# $q$-ROOK POLYNOMIALS AND MATRICES OVER FINITE FIELDS

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Abstract. Connections between $q$-rook polynomials and matrices over finite fields are exploited to derive a new statistic for Garsia and Remmel's $q$-hit polynomial. Both this new statistic *mat* and another statistic for the $q$-hit polynomial $\xi$ recently introduced by Dworkin are shown to induce different multiset Mahonian permutation statistics for any Ferrers board. In addition, for the triangular boards they are shown to generate different families of Euler-Mahonian statistics. For these boards the $\xi$ family includes Denert's statistic *den*, and gives a new proof of Foata and Zeilberger's Theorem that $(exc, den)$ is equi-distributed with $(des, maj)$. The *mat* family appears to be new. A proof is also given that the $q$-hit polynomials are symmetric and unimodal.

## 1. Introduction

Notation: LHS and RHS are abbreviations for "left-hand-side" and "right-hand-side", respectively. $\mathbb{N}$ denotes the nonnegative integers, $\mathbb{Z}$ the integers, $\mathbb{P}$ the positive integers, and $\mathbb{F}_q$ a finite field with $q$ elements.

A *board* is a subset of an $n \times n$ grid of squares. We label the squares of the grid with the same (row,column) coordinates as the squares of an $n \times n$ matrix; the lower-left-hand-corner square has label $(n, 1)$, etc. A *Ferrers board* is a board with the property that $(i, j) \in B \implies (k, p) \in B$ for $1 \le k \le i$ and $j \le p \le n$. Garsia and Remmel [GaRe] introduced the following $q$-rook polynomial;

$$R_k(B) := \sum_C q^{\mathrm{inv}(C,B)}, \tag{1}$$

where the sum is over all placements $C$ of $k$ non-attacking rooks on the squares of the Ferrers board $B$. Non-attacking means no two rooks are in the same column, and no two are in the same row. To calculate the statistic $\mathrm{inv}(C, B)$, cross out all squares which either contain a rook, or are above or to the right of any rook. The number of squares of $B$ not crossed out is $\mathrm{inv}(C, B)$ (see Fig. 1).

Garsia and Remmel showed that the $R_k$ enjoy many of the same properties as the famous rook numbers $r_k$ introduced by Riordan and Kaplansky [KaRi],[Rio]. For example,

$$\sum_{k=0}^{n} [x][x-1]\cdots[x-k+1] R_{n-k}(B) = \prod_{i=1}^{n} [x + c_i - i + 1], \tag{2}$$

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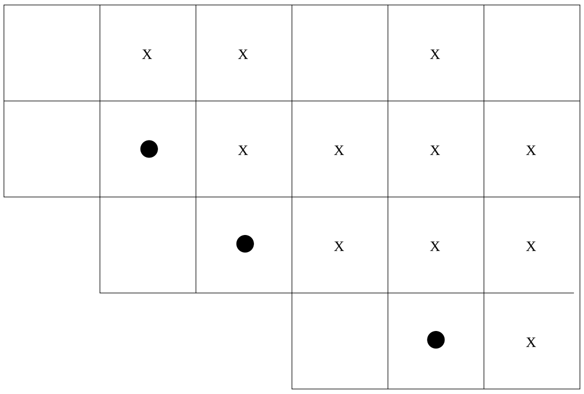

FIGURE 1. A placement of 3 rooks with *inv* statistic 6.

where $[x] := (1-q^x)/(1-q)$ and $c_i :=$ the number of squares in the $i$th column of $B$. Note that our definition of a board requires $c_n \leq n$ (such boards are called *admissible* in the literature). This assumption holds throughout the article, except as noted in Theorem 7. When $q \to 1$ in (2) we get a classic result of Goldman, Joichi, and White [GJW]. As noted by Garsia and Remmel, an interesting consequence of (2) is that two Ferrers boards have the same rook numbers if and only if they have the same $q$-rook numbers, since both of these are determined by the multiset whose elements are the shifted column heights $c_i(B) - i + 1$.

Letting $[k]! := \prod_{1 \leq i \leq k}[i]$, and defining $T_k(B)$ via

$$\sum_{k=0}^{n}[k]!R_{n-k}(B)\prod_{i=k+1}^{n}(x-q^i) = \sum_{k=0}^{n} T_k x^k, \tag{3}$$

another result of Garsia and Remmel is that

$$T_k(B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\operatorname{stat}(C,B)},$$

for some statistic $\operatorname{stat}(C,B) \in \mathbb{N}$. In the sum above $C$ is a placement of $n$ non-attacking rooks on the $n \times n$ grid, with exactly $k$ on $B$. For $q = 1$ it reduces to $t_k(B)$, the hit number of Riordan and Kaplansky, which equals the number of permutations which "hit" $k$ of the "forbidden positions" represented by the squares of $B$.

Garsia and Remmel gave a recursive definition of $\operatorname{stat}(C,B)$ , and left it as an open problem to determine a method of generating $T_k(B)$ directly from the rook placements (as in the definition of $R_k(B)$). This problem has recently been solved by M. Dworkin [Dwo], who shows that

$$T_k(B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\xi(C,B)},$$

where $\xi(C,B)$ is calculated by the following procedure.

First place a bullet under each rook, and an $x$ to the right of any rook. Next, for each rook on $B$, place a circle in the empty cells of $B$ that are below it in the column. Then for each

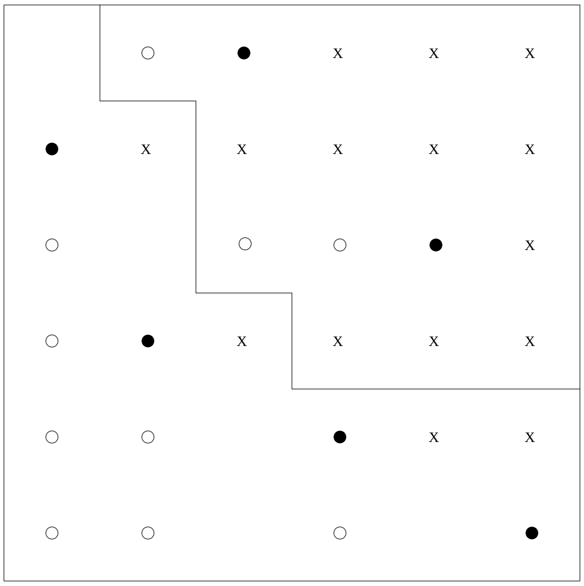

FIGURE 2. A placement of 6 rooks with 2 rooks on $B$: $\xi = 10$.

rook off $B$, place a circle in the empty cells below it in the column, and also in the empty cells of $B$ above it in the column. Then $\xi(C, B)$ is the number of circles. See Fig. 2.

The main result in this article is another solution to Garsia and Remmel's problem, discovered before the author knew of Dworkin's result. This new statistic, which we call *mat*, bears superficial similarities to Dworkin's $\xi$, but the author has been unable to show that one being a solution implies the other is as well. We arrive at *mat* by counting matrices over finite fields subject to certain constraints, while Dworkin first generalizes a recurrence for the hit numbers given by Riordan, then shows $\xi$ satisfies this recurrence.

A permutation $\sigma$ of a multiset $M$ is a linear list $\sigma_1\sigma_2\cdots\sigma_{\#M}$ of the elements of $M$. For any vector $\mathbf{v} = (v_1, v_2, \ldots, v_t)$ of nonnegative integers, let $\{1^{v_1}2^{v_2}\cdots t^{v_t}\}$ denote the multiset having $v_i$ copies of $i$, and let $M(\mathbf{v})$ be the set of permutations of $\{1^{v_1}2^{v_2}\cdots t^{v_t}\}$. If $\mathbf{v} = (1, 1, \ldots, 1)$ is the vector with $n$ ones, we identify the element $\sigma_1\sigma_2\cdots\sigma_n \in M(\mathbf{v})$ with the element $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$ of the symmetric group $S_n$.

A statistic *stat* on permutations in $S_n$ is called *Mahonian* if

$$\sum_{\sigma \in S_n} q^{\mathrm{stat}(\sigma)} = [n]!.$$

It is called *multiset Mahonian* if

$$\sum_{\sigma \in M(\mathbf{v})} q^{\mathrm{stat}(\sigma)} = \begin{bmatrix} & \#M & & \\ v_1, & v_2, & \ldots & , v_t \end{bmatrix}$$

for all vectors $\mathbf{v}$, where $\begin{bmatrix} & \#M & & \\ v_1, & v_2, & \ldots & , v_t \end{bmatrix} := \frac{[\#M]!}{\prod_{i=1}^{t}[v_i]!}$ is the $q$-multinomial coefficient. The study of Mahonian statistics has become a large enterprise in recent years. Dworkin showed that $\xi$ induces a Mahonian statistic for any Ferrers board $B$, and we generalize this to show how $\xi$ and *mat* both induce multiset Mahonian statistics. We should mention that Dworkin gave his definition and results for $\xi$ in the more general setting of *skyline boards*, which are obtained by permuting the columns of a Ferrers board. Unfortunately, if we extend our definition of *mat* in a straightforward way, the resulting statistic is not Mahonian for skyline boards; in fact, it is not even nonnegative. For that reason, we will restrict our attention to Ferrers boards in this article.

For $\sigma \in M(\mathbf{v})$, a descent of $\sigma$ is a value of $i$, $1 \leq i < n$, such that $\sigma_i > \sigma_{i+1}$, where $n = \sum_i v_i$. MacMahon showed the statistic *maj* is multiset Mahonian, where

$$\mathrm{maj}(\sigma) := \sum_{i:\, \sigma_i > \sigma_{i+1}} i.$$

Let $des(\sigma)$ denote the number of descents of $\sigma$. A pair $(stat1, stat2)$ of statistics on permutations in $S_n$ is called *Euler-Mahonian* if it is equi-distributed with $(des, maj)$, i.e. if

$$\sum_{\sigma \in S_n} p^{\mathrm{stat1}(\sigma)} q^{\mathrm{stat2}(\sigma)} = \sum_{\sigma \in S_n} p^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

Dworkin noted that $\xi(B)$ is part of an Euler-Mahonian pair if $B$ is a triangular-shaped board. In section 4 we develop this idea further, and show how $\xi$ and *mat* both induce families of eight Euler-Mahonian pairs of statistics each of the form $(des, stat)$, which are all different from one another.

Let $f(\mathbf{v}) := f_1(\mathbf{v}) \cdots f_n(\mathbf{v})$ be the unique element of $M(\mathbf{v})$ with no descents. An *excedence* of $\sigma \in M(\mathbf{v})$ is a value of $i$ such that $\sigma_i > f_i(\mathbf{v})$, and we denote the number of such excedences by $\mathrm{exc}(\sigma)$. For example, the permutation $\sigma = 2313212$ has 3 excedences, occurring in the first, second, and fourth places of $\sigma$, and so $\mathrm{exc}(\sigma) = 3$.

Not many Euler-Mahonian pairs are known. A general overview of the few that have been discovered can be found in [CSZ]. There the authors classify a solution to the equation

$$\sum_{\substack{\sigma \in S_n \\ \mathrm{exc}(\sigma)=k}} q^{\mathrm{statx}(\sigma)} = \sum_{\substack{\sigma \in S_n \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)}, \tag{4}$$

as a "proper" Euler-Mahonian pair, proper indicating that $exc \neq des$. Using the geometry of the board, it is simple to convert one of our Euler-Mahonian pairs $(des, stat)$ into a solution to (4). When this conversion is applied to the $\xi$ family, we get Denert's statistic *den* [Den], and a new proof of a result of Foata and Zeilberger [FoZe], that $(exc, den)$ is equi-distributed with $(des, maj)$. On the other hand, the *mat* family gives what appears to be a fundamentally new solution to (4).

Garsia and Remmel also obtained a solution to a form of (4), namely

$$\sum_{\substack{\sigma \in S_n \\ \#\{i:\, \sigma_i \geq i\}=k}} q^{\mathrm{naj}(\sigma)} = \sum_{\substack{\sigma \in S_n \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)},$$

involving a statistic *naj* which they defined recursively. They also gave a recursive definition of a Mahonian statistic which involved an arbitrary Ferrers board. It would be interesting to obtain non-recursive versions of the definitions of these statistics, and determine how they relate to other Mahonian statistics and Euler-Mahonian pairs.

In section 5 we show that $T_k(B)$ is a symmetric and unimodal polynomial in $q$ for all $B$, a fact first proved in [Hag1]. The proof is a simple extension of Garsia and Remmel's proof that $T_k(B) \in \mathbb{N}[q]$. For some boards we prove a stronger result by a different method.

## 2. Matrices over Finite Fields

Solomon [Sol] showed how a placement of $k$ non-attacking rooks on a rectangular board can naturally be associated to a rectangular $n \times m$ matrix with entries in $\mathbb{F}_q$ and of rank $k$. Ding has shown that a similar construction involving matrices over the complex numbers in the shape of a Ferrers board has applications to topological questions involving certain algebraic varieties [Din1],[Din2]. In the lemma below we generalize Solomon's result to Ferrers boards; the proof is a straightforward extension of his.

**Definition 1.** *For $B$ a Ferrers board with $n$ columns (some of which may be empty), let $P_k(B)$ be the number of $n \times n$ matrices $A$ with entries in $\mathbb{F}_q$, of rank $k$, and with the restriction that all the entries of $A$ in those squares of $A$ outside of $B$ are zero. For example, if $B$ is the board consisting of squares $(1,2)$,$(1,3)$, and $(2,3)$, then $P_0 = 1$,$P_1 = 2q^2 - q - 1$,$P_2 = q(q-1)^2$, and $P_3 = 0$.*

**Theorem 1.** *For any Ferrers board $B$,*

$$P_k(B) = (q-1)^k q^{\mathrm{Area}(B)-k} R_k(q^{-1}),$$

*where Area$(B)$ is the number of squares of $B$.*

*Proof* : Let $A$ be a matrix of rank $k$, with entries in $\mathbb{F}_q$, and zero outside of $B$. We perform an operation on $A$ which we call the *elimination procedure*. Starting at the bottom of column 1 of $A$, travel up until you arrive at a nonzero square $\beta$ (if the whole first column is zero go to column 2 and iterate). Call this nonzero square a *pivot spot*. Next add multiples of the column containing $\beta$ to the columns to the right of it to produce zeros in the row containing $\beta$ to the right of $\beta$. Also add multiples of the row containing $\beta$ to the rows above it to produce zeros in the column containing $\beta$ above $\beta$. Now go to the bottom of the next column and iterate; find the lowest nonzero square, call it a pivot spot, then zero-out entries above and to the right as before.

If we place rooks on the square $\beta$ and the other pivot spots we end up with $k$ non-attacking rooks. The number of matrices which generate a specific rook placement $C$ is

$$\begin{aligned}&(q-1)^k q^{\#\text{ of squares to the right of or above a rook}}\\ &= (q-1)^k q^{\mathrm{Area}(B)-k-\mathrm{inv}(C,B)}. \qquad \blacksquare\end{aligned}$$

**Corollary 1.** *Let $P_k$ be the number of $n \times n$ upper triangular matrices of rank $k$ with entries in $\mathbb{F}_q$. Then*

$$P_k = (q-1)^k q^{\binom{n+1}{2}-k} S_{n+1,n+1-k}(q^{-1}),$$

*where $S_{n,k}(q)$ is the q-Stirling number of the second kind defined by the recurrences*

$$S_{n+1,k}(q) := q^{k-1} S_{n,k-1}(q) + [k] S_{n,k}(q) \qquad (0 \le k \le n+1),$$

*with the initial conditions $S_{0,0}(q) = 1$ and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.*

*Proof* : It is known [GaRe,p.248] that if $B$ is the triangular board whose $i$th column has height $i$, then

$$R_k(B) = S_{n+1,n+1-k}(q).$$

Now apply Theorem 1. $\blacksquare$

**Corollary 2.** *For any Ferrers board $B$,*

$$\sum_{k=0}^{n}(1-x)(1-xq)\cdots(1-xq^{k-1})P_{n-k}(B)=\prod_{i=1}^{n}(q^{c_i}-xq^{i-1}).$$

*Proof* : This is obtained by replacing by replacing $q$ by $q^{-1}$ in (2), applying Theorem 1, and doing other simple transformations such as replacing $q^x$ by $1/x$. ■

*Remark* : In [Hag1], the following identity was derived as a limiting case of a hypergeometric result:

$$\sum_k R_k(B)(1-q)^k = 1$$

(this can also be obtained by letting $x \to \infty$ in (2)). Using Theorem 1, this is equivalent to the trivial statement

$$\sum_k P_k(B) = q^{\text{Area}(B)}.$$

**Definition 2.** *Let $C$ be a placement of $n$ non-attacking rooks on the $n \times n$ grid, with $k$ rooks on the Ferrers board $B$. Define $cross(C, B)$ to be the number of squares of the $n \times n$ grid satisfying one of the following conditions:*

- *containing a rook or to the right of a rook*
- *above a rook and on $B$*
- *below a rook which is off $B$*

*Furthermore let $mat(C, B) := n(n-k) + Area(B) - cross(C, B)$. See Fig. 3.*

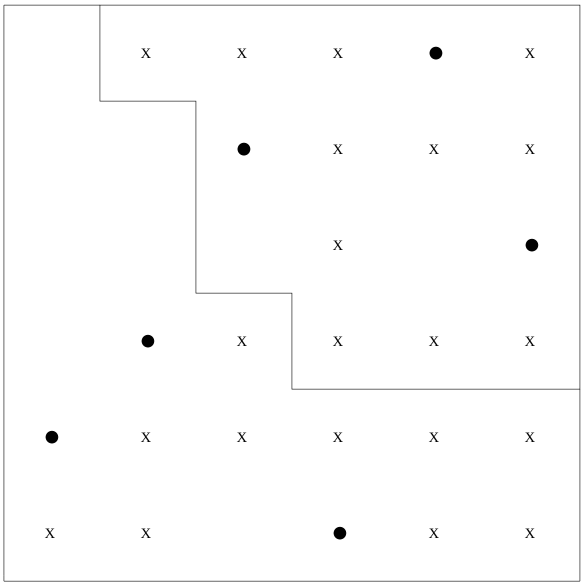


FIGURE 3. A placement of 6 rooks with 3 rooks on $B$: $cross = 27$ so $mat = 3*6+16-27=7$.

**Theorem 2.** *If $B$ is any Ferrers board,*

$$T_k(B) = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} q^{\mathrm{mat}(C,B)}.$$

*Proof* : Replacing $q$ by $q^{-1}$ in (3) and multiplying by $(q-1)^n q^{\mathrm{Area}(B)}$ we get

$$\sum_{k=0}^{n} (q^k-1)\cdots(q^2-1)(q-1)(q-1)^{n-k} q^{\mathrm{Area}(B)-(1+2+\ldots+k-1+k+1+\ldots+n)} R_{n-k}(q^{-1})$$

$$\times \prod_{i=k+1}^{n} (q^i x - 1) = \sum_{k=0}^{n} x^k (q-1)^n q^{\mathrm{Area}(B)} T_k(q^{-1}),$$

or

$$\sum_{k=0}^{n} (q^k-1)\cdots(q-1)(q-1)^{n-k} q^{\mathrm{Area}(B)} R_{n-k}(q^{-1}) q^{-(n-k)} \prod_{i=k+1}^{n} (q^i x - 1)$$

$$= \sum_{k=0}^{n} x^k (q-1)^n q^{\mathrm{Area}(B)+\binom{n}{2}} T_k(q^{-1}), \tag{5}$$

or

$$\sum_{k=0}^{n} (q-1)(q^2-1)\cdots(q^k-1) P_{n-k} \prod_{i=k+1}^{n} (q^i x - 1) = \text{ RHS of (5) } = \sum_{k=0}^{n} x^k Q_k \tag{6}$$

say. We will prove Theorem 2 by showing that

$$Q_k = \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} (q-1)^n q^{\mathrm{Area}(B)+\binom{n}{2}-\mathrm{mat}(C,B)}$$

$$= \sum_{\substack{C \\ n \text{ rooks, } k \text{ on } B}} (q-1)^n q^{\binom{n}{2}+\mathrm{cross}(C,B)-n(n-k)}. \tag{7}$$

Our strategy will be to exploit the combinatorial interpretation of the LHS of (6). Using the following special case of Cauchy's famous *$q$-binomial theorem*;

$$\prod_{j=0}^{m-1} (1+xq^j) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} x^k, \tag{8}$$

where $\begin{bmatrix} m \\ k \end{bmatrix} := \frac{[m]!}{[k]![m-k]!}$ is the $q$-binomial coefficient, the coefficient of $x^s$ in the LHS of (6) can be written as

$$\sum_{k=0}^{n-s} P_{n-k}(q^n - q^{n-k})(q^{n-1} - q^{n-k})\cdots(q^{n-k+1} - q^{n-k})$$

$$\times \begin{bmatrix} n-k \\ s \end{bmatrix} (-1)^{n-k-s} q^{\binom{s}{2}+s(k+1)-k(n-k)}. \tag{9}$$

We want to show that the expression above equals $Q_s$. Let $A$ be a matrix of rank $n-k$, with entries in $\mathbb{F}_q$ and zero outside $B$. We now perform an operation on $A$ which we call the *replacement procedure*. Starting with the last row (the bottom row) of $A$, define row $\alpha_k$ as the bottom-most row linearly dependent on the rows below it (or $\alpha_k = n$ if the last row is zero). Next let row $\alpha_{k-1}$ be the next bottom-most row linearly dependent on the rows below it, etc. Thus we end up with $k$ rows $\alpha_k > \alpha_{k-1} > \cdots > \alpha_1$. We call the rows $\alpha_1, \ldots, \alpha_k$ "dependent rows" and the other rows of $A$ "keeper rows".

Now replace row $\alpha_1$ by any of the $q^n - q^{n-k}$ rows which are linearly independent of the rows of $A$. Call this new row $\overline{\alpha_1}$, and note that $P_{n-k}$ is multiplied by $(q^n - q^{n-k})$ in (9). If $\overline{\alpha_1}$ has any nonzero entries off $B$, we call it a *pivot row*, and the spot where the left-most nonzero entry in $\overline{\alpha_1}$ occurs a *pivot spot* ( when we perform the elimination procedure later, this spot will be a pivot). Next replace row $\alpha_2$ by a new row linearly independent of both the rows of $A$ and the new row $\overline{\alpha_1}$, with the added constraint that if $\overline{\alpha_1}$ is a pivot row, we require the new $\overline{\alpha_2}$ row to have a zero in the column containing the pivot spot in row $\overline{\alpha_1}$. If $\overline{\alpha_1}$ is a pivot row, there are $q^{n-1} - q^{n-k}$ choices for $\overline{\alpha_2}$ (if we look at all linear combinations of $\overline{\alpha_1}, w_1, \ldots, w_{n-k}$ where the $w_i$ are the keeper rows of $A$, then for any fixed $c_2, \ldots, c_{n-k+1}$, the sums

$$c_1\overline{\alpha_1} + c_2 w_1 + \ldots + c_{n-k+1} w_{n-k}$$

produce $q$ different values in the column containing the pivot spot of $\overline{\alpha_1}$ as $c_1$ cycles through its $q$ possible values) and $q^n - q^{n-k+1}$ choices otherwise. In the latter case, we define the *weight* of row $\overline{\alpha_2}$ to be $q^{-1}$ (this is what we need to multiply $q^n - q^{n-k+1}$ by to get the desired factor $q^{n-1} - q^{n-k}$ occurring in (9)). If $\overline{\alpha_1}$ is a pivot row, let the weight of row $\overline{\alpha_2}$ be 1. As before, if $\overline{\alpha_2}$ contains any nonzero entries off $B$ we call it a pivot row, and its left-most nonzero entry a pivot spot.

Now for $\overline{\alpha_3}$, we require there be zeros in the columns containing any pivot spots in rows $\overline{\alpha_1}$ or $\overline{\alpha_2}$. More generally, in $\overline{\alpha_j}$, we require zeros below any of the pivot spots in rows $\overline{\alpha_i}$, $1 \le i < j$, and define the weight of $\overline{\alpha_j}$ to be $q^{-w}$, with $w$ equal to the cardinality of $\{\overline{\alpha_i} : \overline{\alpha_i} \text{ is not a pivot row and } 1 \le i < j\}$.

Let

$$\text{factor}(s,k) := \begin{bmatrix} n-k \\ s \end{bmatrix} (-1)^{n-k-s} q^{\binom{s}{2} + s(k+1) - k(n-k)}$$

(as in (9)). The argument above shows that $Q_s$ equals the number of matrices of rank $n$ which are obtained by starting with matrices which are zero outside of $B$ and performing the replacement procedure, and finally multiplying by the appropriate weight and factor. If we perform the elimination procedure from the proof of Theorem 1 to one of these new matrices, we end up with $n$ pivots, where the pivots off $B$ are exactly those pivots spots defined above from the $\overline{\alpha_i}$.

Let $Q_{s,j}$ be the number of these matrices, counted with weights and factors, with $j$ pivots off $B$.

Case 1: $j = n - s$. In this case, all the weights are 1. The row numbers with pivots off $B$ must have been the original $\alpha_1 < \alpha_2 < \cdots < \alpha_{n-s}$. There are $q^{n-\alpha_{n-s}}$ choices of row $\alpha_{n-s}$ to be dependent on the rows below; we call $q^{n-\alpha_{n-s}}$ the *pre-image term* for this row. There are $q^{n-\alpha_{n-s-1}-1}$ (the pre-image term for this row) choices for row $\alpha_{n-s-1}$ to be dependent on the rows below, etc. Note that the $P_i$ in (9) satisfy $i \ge s$, and only the $i = s$ term can possibly generate matrices with $n-s$ pivots off $B$. By the elimination procedure and (9),

$$Q_{s,n-s} = \sum_{\substack{C \\ n \text{ rooks, } s \text{ on } B}} (q-1)^n q^{n-\alpha_{n-s}+n-\alpha_{n-s-1}-1+\ldots+n-\alpha_1-(n-s-1)}$$

$$\times q^{\#\text{ of squares to the right of a rook, or above a rook and on } B}$$

$$\times q^{-\#\text{ of squares below a rook off } B \text{ and to the right of some rook}}$$

$$\times \text{factor}(s, n-s),$$

where in the sum above $\alpha_1(C) < \cdots < \alpha_{n-s}(C)$ are the row numbers with rooks off $B$. Now

$$\begin{aligned}&\#\text{ of squares to the right of a rook or above a rook and on } B\\ &\qquad -\#\text{ of squares below a rook off } B \text{ and to the right of some rook}\\ &\qquad\qquad + n - \alpha_{n-s} + n - \alpha_{n-s-1} + \ldots + n - \alpha_1 = \text{cross}(C,B) - n,\end{aligned}$$

and plugging this in above, after a short calculation we get

$$Q_{s,n-s} = \sum_{\substack{C\\ n \text{ rooks, } s \text{ on } B}} (q-1)^n q^{\binom{n}{2}+\text{cross}(C,B)-n(n-s)}.$$

In view of (7), Theorem 2 follows if we can show $Q_{s,j} = 0$ if $j < n-s$.

Case 2: $j < n-s$. By an abuse of terminology, if the weight of a row is $q^w$, we sometimes refer to $w$ as the weight. A similar remark applies to the pre-image term.

For each $k$ with $j \le k \le n-s$, the term $P_{n-k}$ in (9) makes a contribution. Say after replacement and elimination, we end up with a placement $C$ of $n$ rooks, with the $j$ pivots off $B$ in rows $\beta_1 < \beta_2 < \cdots < \beta_j$. Then all these rows, and $k-j$ others, must have been the original $\alpha_1, \ldots, \alpha_k$. We have to sum over all choices of the $k-j$ others, taking into account the weights, the pre-image terms $n - \alpha_k + n - \alpha_{k-1} - 1 + \ldots + n - \alpha_1 - (k-1)$, and the factor$(s,k)$ term from (9).

Say there are $\mu_0$ new rows above row $\beta_1$, $\mu_1$ new rows between rows $\beta_1$ and $\beta_2$,…, and $\mu_j$ below row $\beta_j$, with $\mu_i \ge 0$ and $\mu_0 + \mu_1 + \ldots + \mu_j = k - j$. Lets compute the total weight of such an arrangement, using the fact that as we move downwards, the weights of the rows decrease by one each time, unless the row is just below a $\beta_i$, in which case the weight stays the same.

The $\mu_0$ rows above row $\beta_1$ have weights $-0, -1, \ldots, -\mu_0 + 1$.

Row $\beta_1$ has weight $-\mu_0$.

The $\mu_1$ rows between rows $\beta_1$ and $\beta_2$ have weights $-\mu_0, -\mu_0 - 1, \ldots, -\mu_0 - \mu_1 + 1$.

Row $\beta_2$ has weight $-\mu_0 - \mu_1$.

$$\vdots$$

Row $\beta_j$ has weight $-\mu_0 - \mu_1 - \ldots - \mu_{j-1}$.

The $\mu_j$ rows below row $\beta_j$ have weights $-\mu_0 - \mu_1 - \ldots - \mu_{j-1}, \ldots, -\mu_0 - \mu_1 - \ldots - \mu_j + 1$.

For the $\mu_j$ rows below row $\beta_j$, say rows $\gamma_1 > \gamma_2 > \cdots > \gamma_{\mu_j}$, the sum of the pre-image terms will be $n - \gamma_1 + n - \gamma_2 - 1 + \ldots + n - \gamma_{\mu_j} - \mu_j + 1$, and combining this with the weight for these rows gives a total contribution of

$$\begin{aligned}&n - \gamma_{\mu_j} - \mu_j + 1 - \mu_0 - \mu_1 - \ldots - \mu_{j-1} + n - \gamma_{\mu_j - 1} - (\mu_j - 2) - \mu_0 - \mu_1 -\\ &\qquad \ldots - \mu_{j-1} - 1 + \ldots + n - \gamma_1 - \mu_0 - \mu_1 - \ldots - \mu_j + 1,\end{aligned}$$

or

$$n-\gamma_1+n-\gamma_2+\ldots+n-\gamma_{\mu_j}-\mu_j(k-j-1).$$

Lets skip past row $\beta_j$ for the moment and consider rows

$$\gamma_{\mu_j+1}>\gamma_{\mu_j+2}>\cdots>\gamma_{\mu_j+\mu_{j-1}}$$

between rows $\beta_j$ and $\beta_{j-1}$. The pre-image terms will be

$$n-\gamma_{\mu_j+1}-(\mu_j+1)+n-\gamma_{\mu_j+2}-(\mu_j+2)+\ldots+n-\gamma_{\mu_j+\mu_{j-1}}-(\mu_j+\mu_{j-1}).$$

Adding in the weights as before we end up with a contribution of

$$n-\gamma_{\mu_j+1}-1+n-\gamma_{\mu_j+2}-1+\ldots+n-\gamma_{\mu_j+\mu_{j-1}}-1-\mu_{j-1}(k-j-1)$$

for these rows. Continuing in this way, for the $\mu_0$ rows above row $\beta_1$ we get a total contribution of

$$n-\gamma_{\mu_j+\ldots+\mu_1+1}-j+\ldots+\gamma_{\mu_j+\ldots+\mu_0}-j-\mu_0(k-j-1).$$

As we range over all legal choices of the $\gamma_i$ (i.e. $\gamma_i\neq\beta_k$ for all $i,k$), the numbers

$$n-\gamma_1,\ldots,n-\gamma_{\mu_j},n-\gamma_{\mu_j+1}-1,\ldots,n-\gamma_{\mu_j+\mu_{j-1}}-1,\\ \ldots,n-\gamma_{\mu_j+\ldots+\mu_1+1}-j,\ldots,n-\gamma_{\mu_j+\ldots+\mu_0}-j$$

range over all numbers between 0 and $n-j-1$. Thus raising $q$ to the power of all terms above (ignoring the $\beta_i$ weights) gives

$$q^{-(k-j-1)(\mu_0+\ldots+\mu_j)}\times\text{the coefficient of } x^{k-j} \text{ in } \prod_{i=0}^{n-j-1}(1+xq^i). \tag{10}$$

By (8), (10) reduces to

$$q^{-(k-j-1)(k-j)}\begin{bmatrix}n-j\\k-j\end{bmatrix}q^{\binom{k-j}{2}}.$$

Next we add in the contribution from the $\beta_i$. The weight of $\beta_j$ is $-(\mu_0+\ldots+\mu_{j-1})$ and its pre-image term is $n-\beta_j-\mu_j$. For $\beta_2$, the weight is $-(\mu_0+\mu_1)$ and the pre-image term is $n-\beta_2-(j-2)-(\mu_j+\ldots+\mu_2)$. For $\beta_1$, the weight is $-\mu_0$ and the pre-image term is $n-\beta_1-(j-1)-(\mu_j+\ldots+\mu_1)$. The total contribution from the $\beta_i$ is thus $q^\beta$, where

$$\beta:=n-\beta_1+\ldots+n-\beta_j-j(\mu_0+\ldots+\mu_j)-\binom{j}{2}.$$

For fixed $C$ (which also fixes the $\beta_i$) we thus have a contribution to $Q_{s,j}$ of

$$\sum_k(q-1)^n\begin{bmatrix}n-j\\k-j\end{bmatrix}q^{\binom{k-j}{2}-(k-j-1)(k-j)-j(k-j)-\binom{j}{2}}q^{n-\beta_1+\ldots+n-\beta_j}\\ q^{\#\text{ of squares to the right of a rook, or above a rook and on } B}\\ q^{-\#\text{ of squares below any rook off } B \text{ and to the right of some rook}}\\ \times q^{\binom{s}{2}+s(k+1)-k(n-k)}\begin{bmatrix}n-k\\s\end{bmatrix}(-1)^{n-k-s}$$

$$= q^{\mathrm{cross}(C,B)-n+\binom{s+1}{2}} \sum_k \begin{bmatrix} n-j \\ k-j \end{bmatrix} \begin{bmatrix} n-k \\ s \end{bmatrix} (-1)^k q^{k(k/2+s-n+1/2)}.$$

Letting $k = n - s - u$, the sum above reduces to

$$d \begin{bmatrix} n-j \\ s \end{bmatrix} \sum_{u \geq 0} \begin{bmatrix} n-j-s \\ u \end{bmatrix} (-1)^u q^{\binom{u}{2}}$$

(with $d$ independent of $u$) which equals zero for $j < n - s$ by (8). This completes the proof of Theorem 2. ■

## 3. Multiset Mahonian Statistics

A placement $C$ of $n$ rooks on an $n \times n$ grid can be identified with a permutation $\sigma_1\sigma_2\cdots\sigma_n \in S_n$, in a simple way: a rook is on square $(i, j)$ if and only if $\sigma_i = j$. We call this placement $C(\sigma)$ the *graph* of $\sigma$. Hence both $\xi$ and *mat* can be regarded as permutation statistics if we define $\xi(\sigma, B) := \xi(C(\sigma), B)$ and $\mathrm{mat}(\sigma, B) := \mathrm{mat}(C(\sigma), B)$.

Dworkin proved that $\xi$ is Mahonian for all Ferrers boards $B$, i.e.

$$\sum_{\sigma \in S_n} q^{\xi(\sigma,B)} = [n]!,$$

or equivalently

$$\sum_k T_k(B) = [n]!. \tag{11}$$

Eq. (11) can also be obtained by letting $x \to \infty$ in the following known formula [Hag1,p.100], [Dwo,pp.35,38]

$$\sum_k \begin{bmatrix} x+k \\ n \end{bmatrix} T_k(B) = \prod_{i=1}^{n} [x + c_i - i + 1],$$

which in turn follows from (21) and another form of the $q$-binomial theorem.

In this section we show how to construct multiset Mahonian statistics from both $\xi$ and *mat*.

**Definition 3.** *Let $B$ be a Ferrers board. A section $D$ of $B$ of width $d$ is a set of $d$ consecutive columns of the $n \times n$ grid with the property that the height of all the columns of $B$ in $D$ is the same. Say we have a placement $C$ of $n$ rooks on the $n \times n$ grid, with $s$ of the $d$ rooks in $D$ on $B$. We say $C$ is $D$-standard if both of the following hold.*

*1) The $d-s$ rooks off $B$ and in $D$ are in the $d-s$ left-most columns of $D$. Furthermore, these $d-s$ rooks are in "descending" order; if two of these rooks occupy squares $(i, j)$ and $(k, l)$, with $i < k$, then $j < l$.*

*2) The $s$ remaining rooks in $D$ and on $B$ are in "ascending" order; if two of these rooks occupy squares $(i, j)$ and $(k, l)$, with $i < k$, then $j > l$.*

*Call a placement of $n$ rooks on the $n \times n$ grid $B$-standard if it is $D$-standard for all possible sections $D$ of $B$. See Fig. 4.*

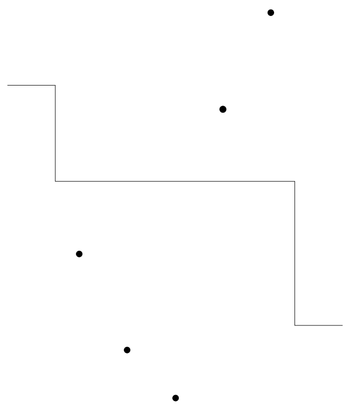

FIGURE 4. A section $D$ of width 5 and a $D$-standard placement of rooks (rooks outside of $D$ are not pictured).

**Lemma 1.** *Let $B$ be a Ferrers board, and let $D$ be a section of $B$ of width $d$. Fix a placement $C$ of $n-d$ rooks in the $n-d$ other columns of the $n \times n$ grid outside of $D$. We say a placement of rooks $C'$ extends $C$ if all the rooks of $C$ are in $C'$. Then the minimum value of $mat(C', B)$, over all placements $C'$ of $n$ rooks extending $C$, occurs when $C'$ is $D$-standard. Furthermore, if $E$ is this $D$-standard extension of $C$,*

$$\sum_{\substack{C' \\ n \text{ rooks, extending } C}} q^{\mathrm{mat}(C',B)} = q^{\mathrm{mat}(E,B)}[d]!.$$

*Proof* : By induction on $d$, the case $d = 1$ being trivial. Let $D'$ be $D$ minus its left-most column, and call an extension $C'$ $D$ *semi-standard* if it is $D'$-standard. Also call the rook in the left-most column of $D$ the *left-rook*. We claim that when we add up $q^{\mathrm{mat}}$ for the $d$ semi-standard extensions, we get $q^{\mathrm{mat}(E,B)}[d]$. Lemma 1 will follow since, if for any of the $d$ choices for the left-rook we let the $d-1$ remaining rooks in $D$ cycle through their $(d-1)!$ possibilities, by induction they generate an extra $[d-1]!$.

Say the rows not attacked by the rooks of $C$ are rows

$$i_s < i_{s-1} < \cdots < i_1,$$

(which intersect $B$ within $D$) and rows

$$j_1 < j_2 < \cdots < j_{d-s},$$

(which do not intersect $B$ within $D$). Note that $j_1 > i_1$. We consider what happens to the statistic *cross* when we move from one semi-standard placement to another, in three special cases.

Case 1: The left-rook changes from row $j_p$ to row $j_{p+1}$ for some $1 \le p < d-s$. Consider Figures 5 and 6. The horizontal line near the middle of the Figure is the boundary of $B$, indicated by the letter **B**. Otherwise, squares which are counted in the definition of cross are indicated by straight lines going through them (ignore contributions from rooks outside of $D$

for the moment). If, in the definition of *cross*, those squares which satisfy two of the three conditions were counted twice, then *cross* would be the same for Figures 5 and 6, since the line segments of lengths $P$,$Q$,$R$, and $S$ are merely shifted around from one figure to the next. But there is one more square in Figure 6 that satisfies two of the three conditions then there is in Figure 5 (note the circled intersections; note also that rooks to the left or the right of $D$ will create the same number of intersections in both placements, hence we are justified in ignoring their contribution when determining how much *cross* changes by, and do not need to include them in our figures). Thus *cross* has decreased by one.

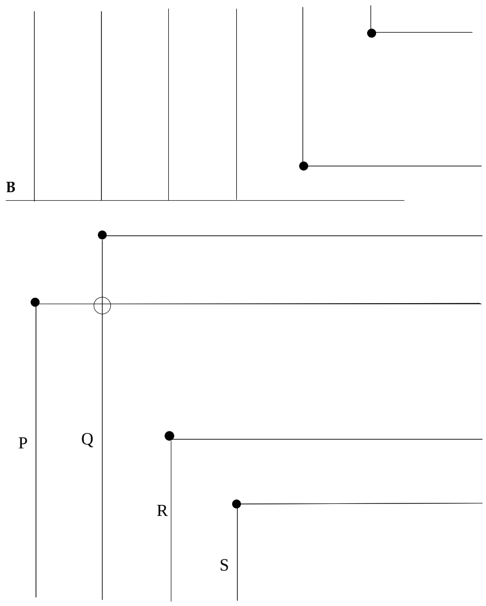


FIGURE 5. A $D$ semi-standard placement of rooks.

Case 2: The left-rook changes from row $j_1$ to row $i_1$. Consider Figures 7 and 8. As in case 1, we need only consider the number of squares which satisfy two of the three conditions. For example, there are $P$ squares below the left-rook in Figure 7, and also $P$ squares below the rook in column two of Figure 8. Similar remarks apply to $Q$,$R$,$S$,$T$,and $U$. Since there are $d-s$ new (circled) intersections, *cross* has decreased by $d-s$.

Case 3: The left-rook changes from row $i_p$ to row $i_{p+1}$ for some $1 \leq p < s$. Figures 9 and 10. As in case 1, *cross* decreases by 1.

Combining cases 1,2, and 3, we see that as the left-rook cycles through rows

$$j_1, j_2, \dots, j_{d-s}, i_1, i_2, \dots, i_s,$$

*cross* decreases by one each time, hence *mat* increases by one each time. Thus

$$\sum_{\substack{\text{semi-standard } C' \\ \text{extending } C}} q^{\operatorname{mat}(C',B)} = q^{\operatorname{mat}(E,B)}[d]$$

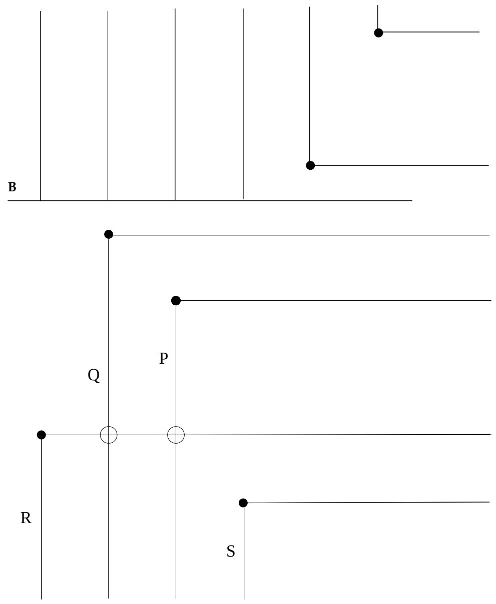


FIGURE 6. Another $D$ semi-standard placement of rooks. The left-rook has moved down to row $j_{p+1}$.

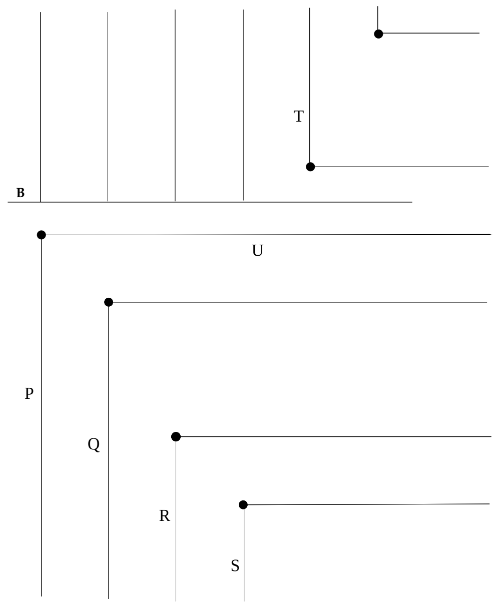


FIGURE 7. A $D$ semi-standard placement of rooks.

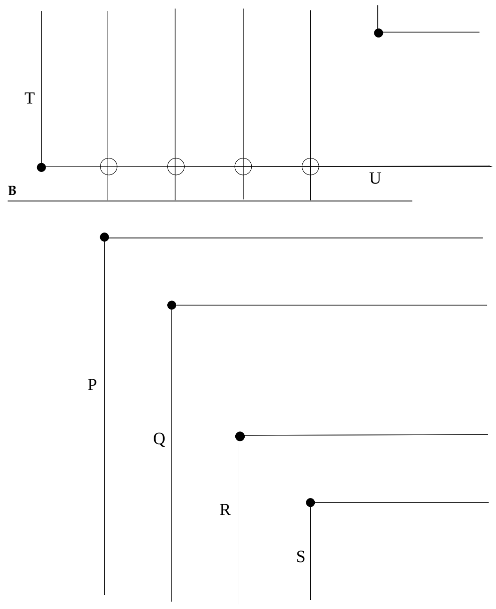


FIGURE 8. Another $D$ semi-standard placement of rooks. The left-rook has moved up to row $i_1$.

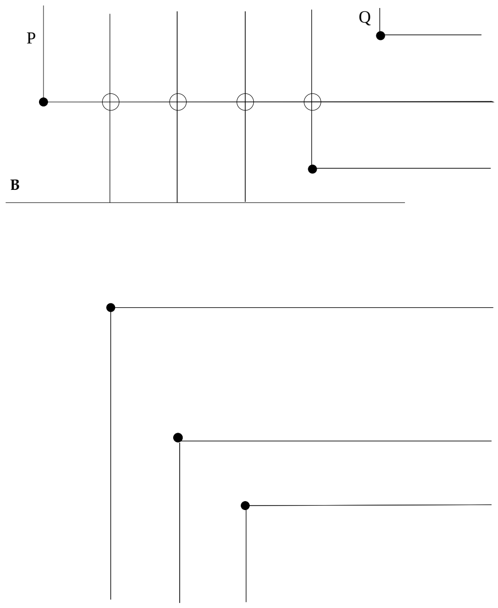


FIGURE 9. A $D$ semi-standard placement of rooks.

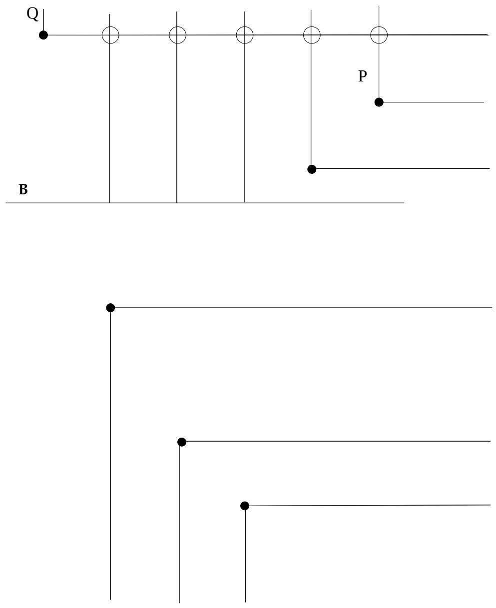


FIGURE 10. Another $D$ semi-standard placement of rooks. The left-rook has moved up to row $i_{p+1}$.

and Lemma 1 follows by induction. ■

Let $B$ be the Ferrers board of Fig. 11, with $\mathbf{d} = (d_1, \dots, d_t) \in \mathbb{P}^t$ a vector satisfying $\sum_i d_i = n$. For technical reasons we allow the $h_i \in \mathbb{N}$ (thus there are in general several different choices for $t$, $\mathbf{h}$, and $\mathbf{d}$ which represent the same board). A placement $C$ of $n$ rooks on squares $(1, \tau_1), \dots, (n, \tau_n)$ can be converted into a multiset permutation $\sigma \in M(\mathbf{d})$ by first forming a sequence $S$ whose $i$th element is $\tau_i$, then replacing numbers 1 through $d_1$ of $S$, wherever they occur, by all 1's, numbers $d_1 + 1$ through $d_2$ of $S$ by all 2's, etc. We call the $\prod_i d_i!$ placements that get mapped to $\sigma \in M(\mathbf{d})$ the *graphs* of $\sigma$. By restricting our attention to $B$-standard placements, we have a bijection between elements $\sigma \in M(\mathbf{d})$ and $B$-standard placements $C(\sigma)$. We call $C(\sigma)$ the *$B$-standard graph* of $\sigma$.

**Definition 4.** *Let $B = B(h_1, d_1; h_2, d_2; \dots; h_t, d_t)$ be the Ferrers board of Fig. 11, where $\sum_i d_i = n$. For a given permutation $\sigma \in M((d_1, d_2, \dots, d_t))$, let*

$$mat(\sigma, B) := mat(C(\sigma), B)$$

*where $C(\sigma)$ is the $B$-standard graph of $\sigma$.*

By iterating Lemma 1 and using (11) we now have

**Theorem 3.** *Let $B := B(h_1, d_1; \dots; h_t, d_t)$ be the Ferrers board of Fig. 11, with $\sum_i d_i = n$. Then mat$(B)$ is multiset Mahonian, i.e.*

$$\sum_{\sigma \in M(\mathbf{d})} q^{\mathrm{mat}(\sigma, B)} = \begin{bmatrix} & n & \\ d_1, & d_2, & \dots & , d_t \end{bmatrix}.$$

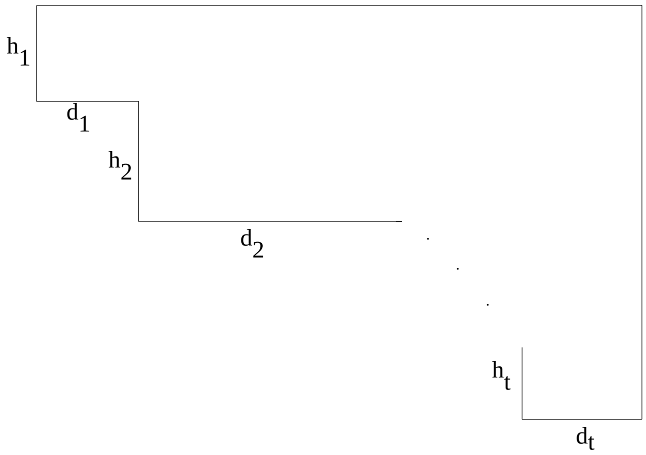


FIGURE 11. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$, where $d_i \in \mathbb{P}, h_i \in \mathbb{N}$ for $1 \leq i \leq t$. The first $d_1$ columns have height $h_1$, the next $d_2$ have height $h_1 + h_2$, etc.

**Definition 5.** *Let $B$ be a Ferrers board, and let $D$ be a section of $B$ of width $d$. Let $C$ be a placement of $n$ rooks on the $n \times n$ grid, with $s$ of the $d$ rooks in $D$ on $B$. We say $C$ is $D$-regular if both of the following hold.*

*1) The $s$ rooks on $B$ and in $D$ are in the $s$ left-most columns of $D$. Furthermore, these $s$ rooks are in "ascending" order; if two of these rooks occupy squares $(i, j)$ and $(k, l)$, with $i < k$, then $j > l$.*

*2) The $d - s$ remaining rooks in $D$ and off $B$ are also in ascending order.*

*Call a placement of $n$ rooks on the $n \times n$ grid $B$-regular if it is $D$-regular for all possible sections $D$ of $B$. See Fig. 12.*

By arguments identical to those given for the statistic $mat$, one can prove the following results for $\xi$.

**Lemma 2.** *Let $B$ be a Ferrers board, and let $D$ be a section of width $d$. Fix a placement $C$ of $n - d$ rooks in the $n - d$ other columns of the $n \times n$ grid outside of $D$. Then the minimum value of $\xi(C', B)$, over all placements $C'$ of $n$ rooks extending $C$, occurs when $C'$ is $D$-regular. Furthermore, if $E$ is this $D$-regular extension of $C$,*

$$\sum_{\substack{C' \\ n \text{ rooks, extending } C}} q^{\xi(C',B)} = q^{\xi(E,B)}[d]!.$$

**Definition 6.** *Let $B = B(h_1, d_1; h_2, d_2; \ldots; h_t, d_t)$ be the Ferrers board of Fig. 11, where $\sum_i d_i = n$. For a given permutation $\sigma \in M((d_1, d_2, \ldots, d_t))$, let*

$$\xi(\sigma, B) := \xi(P(\sigma), B),$$

*where $P(\sigma)$ is the graph of $\sigma$ which is $B$-regular.*

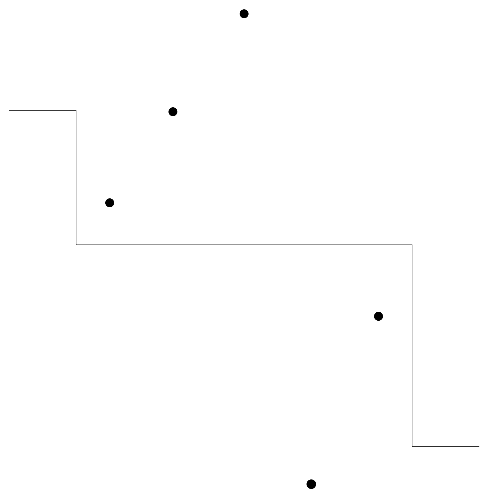

FIGURE 12. A section $D$ of width 5 and a $D$-regular placement of rooks (rooks outside of $D$ are not pictured).

**Theorem 4.** *Let $B := B(h_1, d_1; \ldots ; h_t, d_t)$ be the Ferrers board of Fig. 11, with $\sum_i d_i = n$. Then $\xi(B)$ is multiset Mahonian, i.e.*

$$\sum_{\sigma \in M(\mathbf{d})} q^{\xi(\sigma, B)} = \begin{bmatrix} & n & & \\ d_1, & d_2, & \ldots & , d_t \end{bmatrix}.$$

## 4. Euler-Mahonian Statistics

**Definition 7.** *Let $B(n)$ denote the triangular board of side $n-1$ consisting of all squares $(i,j)$ with $1 \le i < j \le n$.*

In [Hag1] it was shown that

$$\sum_{\substack{\sigma \in S_n \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)} = q^{nk - \binom{n}{2}} T_k(B(n)) \tag{12}$$

and also that

$$\sum_{\substack{\sigma \in S_n \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)} = T_{n-k-1}(B(n)). \tag{13}$$

For any Ferrers board $B(c_1, \ldots , c_n)$, let $B^c$ be the complementary board with column heights $n - c_n, n - c_{n-1}, \ldots , n - c_1$. Dworkin proved a "reciprocity" theorem for $T_k$, namely [Dwo]

$$T_k(B, q^{-1}) = q^{-\binom{n}{2}} T_{n-k}(B^c, q). \tag{14}$$

Combining (13), (14), and a symmetry property of the $T_k$ (Theorem 6) which we prove in section 5, we get

$$\sum_{\substack{\sigma \in S_n \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)} = q^{nk - \binom{n}{2}} T_{k+1}(B(n)^c, q). \tag{15}$$

Similarly, (12) and (14) imply

$$\sum_{\substack{\sigma \in S_n \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)} = T_{n-k}(B(n)^c, q), \tag{16}$$

a fact we will use later.

There is a straightforward way, used by Riordan and Kaplansky [KaRi], to identify a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ having $k$ descents, with a placement $F(\pi)$ of $n$ rooks on the $n \times n$ grid with $k$ rooks on $B(n)$. If $\pi_{j_1} = 1$, let $y_1$ be the cycle $(\pi_1\pi_2\cdots\pi_{j_1})$. If $\alpha$ is the smallest integer not contained in $y_1$, and $\pi_{j_2} = \alpha$, let $y_2$ be the cycle $(\pi_{j_1+1}\pi_{j_1+2}\cdots\pi_{j_2})$, etc. Now let $F(\pi)$ be the placement having a rook on $(i,j)$ if and only if $i$ and $j$ are in the same cycle $y_p$ for some $p$, with $i$ immediately following $j$. Call $F(\pi)$ the *descent graph* of $\pi$. For example, if $\pi = 3521647$, the $y_i$ are the cycles (3521), (64), and (7), and its descent graph is illustrated in in Fig. 13.

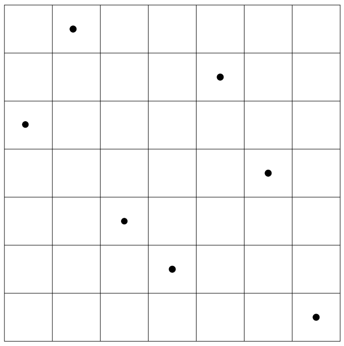

FIGURE 13. The descent graph of the permutation 3521647, whose associated cycles are (3521), (64), and (7). There are three rooks on $B(n)$, corresponding to the three descents of 3521647.

It follows from (12) that if we define

$$\begin{aligned} \mathrm{stat1}(\pi) &:= n\mathrm{des}(\pi) - \binom{n}{2} + \mathrm{mat}(F(\pi), B(n)) \qquad (17) \\ &= n^2 - cross(F(\pi), B(n)), \end{aligned}$$

then $(des, stat1)$ is equi-distributed with $(des, maj)$. To get another such pair $(des, stat2)$ we can reflect the board about the cross diagonal, i.e. relabel square $(i,j)$ as square $(n-j+1, n-i+1)$, which gives us a new rook placement $F'(\pi)$ with the same number of rooks on $B(n)$. For example, if we reflect the placement in Fig. 13 we get the descent graph of 1425763. This placement will have a different value of $mat$, which we can then use to define $stat2$ as in (17) above.

If we reverse a permutation $\pi$ with $k$ descents, we get a new permutation $\beta(\pi) := \pi_n\pi_{n-1}\cdots\pi_1$ with $n-k+1$ descents. By (13), if we let $\mathrm{stat3}(\pi) = \mathrm{mat}(F(\beta(\pi)), B(n))$ , we have an Euler-Mahonian pair $(des, stat3)$. We can also get another pair $(des, stat4)$ by reflection. So far we have four statistics for both $\xi$ and $mat$ which, when combined with $des$, form an Euler-Mahonian pair. For each of these statistics $stat$ we can get another Euler-Mahonian pair by forming $(des, nk - stat)$ (if we let $\zeta_i(\pi) := n - \pi_{n-i+1} + 1$, then $\mathrm{des}(\zeta(\pi)) = \mathrm{des}(\pi)$ and $\mathrm{maj}(\zeta(\pi)) = n\mathrm{des}(\pi) - \mathrm{maj}(\pi)$, hence the LHS of (13) is symmetric about $q^{nk/2}$). Thus both $\xi$ and $mat$ each induce a family of 8 pairs. Table I lists the two families for three sample permutations.

| π | | stat1 | stat2 | stat3 | stat4 | nk-stat1 | nk-stat2 | nk-stat3 | nk-stat4 |
|---|---|---|---|---|---|---|---|---|---|
| 3521647 | | | | | | | | | |
| | ξ | 10 | 14 | 11 | 9 | 11 | 7 | 10 | 12 |
| | mat | 12 | 8 | 11 | 13 | 9 | 13 | 10 | 8 |
| 2537461 | | | | | | | | | |
| | ξ | 13 | 11 | 13 | 10 | 8 | 10 | 8 | 11 |
| | mat | 9 | 11 | 9 | 12 | 12 | 10 | 12 | 9 |
| 1425376 | | | | | | | | | |
| | ξ | 13 | 10 | 11 | 13 | 8 | 11 | 10 | 8 |
| | mat | 9 | 12 | 13 | 11 | 12 | 9 | 8 | 10 |

TABLE I

An examination of Table I shows that none of the sixteen pairs equal each other for all $\pi$. Hence the $\xi$ and $mat$ families are fundamentally different, at least with respect to the simple transformations we have considered here. In addition none of the sixteen pairs are equal to $maj$, and are also unequal to the statistic $mak$ as described in [CSZ] (the pair $(des, mak)$ is known to be Euler-Mahonian [FoZe]).

We can also make use of (15) and (16) to try and generate other Euler-Mahonian pairs. However, examples indicate that the pairs arrived at in this manner are rearrangements of the sixteen pairs above.

Some of our Euler-Mahonian statistics can easily be rephrased as a multiset solution to (4). We utilize the following generalization of (12) [Hag1,p.118];

$$\sum_{\substack{\pi \in M(\mathbf{v}) \\ \mathrm{des}(\pi)=k}} q^{\mathrm{maj}(\pi)} = \frac{T_k(G_{\mathbf{v}})}{\prod_{i=1}^{t} [v_i]!} q^{nk - \mathrm{Area}(G_{\mathbf{v}})}, \tag{18}$$

where $G_{\mathbf{v}}$ is the board of Fig. 14. For $\sigma \in M(\mathbf{v})$, it is easy to see that the number of rooks on $G_{\mathbf{v}}$ in any of the $\prod_i v_i!$ graphs of $\sigma$ is $\mathrm{exc}(\sigma)$. Using this Lemma 1 and (18) imply

$$\sum_{\substack{G_{\mathbf{v}}\text{-standard placements } C \\ k \text{ rooks on } G_{\mathbf{v}}}} q^{nk - \mathrm{Area}(G_{\mathbf{v}}) + \mathrm{mat}(C, G_{\mathbf{v}})} = \sum_{\substack{\sigma \in M(\mathbf{v}) \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)}$$

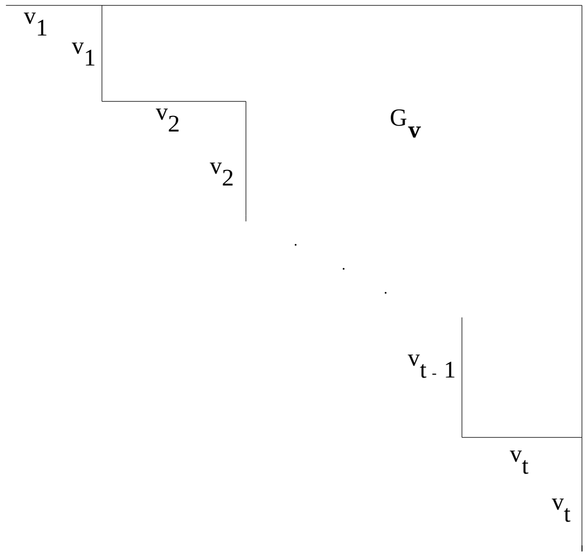


FIGURE 14. The Ferrers board $G_{\mathbf{v}}$. The first $v_1$ columns are empty, the next $v_2$ have height $v_1$, etc.

$$= \sum_{\substack{\sigma \in M(\mathbf{v}) \\ \mathrm{exc}(\sigma)=k}} q^{\mathrm{stat5}(\sigma)}, \tag{19}$$

where $\mathrm{stat5}(\sigma) := nk - \mathrm{Area}(G_{\mathbf{v}}) + \mathrm{mat}(C(\sigma), G_{\mathbf{v}})$, with $C(\sigma)$ the $G_{\mathbf{v}}$-standard graph of $\sigma$. We also have

$$\sum_{\substack{\sigma \in M(\mathbf{v}) \\ \mathrm{exc}(\sigma)=k}} q^{\mathrm{stat6}(\sigma)} = \sum_{\substack{\sigma \in M(\mathbf{v}) \\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)}$$

where $\mathrm{stat6}(\sigma) := \xi(P(\sigma), G_{\mathbf{v}}) + nk - \mathrm{Area}(G_{\mathbf{v}})$, with $P(\sigma)$ the $G_{\mathbf{v}}$-regular graph of $\sigma$.

If we reflect the board $G_{\mathbf{v}}$, the $G_{\mathbf{v}}$-standard graph of $\sigma \in M(\mathbf{v})$ gets sent to one of the $\prod_i v_i!$ graphs of some $\overline{\sigma} \in M((v_t, v_{t-1}, \ldots, v_1))$, with $\mathrm{exc}(\sigma) = \mathrm{exc}(\overline{\sigma})$. Thus we have the identity

$$\sum_{\substack{\sigma \in M((v_t,\ldots,v_1)) \\ \mathrm{exc}(\sigma)=k}} q^{\mathrm{stat7}(\sigma)} = \sum_{\substack{\pi \in M((v_1,\ldots,v_t)) \\ \mathrm{des}(\pi)=k}} q^{\mathrm{maj}(\pi)}, \tag{20}$$

where $\mathrm{stat7}(\sigma) = nk - \mathrm{Area}(G_{\mathbf{v}}) + \mathrm{mat}(C(\overline{\sigma}), G_{\mathbf{v}})$, with $\overline{\sigma}$ and $\sigma$ the reflected images of each other. MacMahon showed the RHS of (20) is invariant under any permutation of the coordinates of $\mathbf{v}$ (the RHS of (2) reduces to $\prod_i \begin{bmatrix} x + v_i - 1 \\ v_i \end{bmatrix} [v_i]!$ when $B = G_{\mathbf{v}}$, which implies the LHS of (2) and hence the $R_k$ are so invariant, and thus also the $T_k$ by (3)). This gives

$$\sum_{\substack{\sigma \in M((v_t,\ldots,v_1)) \\ \mathrm{exc}(\sigma)=k}} q^{\mathrm{stat7}(\sigma)} = \sum_{\substack{\pi \in M((v_t,\ldots,v_1)) \\ \mathrm{des}(\pi)=k}} q^{\mathrm{maj}(\pi)},$$

a proper multiset Euler-Mahonian pair. Clearly we can also replace stat7 above by $\xi(P(\overline{\sigma}), G_{\mathbf{v}}) + nk - \mathrm{Area}(G_{\mathbf{v}})$, where $P(\overline{\sigma})$ is the $G_{\mathbf{v}}$-regular graph of $\overline{\sigma}$.

One could also generate other identities by applying reciprocity to (18), but instead of working with excedences, we would need to work with *rises*, a rise being a value of $i$ such that $\sigma_i \geq f_i(\mathbf{v})$.

It doesn't seem to be as easy to obtain new statistics by reversing the string when working with multiset permutations since if such a permutation has $k$ excedences (or $k$ descents), the number of excedences (or descents) of the reversed string is unpredictable.

In the late 1980's M. Denert introduced an interesting permutation statistic which arose during her research into algebraic number theory. She conjectured that this statistic was Euler-Mahonian when paired with *exc*. Her conjecture was proven by Foata and Zeilberger [FoZe], who named her statistic "Denert's statistic", denoted by *den*. We now show that *den* is part of the $\xi$ family.

It will prove convenient to work with $B(n)^c$ and (16). Let $\sigma \in S_n$, and let $C(\sigma)^T$ denote the transpose of the graph of $\sigma$ (which is the graph of $\sigma^{-1}$). For each rook in $C(\sigma)^T$, put an $x$ on the grid in all the squares to the right and in the row. For each rook off $B(n)^c$, put a circle in the squares below and in the column, and also in the squares above and on $B(n)^c$. Finally, for each rook on $B(n)^c$, put a circle in those squares below and in the column and on $B(n)^c$. Then $\xi$ is the number of squares with circles, minus the number of squares with both circles and $x$'s, or $\#O - \#XO$ say. Now directly from the board we have

$$\#O = \sum_{\sigma_i > i} n - \sigma_i + i + \sum_{\sigma_i \leq i} i - \sigma_i$$

and

$$\begin{aligned}\#XO = \#\{1 \leq i < j \leq n;\ \sigma_i \leq j < \sigma_j\} \\ + \#\{1 \leq i < j \leq n;\ \sigma_i > \sigma_j > j\} \\ + \#\{1 \leq i < j \leq n;\ j \geq \sigma_i > \sigma_j\}.\end{aligned}$$

The formula for $\#XO$ above equals $den(\sigma)$ [FoZe, p.33] and the formula for $\#O$ simplifies to $n \times \mathrm{exc}(\sigma)$, so we get

$$\xi = n \times \mathrm{exc}(\sigma) - \mathrm{den}(\sigma)$$

and the joint distribution of $(exc, den)$ with $(des, maj)$ follows from (16) and the symmetry of the LHS of (13).

It is interesting to compare the statistic *den* with the following result, obtained by performing the above analysis with *mat* instead of $\xi$. We list this in part *a* of the theorem below. As far as the author is aware, this is not equivalent to any known statistic.

**Theorem 5.**

*a) For $\sigma_1 \cdots \sigma_n \in S_n$, define*

$$\begin{aligned}stat(\sigma) := \sum_{\sigma_i > i} \sigma_i - i + \sum_{\sigma_i \leq i} 1 - \sigma_i \\ + \#\{1 \leq i < j \leq n;\ \sigma_i > \sigma_j > j\} + \#\{1 \leq i < j \leq n;\ \sigma_i \leq j \text{ and } \sigma_i < \sigma_j\}.\end{aligned}$$

*Then $(exc, stat)$ is equi-distributed with $(des, maj)$.*

*b) For $\sigma \in M(\mathbf{v})$, with $n = \sum_i v_i$, define*

$$\begin{aligned}statx(\sigma) := \binom{n}{2} + \sum_{\sigma_i \leq f_i(\mathbf{v})} \Big( \sum_{\substack{i<j \\ \sigma_i > \sigma_j}} 1 + \sum_{\substack{m<i \text{ and } m \leq v_1 + \ldots + v_{\sigma_i - 1} \\ \sigma_m < \sigma_i}} 1\Big) \\ + \sum_{\sigma_i > f_i(\mathbf{v})} \sum_{\substack{m<i \\ \sigma_m < \sigma_i}} 1 - \sum_{\sigma_i > f_i(\mathbf{v})} (i-1) - \sum_{\sigma_i \leq f_i(\mathbf{v})} (n - i + \sum_{m < \sigma_i} v_m).\end{aligned}$$

*Then* $(exc, statx)$ *is equi-distributed with* $(des, maj)$.

*Proof* : First we prove part $a$, where $\sigma \in S_n$. For each rook in $C(\sigma)^T$, put an $x$ on the grid in all the squares to the right and in the row. For each rook off $B(n)^c$, put an $x$ in the squares below and in the column, and also in the squares above and on $B(n)^c$. Finally, for each rook on $B(n)^c$, put an $x$ in those squares above and in the column. Then *cross* equals the number of rooks, plus the total number of $x$'s, minus the number of squares with two $x$'s, or cross $= n + \#X - \#XX$ say. Now

$$n + \#X = \binom{n+1}{2} + \sum_{\sigma_i > i} n - \sigma_i + i + \sum_{\sigma_i \leq i} \sigma_i - 1,$$

and

$$\begin{aligned}\#XX = \#\{1 \leq i < j \leq n;\ \sigma_i > \sigma_j > j\} \\ + \#\{1 \leq i < j \leq n;\ \sigma_i \leq j < \sigma_j\} \\ + \#\{1 \leq i < j \leq n;\ \sigma_i < \sigma_j \leq j\}.\end{aligned}$$

Since

$$\operatorname{mat}(C(\sigma)^T, B(n)^c) = n \times \operatorname{exc}(\sigma) + \binom{n+1}{2} - \text{cross},$$

the result follows.

It is desirable to have a multiset version of the explicit formula from part $a$. However, trying to mimic the above argument while using the board $G^c_{\mathbf{v}}$ doesn't seem to lead to a nice formula. Instead we use $C(\sigma)$ and (19), and otherwise proceed exactly as in the proof of part $a$. The result is part $b$ above. The details are left as an exercise to the interested reader. ■

## 5. Unimodality

In this section we show that for any admissible Ferrers board $B$, $T_k(B)$ is symmetric and unimodal. (A different proof that $T_k(B)$ is symmetric can be found in [Dwo,p.52]). For certain boards we prove a stronger result.

**Definition 8.** *Let* $f(q) := \sum_{j=M}^{N} a_j q^j$ *be a polynomial in* $q$, *where* $a_M \neq 0$ *and* $a_N \neq 0$. *We call* $f$ *symmetric if* $a_{M+k} = a_{N-k}$ *for* $0 \leq k \leq N - M$, *and unimodal if there exists* $p$ *such that* $M \leq p \leq N$ *and* $a_m \leq a_{M+1} \leq \cdots \leq a_p \geq a_{p+1} \geq \cdots \geq a_N$. *Let* $darga(f) := M + N$. *We say* $f$ *is* $zsu(d)$ *if* $f$ *is either*

*a) identically zero*

*or*

*b) is* $\in \mathbb{N}[q]$, *and is symmetric and unimodal with* $darga(f) = d$.

*Note that the polynomial* $q^s$ *is* $zsu(2s)$.

**Claim 1.** *If* $f$ *and* $g$ *are polynomials which are both* $zsu(d)$, *then so is* $f + g$.

*Proof* : Trivial. ■

**Claim 2.** *If $f$ is $zsu(d)$ and $g$ is $zsu(e)$, then $fg$ is $zsu(d+e)$.*

*Proof* : (This proof is taken from [Zei]). If either $f$ or $g$ is zero, then so is $fg$. If not, then $f$ can be written as a sum of "atoms" (terms of the form $q^{d-i}+q^{d-i+1}+\ldots+q^i$ for some $d/2 \le i \le d$), and $g$ equals the sum of atoms of the form $q^{e-j}+\ldots+q^j$. The product of two of these atoms is of the form

$$q^{e+d-i-j}+2q^{e+d-i-j+1}+3q^{e+d-i-j+2}+\ldots+3q^{i+j-2}+2q^{i+j-1}+q^{i+j},$$

which is $zsu(d+e)$. Summing over all products of atoms from $f$ and $g$, and applying Claim 1 repeatedly proves the claim. ■

**Definition 9.** *Let $\delta$ be the linear operator such that $\delta x^k := [k]x^{k-1}$ for $k \in \mathbb{Z}$ ; for any formal power series $F(x)$,*

$$\delta F(x) = \frac{F(xq)-F(x)}{xq-x}.$$

**Lemma 3.** *(This appears in [GaRe]). For $0 \le k \le n$,*

$$\delta \frac{x^k}{(1-x)(1-xq)\cdots(1-xq^n)} = \frac{[k]x^{k-1}+[n-k+1]x^k q}{(1-x)(1-xq)\cdots(1-xq^{n+1})}.$$

**Definition 10.**

$$\Phi(x;c_1,c_2,\ldots,c_n) := \frac{\sum_{k=0}^n x^k T_{n-k}(B(c_1,\cdots,c_n))}{(1-x)(1-xq)\cdots(1-xq^n)},$$

*where $B(c_1,c_2,\ldots,c_n)$ is the Ferrers board whose $i$th column has height $c_i$.*

$\Phi$ satisfies the following useful identity [GaRe,p.259]

$$\Phi(x;c_1,c_2,\ldots,c_n) = \sum_{k=0}^{\infty} x^k \prod_{i=1}^n [k+c_i-i+1]. \tag{21}$$

**Theorem 6.** *Let $B := B(c_1,\ldots,c_n)$ be an admissible Ferrers board. Then for $0 \le k \le n$,*

$$T_k(B) \text{ is } zsu(N_k(B)),$$

*where*

$$N_k(B) := Area(B) + n(n-k) - \binom{n+1}{2}. \tag{22}$$

*Proof* : Throughout the proof, $B$ denotes the board $B(c_1,c_2,\ldots,c_n)$. Our proof is a straightforward refinement of the proof in [GaRe,pp.258-263] that $T_k(B) \in \mathbb{N}[q]$. First we show that performing the board transformations RAISE, FLIP, and ADD described below preserve property (22).

RAISE (this operation increases the height of each column by one; it assumes $c_n \le n-1$).

Eq. (21) implies

$$\Phi(x;c_1+1,c_2+1,\ldots,c_n+1) = \Phi(x;c_1,c_2,\ldots,c_n)/x,$$

thus $T_k(B(c_1+1, c_2+1, \ldots, c_n+1)) = T_{k-1}(B)$. Hence if $T_{k-1}(B)$ is $zsu(\mathrm{Area}(B) + n(n-k+1) - \binom{n+1}{2})$, then we have $T_k(B(c_1+1, c_2+1, \ldots, c_n+1))$ is

$$zsu(\mathrm{Area}(B) + n + n(n-k) - \binom{n+1}{2})$$

which is

$$zsu(N_k(B(c_1+1, \ldots, c_n+1))). \quad \blacksquare$$

FLIP (this operation replaces $B$ by $B^*$, where $B^*$ is $B$ reflected about the cross diagonal, the same reflection utilized in section 4).

Since $B^*$ has the same rook numbers as $B$ and hence the same $q$-rook numbers as $B$, $\Phi(x; B^*) = \Phi(x; B)$, and $T_k(B^*) = T_k(B)$. Clearly $\mathrm{Area}(B^*) = \mathrm{Area}(B)$, and so $N_k(B^*) = N_k(B)$. $\blacksquare$

ADD (this operation adds a column of height zero to $B$).

Since $\Phi(x; 0, c_1, c_2, \ldots, c_n) = x\delta x\Phi(x; c_1, c_2, \ldots, c_n)$ [GaRe,p.260], using Lemma 3 we get

$$\Phi(x; 0, c_1, c_2, \ldots, c_n) = x\delta \sum_{k=1}^{n+1} \frac{x^k T_{n-k+1}(B)}{(1-x)(1-xq)\cdots(1-xq^n)}$$

$$= \frac{x}{(1-x)(1-xq)\cdots(1-xq^{n+1})} \sum_{k=1}^{n+1} x^{k-1}[k]T_{n-k+1}(B) + x^k[n-k+1]T_{n-k+1}(B)q^k,$$

or

$$\sum_{k=0}^{n+1} \frac{x^k T_{n+1-k}(B(0, c_1, \ldots, c_n))}{(1-x)(1-xq)\cdots(1-xq^{n+1})}$$
$$= \sum_{k=1}^{n+1} \frac{x^k([k]T_{n+1-k}(B) + [n-k+2]q^{k-1}T_{n-k+2}(B))}{(1-x)(1-xq)\cdots(1-xq^{n+1})}.$$

Comparing numerators we get $T_{n+1}(B(0, c_1, \ldots, c_n)) = 0$, and also (after replacing $k$ by $n+1-k$),

$$T_k(B(0, c_1, \ldots, c_n)) = [n+1-k]T_k(B) + [k+1]q^{n-k}T_{k+1}(B), \qquad 1 \le k \le n+1. \tag{23}$$

Assuming $T_k(B)$ is $zsu(N_k(B))$, and also that $T_{k+1}(B)$ is $zsu(N_{k+1}(B))$, both terms on the RHS of (23) have darga $N_k(B(0, c_1, \ldots, c_n))$, and by Claim 1, the LHS of (23) does also. $\blacksquare$

We now proceed with the proof of Theorem 6 by induction on $\mathrm{Area}(B)$. If $\mathrm{Area}(B) = 0$, then $B$ is the trivial board of width $n$ (i.e. $c_i = 0$ for $1 \le i \le n$). It follows from (3) that

$$\Phi(x; B) = \sum_{k=0}^{n} R_{n-k}(B)[k]! \frac{x^k}{(1-x)\cdots(1-xq^k)}$$

$$= [n]! \frac{x^n}{(1-x)\cdots(1-xq^n)},$$

which implies

$$T_k(B) = \begin{cases} [n]! & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}.$$

Since $N_0(B) = n^2 - \binom{n+1}{2} = \binom{n}{2} = \text{darga}([n]!)$, this shows Theorem 6 is true if $B$ is trivial.

The rest of the proof is precisely as in [GaRe]. Assume $\text{Area}(B) > 0$, and that Theorem 6 is true for all boards of smaller Area than $B$. We now show that $B$ can be obtained from a board of smaller Area by a sequence of RAISE, FLIP, or ADD operations, and Theorem 6 follows by induction.

Case 1) $1 \le c_1 \le c_2 \le \cdots \le c_n$. Applying RAISE to $B(c_1 - 1, c_2 - 1, \ldots, c_n - 1)$ results in $B$, and $\text{Area}(B(c_1 - 1, \ldots, c_n - 1)) < \text{Area}(B)$.

Case 2) $0 \le c_1 \le c_2 \le \cdots \le c_n = n$. After performing FLIP, $B^*$ falls under Case 1.

Case 3) $0 \le c_1 \le c_2 \le \cdots \le c_n \le n - 1$. Let $s := \min\{i : c_i > 0\}$ (if $s$ doesn't exist, $B$ is trivial).

Subcase a) $n - s + 1 \ge c_n$. Let $H := B(c_s - 1, c_{s+1} - 1, \ldots, c_n - 1)$, and note that $\text{Area}(H) < \text{Area}(B)$. Since $n - s + 1 \ge c_n$, performing RAISE to $H$ results in an admissible board, and following this by $s - 1$ ADD operations, we end up with $B$.

Subcase b) $n - s + 1 < c_n$. After performing FLIP, $B^*$ falls under Subcase a. This completes the proof of Theorem 6. ■

Let $B(h_1, d_1; h_2, d_2; \ldots ; h_t, d_t)$ denote the Ferrers board of Fig. 11. Lemma 1 implies that

$$\frac{T_k(B)}{\prod_{i=1}^t [d_i]!} \in \mathbb{N}[q].$$

In certain cases we can show that $T_k(B)/\prod_i [d_i]!$ is symmetric and unimodal; this is a stronger condition then $T_k(B)$ being symmetric and unimodal by Claim 2.

**Definition 11.** *If $m < 0$, extend the definition of the $q$-binomial coefficient in the standard way;*

$$\begin{bmatrix} m \\ k \end{bmatrix} := \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

*We call $m$ the numerator of the $q$-binomial coefficient. Also given numbers $d_i, e_i$ and $h_i$ for $1 \le i \le t$, let $D_i, E_i$, and $H_i$ be abbreviations for the partial sums $d_1 + d_2 + \ldots + d_i$,$e_1 + \ldots + e_i$, and $h_1 + \ldots + h_i$, respectively, with $1 \le i \le t$ and $D_0 = E_0 = H_0 = 0$.*

**Claim 3.** *Given integers $d_i$, $e_i$, and $h_i$, with $0 \le e_i \le d_i$, $d_i \in \mathbb{P}$, $h_i \in \mathbb{N}$ for $1 \le i \le t$, let*

$$P(\mathbf{e}) := \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} \\ e_i \end{bmatrix}.$$

*Set $d_0 = 0$ and assume that either*

*1) $d_{i-1} + d_i \ge h_i$ for $1 \le i \le t$,*

*or*

*2) $D_i \ge H_i$ for $1 \le i \le t$.*

*Then if any of the numerators of the q-binomial coefficients in the definition of $P(\mathbf{e})$ are negative, $P(\mathbf{e}) = 0$.*

*Proof* : If $H_k - D_{k-1} + E_{k-1} < 0$ for some $k$ with $1 \le k \le t$, choose $j$ so that for $i < j$, $H_i - D_{i-1} + E_{i-1} \ge 0$ and $H_j - D_{j-1} + E_{j-1} < 0$. Note that $j \ge 2$. Now $H_j - D_{j-1} + E_{j-1} < 0$ implies $H_j - D_{j-2} + E_{j-2} < d_{j-1} - e_{j-1}$ which implies $H_{j-1} - D_{j-2} + E_{j-2} < d_{j-1} - e_{j-1}$ which implies

$$\begin{bmatrix} H_{j-1} - D_{j-2} + E_{j-2} \\ d_{j-1} - e_{j-1} \end{bmatrix} = 0$$

(since the numerator of this $q$-binomial coefficient is nonnegative by definition of $j$). This implies $P(\mathbf{e}) = 0$. Next assume we have a $j$ for which $D_j + D_{j-1} - H_j - E_{j-1} < 0$, but $D_i + D_{i-1} - H_i - E_{i-1} \ge 0$ for $1 \le i < j$. If condition 2) is true, this is impossible, for $D_j + D_{j-1} - H_j - E_{j-1} \ge D_j + D_{j-1} - H_j - D_{j-1} = D_j - H_j$. So assume 1) holds. Then $D_j + D_{j-1} - H_j - E_{j-1} < 0$ implies $D_j + D_{j-1} - H_j - E_{j-2} < e_{j-1}$ which implies $d_j + d_{j-1} - h_j + D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} < e_{j-1}$ which implies $D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} < e_{j-1}$ which implies

$$\begin{bmatrix} D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} \\ e_{j-1} \end{bmatrix} = 0,$$

since the numerator is nonnegative by definition of $j$ but less than the denominator. ∎

**Claim 4.** *Given integers $d_i$, $e_i$, and $h_i$ as in Claim 3, let*

$$Q(s) := \prod_{i=1}^{t} \begin{bmatrix} s + H_i - D_{i-1} \\ d_i \end{bmatrix}.$$

*Then if $s + H_i - D_{i-1} < 0$ for any $i$ satisfying $1 \le i \le t$, then $Q(s) = 0$.*

*Proof* : Let $j$ be such that $s + H_j - D_{j-1} < 0$, but $s + H_i - D_{i-1} \ge 0$ for $1 \le i < j$. Note that $j \ge 2$ since $s + H_1 - D_0 = s + h_1 \ge 0$. Now $s + H_j - D_{j-1} < 0$ implies $s + H_j - D_{j-2} < d_{j-1}$ which implies $s + H_{j-1} - D_{j-2} < d_{j-1}$ which implies $\begin{bmatrix} s + H_{j-1} - D_{j-2} \\ d_{j-1} \end{bmatrix} = 0$ since the numerator of this $q$-binomial coefficient is nonnegative. ∎

We have previously assumed that $B$ is an admissible board ($c_n \le n$) but in the next theorem we remove that restriction. Note that the definition of $R_k$ makes sense if $c_n > n$ as well, and for such inadmissible boards we define $T_k$ via (3) (in general these $T_k \notin \mathbb{N}[q]$).

**Theorem 7.** *Let $B = B(h_1, d_1; \dots; h_t, d_t)$ be the Ferrers board of Fig. 11, where $H_t$ may be greater than $D_t$ ($B$ inadmissible). Set $L_k(B) = Area(B) + n(n-k) - \sum_{i=1}^{t} D_i d_i$. Then $T_k(B)/\prod_{i=1}^{t}[d_i]!$ is either zero or symmetric with darga $L_k(B)$. In addition, if either*

*1) $d_{i-1} + d_i \ge h_i$ for $1 \le i \le t$,*

*or*

*2) $D_i \ge H_i$ for $1 \le i \le t$,*

*then*

$$\frac{T_k(B)}{\prod_{i=1}^{t}[d_i]!} \text{ is } zsu(L_k(B)).$$

*Proof* : We require the following formulas

$$\frac{T_{n-k}(B)}{\prod_{i=1}^{t}[d_i]!} = \sum_{s=0}^{k} \begin{bmatrix} n+1 \\ k-s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}} \prod_{i=1}^{t} \begin{bmatrix} s + H_i - D_{i-1} \\ d_i \end{bmatrix}, \tag{24}$$

and

$$T_{n-k}(B) = [d_t]! \sum_{k-d_t \le s \le k} T_{n-d_t-s}(B') \begin{bmatrix} H_t - n + d_t + s \\ d_t - k + s \end{bmatrix} \begin{bmatrix} 2n - d_t - H_t - s \\ k - s \end{bmatrix} q^{(k-s)(H_t+k-n)}, \tag{25}$$

where $B' = B(h_1, d_1; \dots ; h_{t-1}, d_{t-1})$ is obtained by truncating the last $d_t$ columns of $B$. The initial conditions are given by $T_s(\emptyset)$ equals 1 if $s = n$ and zero otherwise, where $\emptyset$ denotes the empty board with zero columns.

Eq. (24) is easily derived from (2), (3), and the $q$-Vandermonde convolution [Hag1,p.98], [Dwo,p.39]. Eq. (25) is Theorem 4.3.13 of [Hag1], and can also be obtained by setting $p = t, x = y = 1$ in Corollary 5.10 of [Hag2], where an inductive proof of the result is given.

**Lemma 4.** *With $B$ as above and $P(\mathbf{e})$ as in Claim 3,*

$$T_{n-k}(B) = \prod_{i=1}^{t} [d_i]! \sum_{\substack{e_1+e_2+\ldots+e_t=k \\ 0 \le e_i \le d_i}} \prod_{i=1}^{t} P(\mathbf{e}) q^{e_i(H_i - D_i + E_i)}. \tag{26}$$

*Proof* : By induction, the case $t = 1$ following from (25). For $t > 1$, using (25) and the inductive hypothesis we get

$$T_{n-k}(B) = \prod_{i=1}^{t} [d_i]! \sum_{\substack{0 \le e_t \le k \\ 0 \le e_t \le d_t}} \begin{bmatrix} H_t - D_{t-1} + k - e_t \\ d_t - e_t \end{bmatrix} \begin{bmatrix} D_t + D_{t-1} - H_t - k + e_t \\ e_t \end{bmatrix} q^{e_t(H_t+k-D_t)}$$

$$\times \sum_{\substack{e_1+\ldots+e_{t-1}=k-e_t \\ 0 \le e_i \le d_i}} \prod_{i=1}^{t-1} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} \\ e_i \end{bmatrix} q^{e_i(H_i-D_i+E_i)}$$

which equals the RHS of (26) since $k - e_t = E_{t-1}$.

We now proceed with the proof of Theorem 7. We use the well-known fact (see [GoOH],[Zei] for an amazing constructive proof) that for all $m \in \mathbb{N}$ and $k \in \mathbb{N}$, $\begin{bmatrix} m \\ k \end{bmatrix}$ is $zsu(k(m-k))$. Claim 4 implies that all the terms on the RHS of (24) are polynomials. After a short calculation, we see that they are all symmetric with *darga* $L_k(B)$. This proves the first part of the theorem, but unfortunately the terms on the RHS of (24) alternate in sign and so we cannot conclude that the LHS is unimodal. However, if condition 1) or 2) of Theorem 7 are satisfied, we can apply Claim 3 and conclude all the terms on the RHS of (26) are $\in \mathbb{N}[q]$. They are also all of *darga* $L_k(B)$, and so the second part of Theorem 7 follows by Claim 1. ∎

**Corollary 3.** *For any vector $\mathbf{v}$ of nonnegative integers,*

$$\sum_{\substack{\pi \in M(\mathbf{v}) \\ \mathrm{des}(\pi)=k}} q^{\mathrm{maj}(\pi)} \text{ is } zsu(nk).$$

*Proof* : The board $G_{\mathbf{v}}$ satisfies condition 2) of Theorem 7, and combining this with (18) we have

$$\sum_{\substack{\pi \in M(\mathbf{v}) \\ \mathrm{des}(\pi)=k}} q^{\mathrm{maj}(\pi)}$$

is

$$zsu(\text{Area}(G_{\mathbf{v}}) + n(n-k) - \sum_{i=1}^{t} v_i(v_1 + v_2 + \ldots + v_i) + 2nk - 2\text{Area}(G_{\mathbf{v}})). \tag{27}$$

Now

$$\text{Area}(G_{\mathbf{v}}) = \sum_{i=1}^{t} v_i(v_1 + v_2 + \ldots + v_{i-1}),$$

and since $(v_1 + v_2 + \ldots + v_t)^2 = n^2$, (27) reduces to $zsu(nk)$. ■.

## 6. Final Comments

In [Hag1,p.130], the following more general form of Corollary 3 is derived

$$\sum_{\substack{\pi \in M(\mathbf{v}) \\ k\ r\text{-descents}}} q^{r\text{maj}(\pi)}$$

is $zsu(nk + \sum_{i=1}^{t} v_i(v_{i-r+1} + \ldots + v_{i-1}))$, which involves the $(q-r)$ Simon Newcomb numbers introduced by Rawlings [Raw]. The author hopes to describe connections between these numbers and $q$-rook polynomials more fully elsewhere [Hag3].

Galovich and White have introduced a very general method of generating Mahonian statistics, statistics they call "splittable" [GaWh]. The author would like to thank them for consultations regarding the statistic $mat(B)$, which together with simple examples have led to the conclusion that $mat$ is not splittable, at least not for all boards $B$.

For some time researchers have sought a $q$-analog of the theory of permutations with restricted position. No positive answer to this question has ever been found. Joni and Rota [JoRo] showed how the study of vector spaces over finite fields with restricted bases is relevant to this problem. Later Chen and Rota [ChRo] proved that if you require a $q$-analog to have a certain interpretation in terms of automorphisms with prescribed behavior, then a solution is possible only for a few types of boards. There are interesting similarities between, but no obvious overlap with, some of their results and ours.

One can also try and develop a $q$-analog by finding a way of defining $R_k$ for arbitrary boards (not just Ferrers boards) such that a $q$-analog of (3) holds. Perhaps the connection between matrices over $\mathbb{F}_q$ of fixed rank and rook placements will shed some light on this question.

*Note added in proof:* Using Theorem 6, Theorem 2 implies that the statistic $\text{cross}(C, B) - \binom{n+1}{2}$ also generates $T_k(B)$.

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