CONJECTURED STATISTICS FOR THE q,t-CATALAN NUMBERS

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Abstract. We introduce the distribution function $F_n(q,t)$ of a pair of statistics on Catalan words and conjecture $F_n(q,t)$ equals Garsia and Haiman's q,t-Catalan sequence $C_n(q,t)$, which they defined as a sum of rational functions. We show that $F_{n,s}(q,t)$, defined as the sum of these statistics restricted to Catalan words ending in s ones, satisfies a recurrence relation. As a corollary we are able to verify that $F_n(q,t) = C_n(q,t)$ when $t = 1/q$. We also show the partial symmetry relation $F_n(q,1) = F_n(1,q)$. By modifying a proof of Haiman of a q-Lagrange inversion formula based on results of Garsia and Gessel, we obtain a q-analogue of the general Lagrange inversion formula which involves Catalan words grouped according to the number of ones at the end of the word.

1. Introduction

In [7] Garsia and Haiman introduced a rational function $C_n(q,t)$ which they conjectured always evaluates to a polynomial in q and t with nonnegative coefficients. Later Haiman proved that $C_n(q,t)$ is always a polynomial [11], but with possibly negative coefficients. An elementary proof of this result has been given by Bergeron, et. al. [4], but the nonnegativity remains open. Other conjectures of Haiman have $C_n(q,t)$ related to the Frobenius series of a bigraded S_n module [10].

A Catalan word $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ is a permutation of the multiset $\{0^n\, 1^n\}$ of n 0's and n 1's with the property that for each i, $1 \le i \le 2n$, there are at least as many 0's in the subword $\sigma = \sigma_1\sigma_2\cdots\sigma_i$ as there are 1's. We let $\mathcal{C}_n$ denote the set of all Catalan words of length $2n$; it is well known that the cardinality of $\mathcal{C}_n$ is the n^{th} Catalan number $\binom{2n}{n}/(n+1)$.

Throughout the article we use the standard notation

$$[k] := (1-q^k)/(1-q), \quad [k]! := [1][2]\cdots[k], \quad \begin{bmatrix} m \\ j \end{bmatrix} := [m]!/([j]![m-j]!)$$

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for the q-analogue of the integer k, the q-factorial, and the q-binomial coefficient, and $(x)_n := (1-x)(1-xq)\cdots(1-xq^{n-1})$ for the q-rising factorial.

Garsia and Haiman proved that

$$C_n(q,1) = \sum_{\sigma\in\mathcal{C}_n} q^{\binom{n}{2}-\mathrm{inv}\sigma}, \tag{1}$$

where

$$\mathrm{inv}\sigma := \sum_{\substack{1\le i<j\le 2n\\ \sigma_i>\sigma_j}} 1.$$

They also showed that

$$q^{\binom{n}{2}} C_n(q,q^{-1}) = \frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}. \tag{2}$$

Both $C_n(q,1)$ and $q^{\binom{n}{2}}C_n(q,q^{-1})$ had previously been studied by other authors. In fact, a special case of a result of MacMahon on "lattice permutations" is [12, Vol. 2, pp. 214-215]

$$\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix} = \sum_{\sigma\in\mathcal{C}_n} q^{\mathrm{maj}\sigma},$$

where

$$\mathrm{maj}\sigma := \sum_{\substack{1\le i\le 2n-1\\ \sigma_i>\sigma_{i+1}}} i.$$

See [5] for further background on $C_n(q,1)$ and $q^{\binom{n}{2}}C_n(q,q^{-1})$. Another result of MacMahon we will use later is that if $M=\{0^a1^b\}$ is the multiset of a-0's and b-1's, then

$$\sum_{\pi} q^{\mathrm{inv}\pi} = \begin{bmatrix}a+b\\ a\end{bmatrix}, \tag{3}$$

where the sum is over all multiset permutations π of M.

Because of (1) and (2) Garsia and Haiman called $C_n(q,t)$ the q,t-Catalan sequence. In this article we introduce a statistical refinement of their conjecture, which we now describe.

Given $\sigma\in\mathcal{C}_n$, let $\mathrm{end}(\sigma)$ denote the number of consecutive 1's at the end of σ. We call σ *balanced* if σ is of the form $\sigma = 0^a1^a0^b1^b\cdots 0^s1^s$, where the notation 0^a1^a is shorthand for $\underbrace{00\cdots0}_{a\text{ times}}\underbrace{11\cdots1}_{a\text{ times}}$. To each $\sigma\in\mathcal{C}_n$ we associate a balanced word $b(\sigma)$ by a procedure we call the *balancing algorithm*:

(A) Say $\mathrm{end}(\sigma)=s$. Starting at the end of σ, move left until you find the s^{th} 0, say in spot σ_j. Slide this 0 and the other $s-1$ 0's you passed to the right until they

abut the final s 1's at the end of σ, passing over any intermediary 1's you encounter along the way. We now have the word $b_1(\sigma) = \sigma_1 \cdots \sigma_{j-1} 1^{2n-2s-j+1} 0^s 1^s$.
(B) Apply step A to the first $2n-2s$ letters of $b_1(\sigma)$ (leaving the $0^s 1^s$ part alone) resulting in a word $b_2(\sigma) = \sigma_1 \cdots \sigma_{i-1} 1^{2n-2s-2r-i+1} 0^r 1^r 0^s 1^s$, where $r = \mathrm{end}(\sigma_1 \cdots \sigma_{j-1} 1^{2n-2s-j+1})$. Then apply step A to the first $2n-2s-2r$ letters of $b_2(\sigma)$, resulting in $b_3(\sigma)$, etc.. Iterate this process until a balanced word is obtained; call this $b(\sigma)$.
For example, if $\sigma = 001011000010110111$, we have $b_1(\sigma) = 001011000111000111 = b_2(\sigma)$ and $b_3(\sigma) = 010011000111000111 = b(\sigma)$.

We now define our q,t-Catalan number, $F_n(q,t)$, as

$$F_n(q,t) := \sum_{\sigma \in \mathcal{C}_n} q^{\binom{n}{2} - \mathrm{inv}\sigma} t^{\frac{1}{2}\mathrm{maj} b(\sigma)}.$$

Conjecture 1. *For n a positive integer,*

$$F_n(q,t) = C_n(q,t).$$

Conjecture 1 was discovered after a prolonged investigation of tables of $C_n(q,t)$. It has been verified for $n \leq 14$ by a Maple program.

The distribution of the statistics for $n = 4$ is given in Table I.

σ	$6 - \mathrm{inv}\sigma$	$\frac{1}{2}\mathrm{maj} b(\sigma)$	$b(\sigma)$
00001111	6	0	00001111
00010111	5	1	01000111
00011011	4	2	00110011
00011101	3	3	00011101
00101101	2	4	01001101
00101011	3	2	00110011
00110011	2	2	00110011
00110101	1	5	00110101
01000111	3	1	01000111
01001011	2	3	01010011
01001101	1	4	01001101
00100111	4	1	01000111
01010011	1	3	01010011
01010101	0	6	01010101

Table I: The statistics for $n = 4$.

Define

$$F_{n,s}(q,t) := \sum_{\substack{\sigma \in \mathcal{C}_n \\ \mathrm{end}(\sigma) = s}} q^{\binom{n}{2} - \mathrm{inv}\sigma} t^{\frac{1}{2}\mathrm{maj} b(\sigma)}, \qquad \text{with } F_{n,0}(q,t) := \chi(n=0).$$

In Section 2 we prove a surprising recurrence relation for $F_{n,s}(q,t)$.

Theorem 1.

$$F_{n,s}(q,t) = \sum_{r=0}^{n-s} F_{n-s,r}(q,t) \begin{bmatrix} r+s-1 \\ r \end{bmatrix} q^{\binom{s}{2}} t^{n-s}.$$

As a corollary of this we show

$$q^{\binom{n}{2}} F_n(q,q^{-1}) = q^{\binom{n}{2}} C_n(q,q^{-1}),$$

giving further support to Conjecture 1.

From Garsia and Haiman's rational function definition it is obvious that $C_n(q,t) = C_n(t,q)$. Although the author has been unable to show that $F_n(q,t) = F_n(t,q)$, in Section 3 we do prove that $F_n(q,1) = F_n(1,q)$.

One of the main tools that Garsia and Haiman use to prove identities for special cases of $C_n(q,t)$ is q-Lagrange inversion. In Section 4 we modify a proof of Haiman, of a q-Lagrange inversion formula based on work of Garsia and Gessel, to derive a q-analogue of the general Lagrange inversion formula. As a corollary we obtain an infinite series identity involving $F_{n,s}(q^{-1},1)$.

Notation: LHS and RHS are abbreviations for "left-hand-side" and "right-hand-side", respectively.

2. A Recurrence for $F_{n,s}(q,t)$

In this section we make use of a geometric representation of Catalan words known as Dyck paths. These are lattice paths in the first quadrant of the xy-plane, connecting $(0,0)$ to (n,n), which consist of NORTH and EAST steps and remain weakly above the diagonal (see [7, pp. 201-202] for a more detailed description of Dyck paths). We let SQ(n) denote the square lattice with corners $(0,0)$ and (n,n), and D(σ) the Dyck path corresponding to the Catalan word σ. The NORTH steps of D(σ) correspond to 0's in σ, the EAST steps to 1's. We call the squares below D(σ) and strictly above the diagonal the *area squares* of σ. Note that the number of these squares is is $\binom{n}{2} - \text{inv}(\sigma)$.

The path D$(b(\sigma))$ has a simple description in terms of what could be called "the drawing game of ricochet". Let R denote the closed region consisting of the path D(σ) together with the area squares of σ. In the example of Fig. 1 the area squares are shaded and the path $D(\sigma)$ is in bold.

To play ricochet, start at the upper right corner of SQ(n) and trace a line straight left as far as you can without leaving R, at which point you "ricochet" and trace straight down. Once you hit the diagonal and are about to leave R again you ricochet left, iterating these steps until you finally arrive at $(0,0)$. The line you have traced out is D$(b(\sigma))$.

Lemma 2.1 answers the following question: Given a balanced word $w \in \mathcal{C}_n$, what is the sum of $q^{\binom{n}{2}-\text{inv}\sigma}$, summed over all σ with $b(\sigma) = w$? In the process of proving this lemma we will also prove Theorem 1.

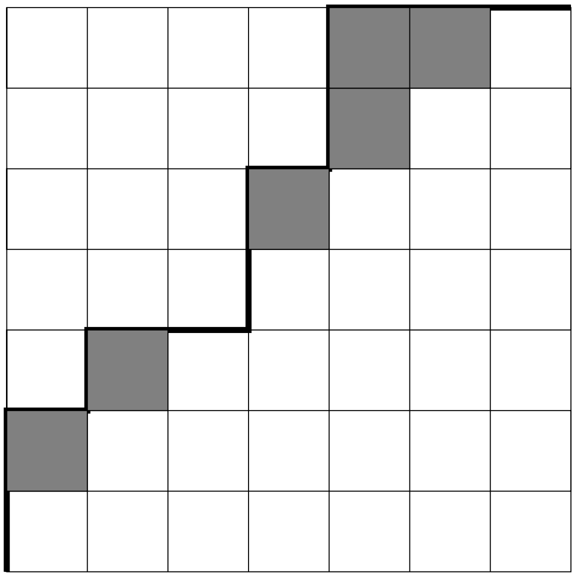

FIGURE 1: THE PATH D(00101100100111) CONTAINED IN SQ(7), TOGETHER WITH THE AREA SQUARES.

Lemma 2.1. *Let $w \in \mathcal{C}_n$ be a balanced word, say $w = 0^{\alpha_1}1^{\alpha_1}0^{\alpha_2}1^{\alpha_2}\cdots 0^{\alpha_k}1^{\alpha_k}$, with $\alpha_1 + \alpha_2 + \ldots + \alpha_k = n$, $\alpha_i > 0$ a composition of n into k positive integers. Then*

$$\sum_{\substack{\sigma\in\mathcal{C}_n \\ b(\sigma)=w}} q^{\binom{n}{2}-\mathrm{inv}\sigma} = q^{\binom{\alpha_1}{2}+\binom{\alpha_2}{2}+\ldots+\binom{\alpha_k}{2}}$$

$$\times \begin{bmatrix} \alpha_1+\alpha_2-1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} \alpha_2+\alpha_3-1 \\ \alpha_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_{k-1}+\alpha_k-1 \\ \alpha_{k-1} \end{bmatrix}. \quad (4)$$

We now give two proofs of both Lemma 2.1 and Theorem 1. Our first proof is rather brief, relying heavily on geometric intuition, while our second is algebraic.
First Proof of Lemma 2.1 *and Theorem* 1*:* The game of ricochet implies that $b(\sigma)$ will end in $0^r1^r0^s1^s$ if and only if the squares just to the left of each downward ricochet (indicated by 0's in Fig. 2) are not below D(σ), while all the squares to the right of these squares and in the same row (indicated by x's in Fig. 2) are.

In addition the area squares of σ can contain any arbitrary subset, in the shape of a partition, of the rectangular region of size $r \times s - 1$ above the horizontal step of length r indicated by the dotted lines. It is well known [2] that the area statistic for a rectangle of this size generates

$$\begin{bmatrix} r+s-1 \\ r \end{bmatrix}.$$

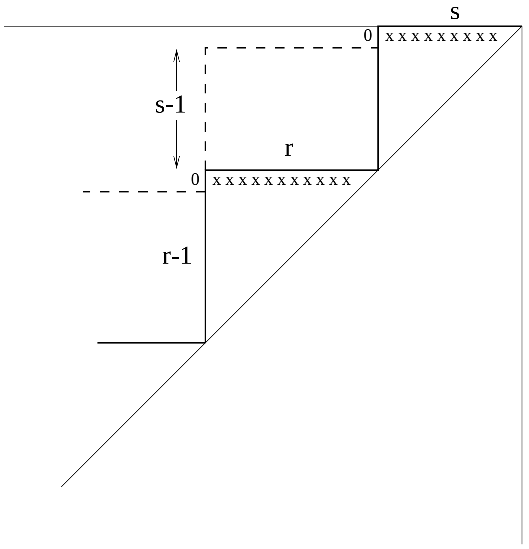

FIGURE 2: PATHS FOR WHICH $b(\sigma)$ ENDS IN $0^r1^r0^s1^s$.

The number of squares below the last horizontal step of $b(\sigma)$ is $\binom{s}{2}$, and it is now clear how the product in Lemma 2.1 arises. Theorem 1 also follows, since the area squares not accounted by the term

$$q^{\binom{s}{2}} \begin{bmatrix} r+s-1 \\ r \end{bmatrix}.$$

correspond to the area squares of an arbitrary word in $\mathcal{C}_{n-s}$ ending in r ones. The factor t^{n-s} accounts for the change in $\frac{1}{2}\mathrm{maj}b(\sigma)$. $\square$

Second Proof of Lemma 2.1 *and Theorem* 1*:* In view of the balancing algorithm, $b(\sigma) = w$ if and only if σ is of the following general form:

$$\sigma = 0^{\alpha_1}\beta_2 0\beta_3 \cdots 0\beta_{k-2}0\beta_{k-1}0\beta_k 01^{\alpha_k}, \tag{5}$$

where β_i is any multiset permutation of $\{0^{\alpha_i-1}1^{\alpha_{i-1}}\}$, $2 \le i \le k$. It follows that the minimal power of invσ among those σ satisfying $b(\sigma) = w$ is

$$\alpha_1(\alpha_3+\alpha_4+\ldots+\alpha_k+1)+\alpha_2(\alpha_4+\ldots+\alpha_k+1)+\ldots+\alpha_{k-2}(\alpha_k+1)+\alpha_{k-1}$$

$$= \binom{\alpha_1+\alpha_2+\ldots+\alpha_k}{2} - \sum_{i=1}^{k}\binom{\alpha_i}{2} - \alpha_1(\alpha_2-1)-\alpha_2(\alpha_3-1)-\ldots-\alpha_{k-1}(\alpha_k-1).$$

When we sum over the β_i, by (3) the term

$$\prod_{i=1}^{k-1}\begin{bmatrix}\alpha_{i-1}+\alpha_i-1 \\ \alpha_{i-1}\end{bmatrix}$$

will account for inversion pairs which occur within the same β_i. Thus

$$\sum_{\substack{\sigma\in\mathcal{C}_n\\ b(\sigma)=w}} q^{\mathrm{inv}\sigma} = q^{\binom{n}{2}-\sum_{i=1}^k\binom{\alpha_i}{2}+\alpha_1+\ldots+\alpha_{k-1}-(\alpha_1\alpha_2+\ldots+\alpha_{k-1}\alpha_k)}\prod_{i=1}^{k-1}\begin{bmatrix}\alpha_{i-1}+\alpha_i-1\\ \alpha_{i-1}\end{bmatrix}.$$

Replacing q by q^{-1} above, multiplying by $q^{\binom{n}{2}}$, and using

$$\begin{bmatrix}\alpha_{i-1}+\alpha_i-1\\ \alpha_{i-1}\end{bmatrix}_{q^{-1}} = \begin{bmatrix}\alpha_{i-1}+\alpha_i-1\\ \alpha_{i-1}\end{bmatrix} q^{-\alpha_{i-1}(\alpha_i-1)}$$

proves the lemma. To prove Theorem 1 start with

$$F_{n,s}(q,t) = \sum_{\alpha_1+\ldots+\alpha_{k-1}=n-s} q^{\binom{\alpha_1}{2}+\ldots+\binom{\alpha_{k-1}}{2}+\binom{s}{2}}\times$$
$$\begin{bmatrix}\alpha_1+\alpha_2-1\\ \alpha_1\end{bmatrix}\cdots\begin{bmatrix}\alpha_{k-2}+\alpha_{k-1}-1\\ \alpha_{k-2}\end{bmatrix}\begin{bmatrix}\alpha_{k-1}+s-1\\ \alpha_{k-1}\end{bmatrix} t^{\alpha_1(k-1)+\alpha_2(k-2)+\ldots+\alpha_{k-1}}$$

$$= \sum_{r=0}^{n-s}\Bigg(\sum_{\alpha_1+\ldots+\alpha_{k-2}=n-s-r} q^{\binom{\alpha_1}{2}+\ldots+\binom{\alpha_{k-2}}{2}+\binom{r}{2}} t^{\alpha_1(k-2)+\ldots+\alpha_{k-2}}$$
$$\times\begin{bmatrix}\alpha_1+\alpha_2-1\\ \alpha_1\end{bmatrix}\cdots\begin{bmatrix}\alpha_{k-2}+r-1\\ \alpha_{k-2}\end{bmatrix}\Bigg)\begin{bmatrix}r+s-1\\ r\end{bmatrix} q^{\binom{s}{2}} t^{n-s}$$

$$= \sum_{r=0}^{n-s} F_{n-s,r}(q,t)\begin{bmatrix}r+s-1\\ r\end{bmatrix} q^{\binom{s}{2}} t^{n-s}. \qquad \square$$

Theorem 2.2. *For $1\le s\le n$,*

$$q^{\binom{n}{2}} F_{n,s}(q,q^{-1}) = \frac{[s]}{[n]}\begin{bmatrix}2n-s-1\\ n-s\end{bmatrix} q^{(s-1)n}.$$

Pf: Since $F_{n,n}(q,t) = q^{\binom{n}{2}}$, Theorem 2.2 holds for $s=n$. If $1\le s<n$, we start with Theorem 1 and then use induction on n;

$$q^{\binom{n}{2}} F_{n,s}(q,q^{-1}) = q^{\binom{n}{2}} q^{-\binom{n-s}{2}}\sum_{r=1}^{n-s} q^{\binom{n-s}{2}} F_{n-s,r}(q,q^{-1}) q^{\binom{s}{2}-(n-s)}\begin{bmatrix}r+s-1\\ r\end{bmatrix}$$

$$= q^{\binom{n}{2}+\binom{s}{2}-(n-s)} q^{-\binom{n-s}{2}}\sum_{r=1}^{n-s}\begin{bmatrix}r+s-1\\ r\end{bmatrix}\frac{[r]}{[n-s]}\begin{bmatrix}2(n-s)-r-1\\ n-s-r\end{bmatrix} q^{(r-1)(n-s)}$$

$$= q^{(s-1)n} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r \end{bmatrix} \frac{[r]}{[n-s]} \begin{bmatrix} 2(n-s)-2-(r-1) \\ n-s-1-(r-1) \end{bmatrix} q^{(r-1)(n-s)}$$

$$= q^{(s-1)n} \frac{[s]}{[n-s]} \sum_{\substack{u=0 \\ (u=r-1)}}^{n-s-1} \begin{bmatrix} s+u \\ u \end{bmatrix} \begin{bmatrix} 2(n-s)-2-u \\ n-s-1-u \end{bmatrix} q^{u(n-s)}$$

$$= q^{(s-1)n} \frac{[s]}{[n-s]} \begin{bmatrix} 2n-2s-2 \\ n-s-1 \end{bmatrix} \sum_{u=0}^{n-s-1} \frac{(q^{s+1})_u}{(q)_u} \frac{(q^{n-s-u})_u}{(q^{2n-2s-1-u})_u} q^{u(n-s)}$$

$$= q^{(s-1)n} \frac{[s]}{[n-s]} \begin{bmatrix} 2n-2s-2 \\ n-s-1 \end{bmatrix} \sum_{u=0}^{n-s-1} \frac{(q^{s+1})_u}{(q)_u} \frac{(q^{1+s-n})_u}{(q^{2+2s-2n})_u} q^{u}$$

$$= q^{(s-1)n} \frac{[s]}{[n-s]} \begin{bmatrix} 2n-2s-2 \\ n-s-1 \end{bmatrix} \frac{(q^{1+s-2n})_{n-s-1}}{(q^{2+2s-2n})_{n-s-1}} q^{(s+1)(n-s-1)} \qquad [8, \text{p. } 236]$$

$$= q^{(s-1)n+(s+1)(n-s-1)} \frac{[s]}{[n-s]} \begin{bmatrix} 2n-2s-2 \\ n-s-1 \end{bmatrix}$$
$$\times \frac{[2n-s-1][2n-s-2]\cdots[2n-s-1-(n-s-1)+1]}{[2n-2s-2][2n-2s-3]\cdots[2n-2s-2-(n-s-1)+1]} q^{\text{pow}}, \qquad (6)$$

where

$$\text{pow} = 2n-2s-2+2n-2s-3+\ldots+n-s-(2n-2s-1+2n-2s-2+\ldots+n+1)$$

$$= n+n-1+\ldots+n-s-(2n-s-1+2n-s-2+\ldots+2n-2s-1)$$

$$= n(s+1) - \binom{s+1}{2} - \left(2n(s+1)-(s+1)^2 - \binom{s+1}{2}\right)$$

$$= n(s+1) - 2n(s+1) + (s+1)^2 = -(n-s-1)(s+1).$$

Thus (6) equals

$$\frac{[s]}{[n-s]} \begin{bmatrix} 2n-s-1 \\ n-s-1 \end{bmatrix} q^{(s-1)n} = \frac{[s]}{[n]} \begin{bmatrix} 2n-s-1 \\ n-s \end{bmatrix} q^{(s-1)n}. \qquad \square$$

Corollary 2.3. *For $n \geq 1$,*

$$q^{\binom{n}{2}} F_n(q, q^{-1}) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Pf: By Theorem 2.2,

$$q^{\binom{n}{2}} \sum_{s=1}^{n} F_{n,s}(q, q^{-1}) = \sum_{s=1}^{n} \frac{[s]}{[n]} \begin{bmatrix} 2n-s-1 \\ n-s \end{bmatrix} q^{(s-1)n}$$

$$= \sum_{\substack{u=0 \\ (u=s-1)}}^{n-1} \frac{[u+1]}{[n]} \begin{bmatrix} 2n-2-u \\ n-1-u \end{bmatrix} q^{nu}$$

$$= \frac{1}{[n]} \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix} \sum_{u=0}^{n-1} [u+1] \frac{[n-u][n-u+1]\cdots[n-1]}{[2n-1-u][2n-u]\cdots[2n-2]} q^{nu}$$

$$= \frac{1}{[n]} \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix} \sum_{u=0}^{n-1} \frac{(q^2)_u (q^{1-n})_u}{(q)_u (q^{2-2n})_u} q^u$$

$$= \frac{1}{[n]} \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix} \frac{(q^{-2n})_{n-1}}{(q^{2-2n})_{n-1}} q^{2(n-1)} \qquad [8, \text{ p. } 236]$$

$$= \frac{1}{[n]} \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix} \frac{[2n][2n-1]}{[n+1][n]} q^{-(2n+2n-1)+(n+1+n)+2(n-1)}$$

$$= \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}. \qquad \square$$

3. Symmetry

In this section we use Lemma 2.1 and a combinatorial argument to prove the following partial symmetry result.

Theorem 3.1. *For $n \geq 0$,*

$$F_n(q, 1) = F_n(1, q).$$

Pf: Given $\sigma \in \mathcal{C}_n$, let $\gamma_i(\sigma)$ denote the number of area squares of σ in the i^{th} row from the top of SQ(n). In the example of Fig. 1, $(\gamma_1(\sigma), \gamma_2(\sigma), \ldots, \gamma_7(\sigma)) = (2, 1, 1, 0, 1, 1, 0)$. Note that $\gamma_n(\sigma) = 0$ for all σ.

From the geometry it is clear that $\gamma_i(\sigma) - \gamma_{i+1}(\sigma) \leq 1, \quad 1 \leq i \leq n-1$. In fact, a sequence $\pi_1\pi_2\cdots\pi_n$ of n nonnegative integers is the γ-sequence $\gamma_1(\sigma)\cdots\gamma_n(\sigma)$ of some $\sigma \in \mathcal{C}_n$ if and only if $\pi_1\cdots\pi_n$ doesn't have a 2-descent (i.e. a value of i for which $\pi_i - \pi_{i+1} > 1$) and $\pi_n = 0$. Let M be a multiset of n nonnegative integers with largest element $k-1$, say $M = \{(k-1)^{\alpha_1}(k-2)^{\alpha_2}\cdots 1^{\alpha_{k-1}}0^{\alpha_k}\}$,

$\alpha_1 + \alpha_2 + \ldots + \alpha_k = n$. How many multiset permutations are there with no 2-descents and which end in 0? For this number to be nonzero, clearly all the α_i must be positive. We can construct such a multiset permutation by first arranging the $\alpha_{k-1} - 1$'s and $\alpha_k - 0$'s in any of $\binom{\alpha_{k-1}+\alpha_k-1}{\alpha_{k-1}}$ patterns, so that each pattern ends in 0. We can then insert the $\alpha_{k-2} - 2$'s into a given pattern in any of $\binom{\alpha_{k-2}+\alpha_{k-1}-1}{\alpha_{k-2}}$ ways, so that any 2 is followed by either another 2 or a 1. Continuing in this way we see that

$$\prod_{i=1}^{k-1} \binom{\alpha_i + \alpha_{i+1} - 1}{\alpha_i}$$

is the answer to our question. For each of these permutations the corresponding Catalan word σ satisfies $\binom{n}{2} - \text{inv}\sigma = \alpha_1(k-1) + \alpha_2(k-2) + \ldots + \alpha_{k-1}$ and so

$$F_n(q,1) = \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=n \\ \alpha_i \geq 1}} q^{\alpha_1(k-1)+\alpha_2(k-2)+\ldots+\alpha_{k-1}} \prod_{i=1}^{k-1} \binom{\alpha_i + \alpha_{i+1} - 1}{\alpha_i}.$$

Since $\alpha_1(k-1) + \alpha_2(k-2) + \ldots + \alpha_{k-1} = \frac{1}{2}\text{maj}(0^{\alpha_1}1^{\alpha_1}\cdots 0^{\alpha_k}1^{\alpha_k})$, Theorem 3.1 follows from the $q = 1$ case of Lemma 2.1. $\square$.

Remark 1*:* One could hope to use the above correspondence to prove $F_n(q,t) = F_n(t,q)$. Unfortunately, it is not weight-preserving, as the following example shows.

Let $n = 4$, $\alpha = 1 + 1 + 2$, $w = 01010011$. There are two σ satisfying $b(\sigma) = w$, namely $\sigma_1 = 01001011$ and $\sigma_2 = 01010011$. We have

$$q^{6-\text{inv}\sigma_1} t^{\frac{1}{2}\text{maj}b(\sigma_1)} + q^{6-\text{inv}\sigma_2} t^{\frac{1}{2}\text{maj}b(\sigma_2)} = q^2t^3 + qt^3. \tag{7}$$

On the other hand, there are also two relevant γ-sequences, namely the permutations of the multiset $\{2\,1\,0^2\}$, with corresponding words $\sigma_3 = 01000111$ and $\sigma_4 = 00011101$. We have

$$q^{6-\text{inv}\sigma_3} t^{\frac{1}{2}\text{maj}b(\sigma_3)} + q^{6-\text{inv}\sigma_4} t^{\frac{1}{2}\text{maj}b(\sigma_4)} = q^3t + q^3t^3, \tag{8}$$

and so interchanging q and t in (7) doesn't give (8).

Remark 2*:* Let $d_{n,u,v}$ denote the coefficient of $q^u t^v$ in $F_n(q,t)$. By Lemma 2.1,

$$d_{n,u,v} = \sum_{k=1}^{n} \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=n \\ \alpha_1(k-1)+\alpha_2(k-2)+\ldots+\alpha_{k-1}=v}} \text{the coefficient of } q^u \text{ in}$$

$$q^{\binom{\alpha_1}{2}+\ldots+\binom{\alpha_k}{2}} \prod_{i=1}^{k-1} \begin{bmatrix} \alpha_i + \alpha_{i+1} - 1 \\ \alpha_i \end{bmatrix}.$$

Showing $F_n(q,t) = F_n(t,q)$ is thus equivalent to showing the RHS above is symmetric in u and v.

4. q-Lagrange Inversion and $F_{n,s}(q,1)$.

Given $\sigma \in \mathcal{C}_n$, let $\lambda(\sigma) = \lambda_1(\sigma)\lambda_2(\sigma)\cdots$ denote the partition consisting of the horizontal step lengths of $\mathrm{D}(\sigma)$ (i.e. the lengths of the blocks of consecutive 1's in σ, arranged in nonincreasing order). Let e_k denote the k^{th} elementary symmetric function in some set of variables, and $e_{\lambda(\sigma)} := \prod_{i\ge 1} e_{\lambda_i(\sigma)}$. Define $H(z)$ via the equation $1/H(-z) := \sum_{k=0}^{\infty} e_k z^k$, and let μ_i, $i \ge 1$ be variables.

The $\mu_i = 1$, $i \ge 1$ case of the following lemma reduces to a q-Lagrange inversion formula proven in [10,pp. 47-48]. Haiman also includes a discussion of the connection of that inversion formula to work of Andrews, Garsia, and Gessel [1], [6], [9]. Further background on their work is contained in [13].

Lemma 4.1. *Define $h_n^*(\mu, q)$, $n \ge 0$ via the equation*

$$\sum_{k=0}^{\infty} e_k \mu_k z^k = \sum_{k=0}^{\infty} q^{-\binom{k}{2}} h_k^*(\mu, q) z^k H(-q^{-1}z)H(-q^{-2}z)\cdots H(-q^{-k}z). \tag{9}$$

Then for $n \ge 1$, $h_n^*(\mu, q)$ has the explicit expression

$$h_n^*(\mu, q) = \sum_{\sigma \in \mathcal{C}_n} q^{\binom{n}{2} - \mathrm{inv}\,\sigma} e_{\lambda(\sigma)} \mu_{\mathrm{end}(\sigma)}. \tag{10}$$

Pf: Our proof follows Haiman's proof of the $\mu_i = 1$ case closely. Set $H^*(z, \alpha; q) := \sum_{n=0}^{\infty} h_n^*(\mu, q) z^k$, and $H^*(z; q) := \sum_{n=0}^{\infty} h_n^*(q) z^k$, where $h_n^*(q)$ is obtained by setting $\mu_i = 1$, $i \ge 1$ in $h_n^*(\mu; q)$. From [6] we have that if

$$(F \circ_q G)(z) = \sum_n f_n G(z)G(qz)\cdots G(q^{n-1}z)$$

denotes the q-functional composition of F and G, where $F = \sum_n f_n z^n$, then

$$F \circ_q G = z \qquad \text{and} \qquad G \circ_{q^{-1}} F = z$$

are equivalent to each other and also to

$$(\Phi \circ_{q^{-1}} F) \circ_q G = \Phi = (\Phi \circ_q G) \circ_{q^{-1}} F \qquad \text{for all } \Phi.$$

Let $\Phi = H^*(zq, \mu; q)$, $F = zH(-z)$, and $G = zH^*(z; q)$. Replacing z by zq in (9) gives

$$\begin{aligned}\sum_{k=0}^{\infty} e_k \mu_k q^k z^k &= \sum_{k=0}^{\infty} q^k h_k^*(\mu, q) z H(z) z q^{-1} H(-q^{-1}z)\cdots z q^{1-k} H(-q^{1-k}z) \\ &= \Phi \circ_{q^{-1}} F.\end{aligned}$$

The $\mu_i = 1$, $i \geq 1$ case of Lemma 4.1 (recall this is proven in [10]) implies that $G \circ_{q^{-1}} F = z$, and so

$$\Phi = (\sum_{k=0}^{\infty} e_k \mu_k q^k z^k) \circ_q G. \tag{11}$$

Comparing coefficients of z^n in (11) and simplifying we get

$$h_n^*(\mu, q) = \sum_{k=1}^{n} q^{\binom{k}{2}} e_k \mu_k \sum_{\substack{n_1+\ldots+n_k=n-k \\ n_i \geq 0}} \prod_{i=1}^{k} q^{(i-1)n_i} h_{n_i}^*(q). \tag{12}$$

To show the RHS of (10) equals the RHS of (12) use the "factorization of Dyck paths" as discussed in [10]. The terms multiplied by μ_k correspond to $\sigma \in \mathcal{C}_n$ with $\mathrm{end}(\sigma) = k$. $\square$

If in Lemma 4.1 we begin by setting $e_k = 1$, $k \geq 1$, $\mu_s = 1$, $\mu_j = 0$ for $j \neq s$ and then replace q by q^{-1} and z by z/q, we get the following.

Corollary 4.2. *For $s \geq 1$,*

$$z^s = \sum_{n \geq s} q^{\binom{n}{2}-n+s} F_{n,s}(q^{-1}, 1) z^n (1-z)(1-qz)\cdots(1-q^{n-1}z).$$

Lemma 4.1 is a q-analogue of the general Lagrange inversion formula [3, p. 629]

$$f(x) = f(0) + \sum_{k=1}^{\infty} \frac{x^k}{k!\phi(x)^k} \left[\frac{d^{k-1}}{dx^{k-1}} (f'(x)\phi^k(x)) \right]_{x=0}, \tag{13}$$

where ϕ and f are analytic in a neighborhood of 0, with $\phi(0) \neq 0$. To see why, set $f(x) = f(0) + \sum_{k=1}^{\infty} \mu_k x^k$ and $\phi = \frac{1}{H(-x)} = \sum_{k=0}^{\infty} e_k x^k$ in (13) to get

$$\begin{aligned}
\sum_{k=0}^{\infty} \mu_k x^k &= \sum_{k=1}^{\infty} \frac{x^k}{k!} H(-x)^k \sum_{p \geq 1} p\mu_p (k-1)! \times \text{the coefficient of } x^{k-p} \text{ in } (\sum_{n=0}^{\infty} e_n x^n)^k \\
&= \sum_{k=1}^{\infty} x^k \frac{H(-x)^k}{k} \sum_{p \geq 1} p\mu_p \sum_{\substack{j_1+j_2+\ldots+j_k=k-p \\ j_i \geq 0}} e_{j_1} e_{j_2} \cdots e_{j_k} \\
&= \sum_{k=1}^{\infty} x^k \frac{H(-x)^k}{k} \sum_{p \geq 1} p\mu_p \sum_{|\beta|=k-p} e_\beta \binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots},
\end{aligned}$$

where $m_i(\beta)$, $i \geq 1$ is the multiplicity of i in the partition β, $\ell(\beta) = \sum_{i \geq 1} m_i(\beta)$, and $|\beta| = \sum_{i \geq 1} \beta_i$.

The equivalence of the above equation to the $q = 1$ case of Lemma 4.1 (with μ_k replaced by μ_k/e_k, $k \geq 1$) will follow if we can show that

$$\sum_{\sigma \in \mathcal{C}_k} e_{\lambda(\sigma)} \frac{\mu_{\mathrm{end}(\sigma)}}{e_{\mathrm{end}(\sigma)}} = \sum_{p \geq 1} \frac{p\mu_p}{k} \sum_{|\beta|=k-p} e_\beta \binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots}.$$

This identity is equivalent to Lemma 4.3 below.

Lemma 4.3. *Given integers k, p with $k > p \geq 1$, and a partition β of $k-p$,*

$$\sum_{\substack{\sigma \in \mathcal{C}_k \\ \mathrm{end}(\sigma)=p \\ \lambda(\sigma)-\mathrm{end}(\sigma)=\beta}} 1 = \frac{p}{k}\binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots}, \tag{14}$$

where $\lambda(\sigma) - end(\sigma)$ denotes the partition $\lambda(\sigma)$ with a part of size $end(\sigma)$ removed.

Pf: Let $g(\beta, p, k)$ denote the LHS above, i.e. the number of Catalan words whose path $\mathrm{D}(\sigma)$ ends in a horizontal step of length p and whose other steps are the parts of the partition β. If a given $\sigma \in \mathcal{C}_k$ counted by $g(\beta, p, k)$ has more than one zero immediately preceding the p ones at the end of σ, remove one of these zeros and one of the p ones to get a σ' counted by $g(\beta, p-1, k-1)$. If instead this σ has exactly one zero preceding the p ones, and a block of π 1's preceding this 0, remove the 0 and one of the p ones to get a σ' counted by $g(\beta - \pi, \pi + p - 1, k - 1)$. This procedure shows that g satisfies the recurrence

$$g(\beta, p, k) = g(\beta, p-1, k-1) + \sum_{\substack{\pi \\ m_\pi(\beta)>0}} g(\beta - \pi, \pi + p - 1, k - 1).$$

Proceeding by induction on k,

$$\begin{aligned}
g(\beta, p, k) &= \frac{p-1}{k-1}\binom{k-1}{k-1-\ell(\beta), m_1(\beta), m_2(\beta), \ldots} \\
&\quad + \sum_{\substack{\pi \\ m_\pi(\beta)>0}} \frac{\pi + p - 1}{k-1}\binom{k-1}{k-1-(\ell(\beta)-1), m_1(\beta), \ldots, m_\pi(\beta)-1, \ldots}
\end{aligned}$$

$$= \frac{p}{k}\binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots}\left(\frac{p-1}{p}\frac{(k-\ell(\beta))}{(k-1)} + \sum_\pi \frac{(\pi+p-1)m_\pi(\beta)}{p(k-1)}\right)$$

$$\begin{aligned}
&= \frac{p}{k}\binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots}\frac{1}{p(k-1)} \\
&\qquad \times \left((p-1)k - (p-1)\ell(\beta) + \sum_\pi \pi m_\pi(\beta) + (p-1)\ell(\beta)\right) \\
&= \frac{p}{k}\binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots}\frac{1}{p(k-1)}((p-1)k + k - p) \\
&= \frac{p}{k}\binom{k}{k-\ell(\beta), m_1(\beta), m_2(\beta), \ldots}. \qquad \square
\end{aligned}$$

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