(c_1, \dots, c_n)$ be a Ferrers board. Let C be a given placement of rooks on B. We will use s_i to denote the unique square in column i (when it exists) on which, considering only rooks from C in columns 1 through $i-1$ on B, a rook can be placed to create a new cycle in the corresponding digraph. Recall by Lemma 4.1 that such a square exists in column i of B if and only if $c_i \geq i$. We then let $\mathrm{E}(C, B)$ denote the number of i such that $c_i \geq i$ and there is no rook from C in column i on or above square s_i.

We use the notation $\mathrm{cyc}(C)$ as defined on page 33 and $\mathrm{inv}(C, B)$ as defined on page **??** REFERENCE NEEDED. We also keep the notation $[y] = (1-q^y)/(1-q)$. Then the *cycle-counting q-rook numbers* are given by

$$R_k(y, B) = \sum_{\substack{C \\ k \text{ rooks on } B}} [y]^{\mathrm{cyc}(C)} q^{\mathrm{inv}(C,B) + \mathrm{E}(C,B)(y-1)}. \tag{4.23}$$

We get the identity

$$\sum_{k=0}^{n} R_{n-k}(y, B)[z][z-1]\cdots[z-k+1] = PR[z, y, B], \tag{4.24}$$

where $PR[z, y, B]$ denotes the obvious q-version of (4.10)

$$PR[z, y, B] = \prod_{c_i \geq i} [z + c_i - i + y] \prod_{c_i < i} [z + c_i - i + 1]. \tag{4.25}$$

The proof of (4.24) is left as an exercise. It makes use of Lemma 4.1, and the fact that for any natural numbers j and k with $j \leq k$,

$$1 + q + \cdots + q^j[y] + q^{y+j} + \cdots + q^{y+k-1} = [y+k]. \tag{4.26}$$

Exercise 4.4. Prove (4.24), using Lemma 4.1 and (4.26).

A q-version of the $t_k(y, B)$ can then be defined algebraically as

$$\sum_{k=0}^{n} R_k(y, B)[y][y+1]\cdots[y+k-1] \prod_{i=k+1}^{n} (z - q^{y+i-1}) = \sum_{k=0}^{n} T_k(y, B) z^k. \tag{4.27}$$

Notice that by letting $q \to 1$ in (4.27) we obtain (4.5), so these algebraically defined $T_k(y, B)$ are a generalization of the $t_k(y, B)$ at least in this rudimentary sense. We also have the following q-versions of (4.12) and (4.13). Recall the definition of $\begin{bmatrix} x \\ k \end{bmatrix}$ in **??** REFERENCE NEEDED.

Lemma 4.5. *For any regular Ferrers board B,*

$$[k]! R_{n-k}(y, B) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j, y, B] \tag{4.28}$$

and

$$T_{n-k}(y, B) = \sum_{j=0}^{k} \begin{bmatrix} n+y \\ k-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j, y, B]. \tag{4.29}$$

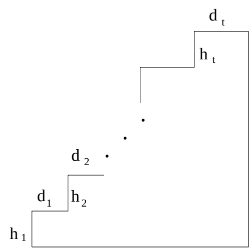


FIGURE 2. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$.

Haglund's proof of Lemma 4.5 given in [**Hag96**] follows that of Corollary 4.3.1 almost exactly, so the details are omitted. The main difference in the proof of Lemma 4.5 is that, in place of the Vandermonde convolution (4.9), the Heine transformation

$$\sum_{k=0}^{\infty} \frac{(x;q)_k (b;q)_k}{(q;q)_k (c;q)_k} z^k = \frac{(b;q)_\infty (xz;q)_\infty}{(c;q)_\infty (z;q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b:q)_k (z;q)_k}{(q;q)_k (xz;q)_k} b^k \tag{4.30}$$

is used. Here $(w;q)_k = (1-w)(1-wq)\cdots(1-wq^{k-1})$, and $(w;q)_\infty = \prod_{k\geq 0}(1-wq^k)$.

Up until this point, we have used the notation $B(c_1, \ldots, c_n)$ to specify a Ferrers board by its column heights. Another way in which a Ferrers board can be described is using the step heights and depths. A Ferrers board with step heights $h_1, \ldots, h_t$ and step depths $d_1, \ldots d_t$ as in Figure 2 will be denoted $B(h_1, d_1; \ldots; h_t, d_t)$.

For the Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$ and any $1 \leq p \leq t$, we will use the notations $H_p := h_1 + \cdots + h_p$ and $D_p := d_1 + \cdots + d_p$. We define a *regular Ferrers board* as a Ferrers board in which $c_i \geq i$ for all i (or equivalently $H_1 \geq D_1, H_2 \geq D_2, \ldots H_t \geq D_t$ as was defined in [**Hag96**]). The Ferrers board obtained from $B = B(h_1, d_1; \ldots; h_t, d_t)$ by decreasing the pth step height and depth by 1 (giving the board $B(h_1, d_1; \ldots, h_p - 1, d_p - 1; \ldots h_t, d_t)$) will be denoted $B - h_p - d_p$. The following recurrence is essential in the next section, where a combinatorial interpretation of the $T_k(y, B)$ is given.

LEMMA 4.6. *For any regular Ferrers board $B = B(h_1, d_1; \ldots; h_t, d_t)$ and $1 \leq p \leq t$,*

$$T_{n-k}(y, B) = [y + k + H_p - D_{p-1} - 1] T_{n-k-1}(y, B - h_p - d_p) \tag{4.31}$$

$$+ q^{y+k+H_p-D_{p-1}-2}[n - k - H_p + D_{p-1} + 1] T_{n-k}(y, B - h_p - d_p).$$

PROOF. By (4.29), $T_{n-k}(y, B)$ is equal to

$$\sum_{j=0}^{k} \begin{bmatrix} n+y \\ k-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j, y, B] \tag{4.32}$$

(4.33)

$$= \sum_{j=0}^{k} \begin{bmatrix} n+y \\ k-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} [j + y + H_p - D_{p-1} - 1] PR[j, y, B - h_p - d_p]$$

$$= \sum_{j=0}^{k} \begin{bmatrix} n+y \\ k-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j,y,B-H_p-d_p] \tag{4.34}$$

$$\times \big\{ [k+y+H_p-D_{p-1}-1] - q^{j+y+H_p-D_{p-1}-1}[k-j] \big\}$$

(4.35)

$$= [k+y+H_p-D_{p-1}-1] \sum_{j=0}^{k} \begin{bmatrix} n+y \\ k-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j,y,B-h_p-d_p]$$

$$-q^{y+H_p-D_{p-1}-1} \sum_{j=0}^{k-1} [n+y] \begin{bmatrix} n+y-1 \\ k-1-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}+j} PR[j,y,B-h_p-d_p]$$

(4.36)

$$= [k+y+H_p-D_{p-1}-1] \sum_{j=0}^{k} \left\{ \begin{bmatrix} n+y-1 \\ k-j \end{bmatrix} q^{\binom{k-j}{2}} + \begin{bmatrix} n+y-1 \\ k-1-j \end{bmatrix} q^{\binom{k-1-j}{2}+n+y-1} \right\}$$

$$\times \begin{bmatrix} y+j-1 \\ j \end{bmatrix} PR[j,y,B-h_p-d_p]$$

$$-q^{y+H_p-D_{p-1}-1} \sum_{j=0}^{k-1} [n+y] \begin{bmatrix} n+y-1 \\ k-1-j \end{bmatrix} \begin{bmatrix} y+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-1-j}{2}+k-1} PR[j,y,B-h_p-d_p]$$

(since $\begin{bmatrix} n+y \\ k-j \end{bmatrix} = \begin{bmatrix} n-1+y \\ k-j \end{bmatrix} + q^{n+y-k+j} \begin{bmatrix} n-1+y \\ k-1-j \end{bmatrix}$)

$$= [k+y+H_p-D_{p-1}-1] T_{n-k-1}(y, B-h_p-d_p) \tag{4.37}$$

$$+T_{n-k}(y,B-h_p-d_p)(-q^{n+y-1}[k+y+H_p-D_{p-1}-1] + q^{k+y+H_p-D_{p-1}-2}[n+y])$$

by (4.5) with $B = B-h_p-d_p$, which equals the RHS of (4.31) when simplified. □

Note that by letting $y=1$ in Lemma 4.6, (4.31) can be specified to the recurrence

$$T_{n-k}(B) = [k+H_p-D_{p-1}] T_{n-k-1}(B-h_p-d_p) \tag{4.38}$$

$$+q^{k+H_p-D_{p-1}-1}[n-k-H_p+D_{p-1}+1] T_{n-k}(B-h_p-d_p)$$

for the $T_k(B)$ when B is a regular Ferrers board.

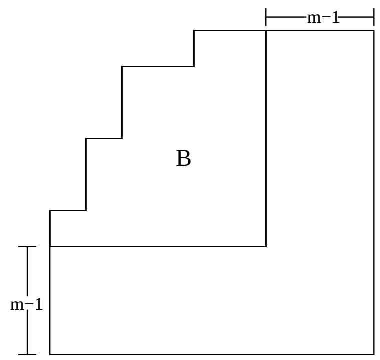


FIGURE 3. The Ferrers board $B * m$.

A Combinatorial Interpretation of the Cycle-Counting q-Hit Numbers

In this section we sketch the derivation of a combinatorial interpretation for the $T_k(y, B)$ found in by Butler [**?**] REFERENCE NEEDED. For a positive integer m, the Ferrers board obtained by increasing the height of the first step by $m-1$ and the depth of the last step by $m-1$ (giving the board $B(h_1+m-1, d_1; \dots h_t, d_t+m-1)$) will be denoted by $B*m$; see figure 3. Note that if B is a subset of an $n \times n$ board, then $B * m$ will be a subset of an $(n + m - 1) \times (n + m - 1)$ board.

We have the following proposition relating the $T_k(y, B)$ when y is a positive integer to the ordinary q-hit numbers of the larger board $B*m$. This proposition will allow us to use Haglund's combinatorial interpretation of $T_k(B)$ defined in Chapter **??** BETTER REFERENCE NEEDED to find a combinatorial interpretation of the $T_k(y, B)$.

PROPOSITION 4.6.1. *Let B be a regular Ferrers board contained in the $n \times n$ square board, m a positive integer. Then*

$$T_k(m, B) = \frac{T_{k+m-1}(B * m)}{[m-1]!} \tag{4.39}$$

PROOF. The proof is by induction on the area of the board B (which is the number of squares of B) . The only regular Ferrers board of area 1 is the 1×1 square board, so B is this board and $B * m$ is the $m \times m$ square board. An easy calculation shows in this case that that $T_1(m, B) = [m]$ and $T_k(m, B) = 0$ for all $k \neq 1$. Using the definition of Haglund's statistic $\beta(C, B * m)$ on any placement C of rooks on $B * m$, we see that $T_m(B * m) = [m]!$ and $T_k(B * m) = 0$ for $k \neq m$ so the proposition holds in this case.

Now assume the proposition holds for all regular Ferrers boards or area less than A, and suppose B is a regular Ferrers board of area A. Since B is regular we see that $H_t = D_t = n$, hence $H_t - D_{t-1} = H_t - D_t + d_t = n - n + d_t = d_t$. Now by Lemma 4.6 with $p = t$ and $k = n - k$,

$$T_k(m, B) = [m + n - k + d_t - 1]T_{k-1}(m, B - h_t - d_t) \tag{4.40}$$

$$+q^{m+n-k+d_t-2}[k - d_t + 1]T_k(m, B - h_t - d_t).$$

By induction, $T_{k-1}(m, B - h_t - d_t) = T_{k-1+m-1}((B - h_t - d_t) * m)/[m-1]!$ and $T_k(m, B - h_t - d_t) = T_{k+m-1}((B - h_t - d_t) * m)/[m-1]!$. Since $(B - h_t - d_t)_m$ is the board $B(h_1 + m - 1, d_1; \ldots h_t - 1, d_t - 1 + m - 1)$, we get that $T_k(m, B)$ is equal to

(4.41)
$$\{[m + n - k + d_t - 1]T_{k-1+m-1}(B(h_1 + m - 1, d_1; \ldots h_t - 1, d_t - 1 + m - 1))$$

$$+q^{m+n-k+d_t-2}[k - d_t + 1]\times$$

$$T_{k+m-1}(B(h_1 + m - 1, d_1; \ldots h_t - 1, d_t - 1 + m - 1))\}/[m-1]!.$$

Now by (4.38) with $n = n + m - 1$, $p = t$, $k = n - k$, and $B = B(h_1 + m - 1, d_1; \ldots h_t, d_t + m - 1)$, (4.41) is equal to $T_{k+m-1}(B(h_1 + m - 1, d_1; \ldots h_t, d_t + m - 1))/[m-1]!$, which is exactly equal to $T_{k+m-1}(B * m)/[m-1]!$ as desired.

$\square$

The remainder of the derivation is as follows. Through a series of lemmas (see [?] for details) making use of Haglund's statistic β, it is shown that the terms of $T_{k+m-1}(B * m)/[m-1]!$ can be reorganized into the form

$$\sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} [m]^{\mathrm{cyc}(C)} q^{(n-\mathrm{cyc}(C))(m-1)+\mathrm{s}(C,B)+\mathrm{E}(C,B)}. \tag{4.42}$$

Here we're assuming B is a regular Ferrers board contained within the $n \times n$ square board, m is a positive integer, $B * m$ is the board defined above, k is an integer between 0 and n, C is a placement of n rooks on the $n \times n$ board with k rooks on B, and $\mathrm{cyc}(C)$ and $\mathrm{E}(C, B)$ are as defined earlier. The statistic $\mathrm{s}(C, B)$ is the number of squares on the $n \times n$ board which neither contain a rook from C nor are crossed out after applying the following cancellation scheme:

(1) Each rook cancels all squares to the right in its row;
(2) Each rook on B cancels all squares above it in its column;
(3) Each rook on B on square s_i for some i (as defined on page 37) also cancels all squares below it in its column;
(4) Each rook off B cancels all squares below it in its column, but above B.

Combining Proposition 4.6.1 and (4.42), we obtain the following.

THEOREM 4.7. *For any regular Ferrers board B, we have that*

$$T_k(y, B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} [y]^{cyc(C)} q^{(n-cyc(C))(y-1)+s(C,B)+E(C,B)}. \tag{4.43}$$

PROOF. Both $T_k(y, B)$ and the RHS of (4.43) are polynomials in the variable q^y over the field $\mathbb{Q}(q)$ of fixed degree. By (4.6.1) and (4.42), these two polynomials are equal for all positive integer values of y. Hence these two polynomials have infinitely many common values, so must be equal for all values of y. $\square$

Note that if we let $y = 1$ in (4.43), then we obtain a statistic to generate the q-hit numbers as defined in (1.39). That is, we have

$$T_k(B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\mathrm{s}(C,B)+\mathrm{E}(C,B)} \tag{4.44}$$

While this new statistics is equal to neither Haglund's $\beta(C, B)$ [**Hag98**] nor Dworkin's $\xi(C, B)$ [**Dwo98**], it is a member of the family of statistics discussed by Haglund and Remmel [**HR01**, p. 479].

Additionally, Theorem 4.7 provides a generalization of the familiar notion of a Mahonian permutation statistic. For a permutation $\pi = (\pi_1\pi_2\cdots\pi_n) \in S_n$, a statistic $\mathrm{stat}(\pi)$ is called *Mahonian* if

$$\sum_{\pi\in S_n} q^{\mathrm{stat}(\pi)} = [n]!. \tag{4.45}$$

We shall say a pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ is *cycle-Mahonian* if

$$\sum_{\pi\in S_n} [y]^{\mathrm{stat}_1(\pi)} q^{\mathrm{stat}_2(\pi)} = [y][y+1]\cdots[y+n-1]. \tag{4.46}$$

Note that the statistic stat_2 may depend on both π and y. This notion of cycle-Mahonian generalizes that of a Mahonian statistic, since letting $y = 1$ in (4.46) yields (4.45).

As noted in Chapter **??**, a permutation $\pi = (\pi_1\pi_2\cdots\pi_n) \in S_n$ can be identified with a placement $P(\pi)$ of n rooks on the $n \times n$ board by letting a rook on (i, j) correspond to $\pi_i = j$. Thus any statistic for placements of n rooks on the $n \times n$ board can also be considered as a permutation statistic via the above identification. In particular, for any permutation $\pi \in S_n$ and regular Ferrers board B contained in the $n \times n$ board, we can define

$$\mathrm{s}(\pi, B) = \mathrm{s}(P(\pi), B). \tag{4.47}$$

Exercise 4.8. Show that for any regular Ferrers board B, the pair of permutation statistics $(\mathrm{cyc}, \mathrm{s}(\pi, B))$ is cycle-Mahonian.

Cycle-Counting q-Eulerian Numbers

We define the *cycle-counting q-Eulerian numbers* via the equation

$$A_{k+1}(n, y, q) = \sum_{\substack{\pi\in S_n \\ k \text{ descents}}} [y]^{\mathrm{lrmin}(\pi)} q^{(n-\mathrm{lrmin}(\pi))(y-1)+\mathrm{maj}(\pi)}. \tag{4.48}$$

Here $\mathrm{lrmin}(\pi)$ denoted the number of left-to-right minima of π as discussed on page **??** REFERENCE NEEDED, and $\mathrm{maj}(\pi)$ denotes the major index of π. The $A_{k+1}(n, y, q)$ generalize the classical Eulerian numbers discussed on page **??** REFERENCE NEEDED, along with their q-analog [] REFERENCE NEEDED.

The $A_{k+1}(n, y, q)$ can be easily shown to satisfy the useful recurrence

$$A_{k+1}(n, y, q) = [y + k]A_{k+1}(n - 1, y, q) + q^{y+k-1}[n - k]A_k(n - 1, y, q). \tag{4.49}$$

The argument mimics the well-known proof of the analogous recurrence for the q-Eulerian numbers given in [] REFERENCE NEEDED as follows. Any permutation $\sigma \in S_n$ (written in one line notation) with k descents can be built from a permutation in $\sigma' \in S_{n-1}$ with either k or $k - 1$ descents by inserting n into appropriate positions.

If σ' has k descents in positions $i_1 < i_2 < \ldots < i_k$, then placing n in any of positions $i_1+1, i_2+1, \ldots, i_k+1$ will result in a permutation in S_n with k descents. Placing n in position i_k+1 will have no effect on $\mathrm{lrmin}(\sigma')$, but it will move all descents one position to the right, increasing $\mathrm{maj}(\sigma')$ by k. Thus in this case

$$[y]^{\mathrm{lrmin}(\sigma)}q^{(n-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)} \tag{4.50}$$

$$= q^{y-1+k}[y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}.$$

We continue in this manner, placing n is positions $i_2+1, \ldots i_k+1$, keeping track of the factor by which $[y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}$ increases in each case. The final position in which we can place n without affecting the number of descents is at the end of σ'. Doing this will not change $\mathrm{maj}(\sigma')$, but will increase $\mathrm{lrmin}(\sigma')$ by 1. So in this case

$$[y]^{\mathrm{lrmin}(\sigma)}q^{(n-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)} \tag{4.51}$$

$$= [y] \times [y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}.$$

Summing over all positions in which n can be placed without creating a new descent yields

$$\{[y] + q^y + q^{y+1} + \cdots + q^{y+k-1}\}[y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')} \tag{4.52}$$

$$= [y+k][y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}.$$

Summing over all $\sigma' \in S_{n-1}$ with k descents yields the first term in the recurrence.

We proceed similarly if $\sigma' \in S_{n-1}$ has $k-1$ descents in positions $i_1 < i_2 < \ldots < i_{k-1}$. Now we will successively place n into positions in σ' that will create a new descent. These are any except positions $i_1+1, i_2+1, \ldots, i_{k-1}+1$. Putting n in the first position will create a new descent which adds 1 to $\mathrm{maj}(\sigma')$, and moves each of the $k-1$ other descents one position to the right, adding a total of k to $\mathrm{maj}(\sigma')$. Doing this will not change $\mathrm{lrmin}(\sigma')$, so in this case

$$[y]^{\mathrm{lrmin}(\sigma)}q^{(n-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)} \tag{4.53}$$

$$= q^{y-1+k}[y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}.$$

We continue in this manner, skipping over the positions $i_1+1, i_2+1, \ldots, i_{k-1}+1$. Summing over all positions in which n can be placed to create a new descent yields

$$\{q^{y+k-1} + q^{y+k} + \cdots + q^{y+n-2}\}[y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')} \tag{4.54}$$

$$= q^{y+k-1}[n-k][y]^{\mathrm{lrmin}(\sigma')}q^{(n-1-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}.$$

Summing over all $\sigma' \in S_{n-1}$ with $k-1$ descents yields the second term in the recurrence.

Now we will tie this discussion to rook theory. Recall (1.4), which says that $t_k(B_n)$ is equal to the $(k+1)$st Eulerian number, where B_n denotes the Ferrers with column heights $0, 1, \ldots, n-1$ contained in the $n \times n$ board. Note that any placement of n rooks on the $n \times n$ grid with k rooks on B_n will also be a placement with $n-k$ rooks off B_n. So if we let B_n^c denote the complimentary board of B_n in the $n \times n$ grid, rotated π radians (so the resulting board is Ferrers), then we can see that

$$t_{n-k}(B_n^c) = A_{k+1}(n) \tag{4.55}$$

as well. Identity (4.55) is more useful to us at this point than (1.4) because B_n^c is a regular Ferrers board, so Theorem 4.7 applies. Since letting $y = 1$ and $q \to 1$ in $T_{n-k}(y, B_n^c)$ yields $t_{n-k}(B_n^c)$ (which is equal to $A_{k+1}(n)$ by (4.55)), we see that $T_{n-k}(y, B_n^c)$ also gives a sort of cycle-counting q-version of the Eulerian numbers.

Recall the earlier discussion relating hit numbers and Eulerian numbers from Chapter **??** BETTER REFERENCE NEEDED. In it, we noted that a permutation π in S_n with k descents could be associated to a permutation $\beta(\pi)$ with k cycles by viewing each left-to-right minimum of π as as the last element of a cycle of $\beta(\pi)$. We can now associate to $\beta(\pi)$ a placement $P^o(\beta(\pi))$ by placing a rook on square (i, j) if and only if i follows j in a cycle of $\beta(\pi)$. Note this is slightly different than the bijection used in Chapter **??**, since we are now working with the board B_n^c instead of B_n.

With the above discussion in mind, Theorem 4.7 gives that $T_{n-k}(y, B_n^c)$ is equal to

$$\sum_{\substack{\pi \in S_n \\ k \text{ descents}}} [y]^{\mathrm{cyc}(P^o(\beta(\pi)))} q^{(n-\mathrm{cyc}(P^o(\beta(\pi))))(y-1)+\mathrm{s}(P^o(\beta(\pi)),B_n^c)+\mathrm{E}(P^o(\beta(\pi)),B_n^c)}. \tag{4.56}$$

Note that by (4.31) with $B = B_n^c$, $H_p = H_t = n$, and $D_{p-1} = D_{t-1} = n-1$, we get that $T_{n-k}(y, B_n^c)$ satisfies the recurrence

$$T_{n-k}(y, B_n^c) = [y+k]T_{n-k-1}(y, B_{n-1}^c) + q^{y+k-1}[n-k]T_{n-k}(y, B_{n-1}^c). \tag{4.57}$$

Combining (4.57) with (4.49) yields the following.

Theorem 4.9. *For any n and k in $\mathbb{N}$, we have*

$$A_{k+1}(n, y, q) = T_{n-k}(y, B_n^c). \tag{4.58}$$

Proof. By their definitions, it is easy to verify that $A_1(1, y, q) = [y]$ and $T_1(y, B_1^c) = [y]$. Thus the $A_{k+1}(n, y, q)$ and $T_{n-k}(y, B_n^c)$ satisfy the same initial conditions, and by (4.57) with (4.49) they satisfy the same recurrence. Thus they are equal for all n and k in $\mathbb{N}$. □

Corollary 4.9.1. *The pair* (lrmin, $(n-\mathrm{lrmin})(y-1)+\mathrm{maj}$) *is cycle-Mahonian.*

Proof. We have

$$\begin{aligned}\sum_{\pi \in S_n} [y]^{\mathrm{lrmin}(\pi)} q^{(n-\mathrm{lrmin}(\pi))(y-1)+\mathrm{maj}(\pi)} &= \sum_{k=0}^{n-1} A_{k+1}(n, y, q) \\ &= \sum_{k=0}^{n-1} T_{n-k}(y, B_n^c) \\ &= [y][y+1]\cdots[y+n-1]\end{aligned}$$

by Exercise 4.8, since $T_0(y, B_n^c) = 0$. □

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