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A Positivity Result in the Theory of Macdonald Polynomials

by

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Abstract. We outline here a proof that a certain rational function $C_n(q,t)$ which has come to be known as the "q,t-$Catalan$" is in fact a polynomial with positive integer coefficients. This has been an open problem since 1994, (see Garsia-Haiman J. Algebraic Combin. **5** (1996), no. 3, 191–244). Since $C_n(q,t)$ evaluates to the Catalan number at $t = q = 1$, it has also been an open problem to find a pair of statistics a, b on the collection $\mathcal{D}_n$ of Dyck paths Π of length $2n$ yielding $C_n(q,t) = \sum_\pi t^{a(\Pi)} q^{b(\Pi)}$. Our proof is based on a recursion for $C_n(q,t)$ suggested by a pair of statistics recently proposed by J. Haglund. One of the byproducts of our results is a proof of the validity of Haglund's conjecture. It should also be noted that our arguments rely and expand on the plethystic machinery developed in Methods and Applications of Analysis, VII, **3**, (**99**), 363–420.

1. Preliminaries

At the 1988 Alghero meeting of the Lotharingian Seminar Macdonald introduced a 2-parameter symmetric function basis $\{J_\mu[X;q,t]\}_\mu$ which has since proved to be fundamental in the Theory of Symmetric Functions. In recent years the Theory of Symmetric Functions has acquired particular importance because of its relation to the Representation Theory of Hecke algebras and the Symmetric Groups, and has been shown to have applicability in a wide range of scientific and mathematical disciplines. In many of these developments the Macdonald polynomials and some of their specializations have played a central role. In the original paper [1] and in subsequent work ([2], [3], [4], [5], [6], [7]) a number of conjectures have been formulated which assert that certain rational functions in q,t are in fact polynomials with positive integer coefficients. For a decade these conjectures have resisted several various attempts of proof by a wide range of approaches. Although these conjectures lie squarely within the Theory of Symmetric Functions, the approaches range from diagonal actions of the symmetric group on polynomial rings in two sets of variables [2],[3], [5] to the Algebraic Geometry of Hilbert schemes [8]. Efforts to resolve these conjectures within the Theory of Symmetric Functions, have led to the discovery of a variety of new methods to deal with symmetric function identities [3], [4], [6], [8]. In this paper we outline an argument that yields a purely symmetric function proof of one of these conjectures. To state the result we need some definitions and notational conventions.

A partition μ will always be identified with its Ferrers diagram. The partition conjugate to μ will be denoted "μ' ". By the French convention, if the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, the Ferrers diagram has μ_i lattice cells in the i^{th} row (from the bottom up). Here $|\mu|$ and $l(\mu)$ denote respectively the sum of the parts and the number of nonzero parts of μ. The symbol "$\mu \vdash n$" will also be used to indicate that $|\mu| = n$. Following Macdonald, the *arm, leg, coarm* and *coleg* of a lattice square s are the parameters $a_\mu(s), l_\mu(s), a'_\mu(s)$ and $l'_\mu(s)$ giving the number of cells of μ that are respectively *strictly* EAST, NORTH, WEST and SOUTH of s in μ.

Here and after, for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ we set

$$T_\mu = \prod_{s\in\mu} t^{l'_\mu(s)} q^{a'_\mu(s)} \,, \qquad B_\mu(q,t) = \sum_{s\in\mu} t^{l'_\mu(s)} q^{a'_\mu(s)} \,, \qquad \Pi_\mu(q,t) = \prod_{s\in\mu} \big(1 - t^{l'_\mu(s)} q^{a'_\mu(s)}\big) \qquad 1.1$$

and

$$\tilde{h}_\mu(q,t) \,=\, \prod_{s\in\mu} \big(q^{a'_\mu(s)} - t^{l'_\mu(s)+1}\big) \quad , \quad \tilde{h}'_\mu(q,t) \,=\, \prod_{s\in\mu} \big(t^{l'_\mu(s)} - q^{a'_\mu(s)+1}\big) \,. \qquad 1.2$$

This given we can show

Theorem 1.1

For every $n \geq 1$ the rational function

$$C_n(q,t) \;=\; \sum_{\mu\vdash n} \frac{T_\mu^2\,(1-t)(1-q)B_\mu(q,t)\,\Pi_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \qquad 1.3$$

evaluates to a polynomial with positive integer coefficients.

To show how this relates to Macdonald polynomials and to outline our proof we need to introduce plethystic notation. This is a very powerful notational device which considerably facilitates the manipulation of symmetric function identities. This device can also easily implemented in software such as MAPLE or MATHEMATICA when we express symmetric functions in terms of the power sum sequence $\{p_k\}_{k\geq 1}$. To begin with, if $E = E[t_1, t_2, t_3, \ldots]$ is a formal Laurent series in the variables $t_1, t_2, t_3, \ldots$ (which may include the parameters q, t) we set

$$p_k[E] \;=\; E[t_1^k, t_2^k, t_3^k, \ldots]\ .$$

More generally, if a certain symmetric function F is expressed as the formal power series

$$F \;=\; Q[p_1, p_2, p_3, \ldots]$$

then we simply let

$$F[E] \;=\; Q[p_1, p_2, p_3, \ldots]\Big|_{p_k\to E[t_1^k, t_2^k, t_3^k, \ldots]}\ , \qquad 1.4$$

and refer to it as "*plethystic substitution*" of E into the symmetric function F. We also adopt the convention that inside the plethystic bracket X and X_n stand for $X = x_1 + x_2 + \cdots$ and $X_n = x_1 + x_2 + \cdots + x_n$. In particular a symmetric polynomial $P = P(x_1, x_2, \ldots, x_n)$ may be simply written in the form $P = P[X_n]$. We should mention that the present breakthrough would not have been possible without the insight provided by this notational device.

This given, we will work here with the modified Macdonald polynomial $\tilde{H}_\mu[X; q, t]$ obtained by setting

$$\tilde{H}_\mu[X; q, t] \;=\; t^{n(\mu)} J_\mu[\tfrac{X}{(1-1/t)}; q, 1/t] \qquad \big(\text{with} \quad n(\mu) = \textstyle\sum_{s\in\mu} l'_\mu(s)\big) \qquad 1.5$$

Another important ingredient here is the linear operator ∇ defined by setting for the basis $\big\{\tilde{H}_\mu[X; q, t]\big\}_\mu$:

$$\nabla \tilde{H}_\mu[X; q, t] \;=\; T_\mu \tilde{H}_\mu[X; q, t]. \qquad 1.6$$

Now it was shown in [6] that the elementary symmetric function e_n has the expansion

$$e_n[X] = \sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X; q, t](1-t)(1-q)B_\mu(q,t)\,\Pi_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)} \qquad 1.7$$

so that 1.6 gives

$$\nabla e_n[X] = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu[X;q,t](1-t)(1-q)B_\mu(q,t)\,\Pi_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}\,. \tag{1.8}$$

Equating coefficients of the Schur function S_λ gives

$$\nabla e_n[X]\Big|_{S_\lambda} = \sum_{\mu \vdash n} \frac{T_\mu \tilde{K}_{\lambda\mu}(q,t)(1-t)(1-q)B_\mu(q,t)\,\Pi_\mu(q,t)}{\tilde{h}_\mu(q,t)\,\tilde{h}'_\mu(q,t)}\,, \tag{1.9}$$

where from 1.5 we derive that $\tilde{K}_{\lambda\mu}(q,t)$ is related to the Macdonald q,t-Kostka coefficient $K_{\lambda\mu}(q,t)$ by the simple reversion

$$\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,1/t)\,.$$

In particular, it follows from Macdonald's work [9] that (for $\mu \vdash n$)

$$\tilde{K}_{1^n,\mu}(q,t) = T_\mu\,. \tag{1.10}$$

Using this in 1.9 for $\lambda = 1^n$ 1.3 becomes

$$C_n(q,t) \quad = \quad \nabla e_n[X]\Big|_{S_{1^n}}\,. \tag{1.11}$$

Our proof of Theorem 1.1 is based on this identity. The reader is referred to [6] and [7] for several conjectures concerning the expressions in 1.9.

Here it suffices to know that it was shown in [4] that ∇ acts integrally on Schur functions. This implies that all the expressions in 1.9, and in particular $C_n(q,t)$, evaluate to polynomials in q,t with integer coefficients. Our proof of Theorem 1.1 gives the positivity of the latter coefficients as well as a combinatorial interpretation of their values. This is obtained by means of a recursion satisfied by the 2-parameter family of polynomials

$$Q_{n,s}(q,t) \quad = \quad t^{n-s} q^{\binom{s}{2}}\, \nabla e_{n-s}\big[X \tfrac{1-q^s}{1-q}\big]\Big|_{S_{1^{n-s}}}\,. \tag{1.12}$$

More precisely we show that

Theorem 1.2

For any pair of integers $n \geq s \geq 1$ we have

$$Q_{n,s}(q,t) \quad = \quad t^{n-s} q^{\binom{s}{2}} \sum_{r=0}^{n-s} \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q Q_{n-s,r}(q,t) \quad , \tag{1.13}$$

where $\left[{n \atop k}\right]_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$ denotes the q-binomial coefficient.

Note that since $\nabla 1 = 1$, 1.12 gives the initial conditions

$$Q_{n,n}(q,t) \quad = \quad q^{\binom{s}{2}} \,. \tag{1.14}$$

It is then easily seen that 1.13 yields

$$Q_{n,s}(q,t) \in \mathbf{N}[q,t] \qquad \forall\, n \geq s \geq 1 \ .$$

Moreover, 1.12 and 1.13 with $n \to n+1$ and $s \to 1$ give

$$C_n(q,t) = \nabla e_n \Big|_{S_{1^n}} = \sum_{r=1}^{n} Q_{n,1}(q,t) \in \mathbf{N}[q,t] \ . \tag{1.15}$$

A remarkable corollary of 1.13 and 1.15 is the combinatorial formula

$$C_n(q,t) \quad = \sum_{\Pi \in \mathcal{D}_n} t^{area(\Pi)} q^{maj(\beta(\Pi))} \tag{1.16}$$

where $\mathcal{D}_n$ is the collection of all Dyck paths of length $2n$, "$area(\Pi)$" denotes the area under the path, and "$maj(\beta(\Pi))$" denotes the "*major index*" of a certain path $\beta(\Pi)$ associated to Π. The reader is referred to [6] for a more detailed description of these combinatorial structures.

We must mention that 1.16 had been previously conjectured by J. Haglund in an article to appear in the journal Advances in Mathematics and was in fact the starting point of the investigation that led to the present results.

2. Outline of the argument.

Since it can be shown that

$$\begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \quad = \quad h_r\big[\tfrac{1-q^s}{1-q}\big] \tag{2.1}$$

we see that 1.13 simply states that the equality

$$\nabla e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{S_{1^m}} \quad = \quad \sum_{r=1}^{m} h_r\big[\tfrac{1-z}{1-q}\big]\, t^{m-r} q^{\binom{r}{2}} \nabla e_{m-r}\big[X\tfrac{1-q^r}{1-q}\big]\Big|_{S_{1^{m-r}}} \tag{2.2}$$

must hold true for $z = q^s$ and all pairs $m, s \geq 1$. This of course implies (and is, in fact, equivalent to) the equality of the two polynomials on both sides of 1.2.

Now the polynomials

$$h_r\big[\tfrac{1-z}{1-q}\big] = \tfrac{(z;q)_n}{(q;q)_n} \tag{2.3}$$

have the "Taylor" expansion formula:

$$P(z) \quad = \quad \sum_{r \geq 0} \frac{(z;q)_r}{(q;q)_r}\, q^r \big(\delta_q^r P(z)\,\big|_{z=1}\big)\,, \tag{2.4}$$

with δ_q the q-difference operator

$$\delta_q P(z) \;=\; \frac{P(z)-P(z/q)}{z}\,. \tag{2.5}$$

Using 2.4 we immediately derive that 1.13 holds true if and only if we have

$$\delta_q^k \nabla e_m\big[X\tfrac{1-z}{1-q}\big]\Big|_{S_{1^m}}\Big|_{z=1} \;=\; \frac{t^{m-k}}{q^k}\, q^{\binom{k}{2}}\, \nabla e_{m-k}\big[X\tfrac{1-q^k}{1-q}\big]\Big|_{S_{1^{m-k}}} \qquad \forall\;\; k=1,2,\ldots,m\,. \tag{2.6}$$

This identity is made more amenable to symmetric function manipulations by means of the expansions

$$e_m\big[X\tfrac{1-q^k}{1-q}\big] = (1-q^k)\sum_{\mu\vdash m}\frac{\tilde H_\mu[X;q,t]\Pi_\mu(q,t)h_k\big[(1-t)B_\mu(q,t)\big]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)} \tag{2.7}$$

$$h_m\big[X\tfrac{1-q^k}{1-q}\big] = (-t)^{m-k}q^{k(m-1)}(1-q^k)\sum_{\mu\vdash m}\frac{\tilde H_\mu[X;q,t]\Pi_\mu(q,t)e_k\big[(1-t)B_\mu(\tfrac1q,\tfrac1t)\big]}{\tilde h_\mu(q,t)\tilde h'_\mu(q,t)}\,. \tag{2.8}$$

Using these relations we were able to derive from 2.6 that Theorem 1.1 is equivalent to the following

Theorem 2.1

For all $1\le k\le m$ *we have*

$$\begin{aligned}\sum_{i=1}^{k}\Big[\begin{matrix}k\\ i\end{matrix}\Big]_q q^{\binom{i}{2}-i}\,(1-q^i)\,t^{m-i}\sum_{\mu\vdash m}\frac{T_\mu^2\,\Pi_\mu}{\tilde h_\mu\tilde h'_\mu}\,e_i[(1-t)B_\mu(\tfrac1q,\tfrac1t)] \;&=\\ =\;\frac{t^{m-k}}{q^k}\,q^{\binom{k}{2}}\,(1-q^k)\sum_{\nu\vdash m-k}&\frac{{T_\nu}^2\Pi_\nu h_k\big[(1-t)\,B_\nu(q,t)\big]}{\tilde h_\nu(q,t)\tilde h'_\nu(q,t)}\,.\end{aligned} \tag{2.9}$$

Note that 2.9 for $k=1$ reduces to

$$\sum_{\mu\vdash m}\frac{{T_\mu}^2\Pi_\mu}{\tilde h_\mu\tilde h'_\mu}\,B_\mu(\tfrac1q,\tfrac1t) = \sum_{\nu\vdash m-1}\frac{{T_\nu}^2\Pi_\nu}{\tilde h_\nu\tilde h'_\nu}\,B_\nu(q,t)\;, \tag{2.10}$$

and for $k=2$

$$\begin{aligned}t(1-t)\sum_{\mu\vdash m}\frac{{T_\mu}^2\Pi_\mu}{\tilde h_\mu\tilde h'_\mu}B_\mu(\tfrac1q,\tfrac1t)+\sum_{\mu\vdash m}\frac{{T_\mu}^2\Pi_\mu}{\tilde h_\mu\tilde h'_\mu}\,e_2\big[(1-t)B_\mu(\tfrac1q,\tfrac1t)\big] \;&=\\ =\sum_{\nu\vdash m-2}\frac{{T_\nu}^2\Pi_\nu}{\tilde h_\nu\tilde h'_\nu}\,h_2\big[(1-t)B_\nu(q,t)\big]\,.&\end{aligned} \tag{2.11}$$

To establish these identities we need a basic mechanism for converting sums over partitions of size m to sums over partitions of smaller size. Now, it develops that this can be achieved by summation formulas involving "*Pieri*" coefficients. The latter are the rational functions $d^{f}_{\mu\nu}(q,t)$ occurring in multiplication rules of the form.

$$f[X]\tilde{H}_{\nu}[X;q,t\,] \;=\; \sum_{\substack{\mu\supseteq\nu\\ k\leq|\mu|\leq k+d}} d^{f}_{\mu\nu}(q,t)\;\tilde{H}_{\mu}[X;q,t\,]\,, \qquad 2.12$$

when $\nu\vdash k$ and $f[X]$ is a symmetric function of degree d. Stanley for the Jack symmetric functions case [10] and Macdonald in [1] give explicit formulas for $d^{f}_{\mu\nu}(q,t)$ when $f=h_d$ or $f=e_d$ for some $d\geq 1$. These formulas may be used to settle 2.10 and 2.11. They should also yield what is needed for 2.9 as well, since in principle the multiplication rules for any f may be obtained by combining successive multiplications by the elementary (or homogeneus) symmetric functions. However, to carry this out in full generality we run into a task of forbidding complexity.

The breakthrough was the discovery that the necessary summation formulas may be directly obtained through the operator ∇. This shows once more that this remarkable operator somehow encodes within its action a great deal of the combinatorial complexity of Macdonald polynomials (see [2], [3], [4], [11]). To state our summation formulas we need further notation.

Let us recall that the so called "*Hall*" scalar product $\langle\;,\;\rangle$ is defined by setting for the power basis

$$\big\langle\, p_{\mu}\;,\;p_{\nu}\big\rangle \;=\; \chi(\mu=\nu)\, z_{\mu}\,, \qquad 2.13$$

where for a partition $\mu=1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots$ we set $z_{\mu}=1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^{\alpha_3}\alpha_3!\cdots$ Our versions of the Macdonald polynomials $\tilde{H}_{\mu}$ are orthogonal with respect to the scalar propduct $\langle\;,\;\rangle_{*}$ defined by setting

$$\big\langle\, p_{\mu}\;,\;p_{\nu}\big\rangle_{*} \;=\; (-1)^{|\mu|-l(\mu)}\chi(\mu=\nu)\, z_{\mu}p_{\mu}[(1-t)(1-q)]\,. \qquad 2.14$$

To be precise we have

$$\langle\,\tilde{H}_{\mu}\;,\;\tilde{H}_{\nu}\rangle_{*} \;=\; \tilde{h}_{\mu}\tilde{h}'_{\mu}\;\chi(\mu=\nu)\,. \qquad 2.15$$

Now, companions to the Pieri rules in 2.12 are their dual forms

$$f^{\perp}\,\tilde{H}_{\mu}[X;q,t\,] \;=\; \sum_{\substack{\nu\subseteq\mu\\ m-d\leq|\nu|\leq m}} c^{f^{\perp}}_{\mu\nu}(q,t)\,\tilde{H}_{\nu}[X;q,t\,] \qquad 2.16$$

where $\mu\vdash m$, f is any symmetric function of degree d and $f^{\perp}$ denotes the operator that is the Hall-adjoint of multiplication by f. We should note that the Pieri coefficients and their dual counterparts are related by the identity

$$c^{f^{\perp}}_{\mu\nu}(q,t)\;\tilde{h}_{\nu}\tilde{h}'_{\nu} \;=\; d^{\omega f^{*}}_{\mu\nu}(q,t)\,\tilde{h}_{\mu}\tilde{h}'_{\mu}\;\;, \qquad 2.17$$

where ω as customary denotes the fundamental involution of symmetric functions and for any symmetric polynomial f we set

$$f^*[X] \;=\; f[\tfrac{X}{(1-t)(1-q)}]\,. \qquad 2.18$$

This is an easy consequence of 2.15 and the definitions in 2.12 and 2.16.

This given, our proof of the recursion in 1.13 is based on the following two remarkable summation formulas.

Theorem 2.2

For g a symmetric polynomial of degree d and $\mu \vdash m$ we have

$$\sum_{\substack{\nu\subseteq\mu \\ m-d\le|\nu|\le m}} c_{\mu\nu}^{(\omega g)^{\perp}}(q,t)\,T_\nu \;=\; T_\mu\, G\big[(1-\tfrac{1}{t})(1-\tfrac{1}{q})B_\mu(\tfrac{1}{q},\tfrac{1}{t})-1\big] \qquad 2.19$$

with

$$G[X] \;=\; \omega\nabla\Big(g\big[\tfrac{X+1}{\widetilde{M}}\big]\Big)\,. \qquad 2.20$$

Theorem 2.3

For f a symmetric polynomial of degree d and $\nu \vdash k$ we have

$$\sum_{\substack{\mu\supseteq\nu \\ k\le|\mu|\le k+d}} d_{\mu,\nu}^{f}T_\mu\Pi_\mu \;=\; T_\nu\Pi_\nu\,\big(\nabla f\big)\big[MB_\nu\big]\,. \qquad 2.21$$

We should mention that both 2.19 and 2.21 are ultimate consequences of the following result (proved in [11])

Theorem 2.4

For a given symmetric function P set

$$\mathbf{\Pi}_P[X;q,t] \;=\; \nabla^{-1}P[X-\epsilon]\,\big|_{\epsilon=-1}.$$

Then for all partitions μ we get

$$\big\langle P\;,\;\tilde{H}_\mu[X+1;q,t]\big\rangle_* \;=\; \mathbf{\Pi}_P\big[(1-t)(1-q)B_\mu(q,t)-1\,;q,t\big]\,.$$

To give a idea of the manner in which 2.19 and 2.21 are used to obtain 2.9, we shall use them to prove 2.10 and 2.11.

To begin, the case $\omega g = h_1$ of 2.19 gives

$$T_\mu B_\mu(\tfrac{1}{q},\tfrac{1}{t}) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}^{h_1^{\perp}}(q,t)\,T_\nu$$

where the symbol "$\nu{\rightarrow}\mu$" is to indicate that ν is obtained by removing one of the corners of μ. Substituting this in the left hand side of 2.10 gives

$$\begin{aligned}\sum_{\mu\vdash m}\frac{{T_\mu}^2\Pi_\mu}{\tilde h_\mu \tilde h'_\mu}B_\mu\big(\tfrac1q,\tfrac1t\big) &= \sum_{\mu\vdash m}\frac{T_\mu\Pi_\mu}{\tilde h_\mu \tilde h'_\mu}\sum_{\nu\to\mu}c_{\mu\nu}^{h_1^\perp}(q,t)\,T_\nu\\ &= \sum_{\nu\vdash m-1}\frac{T_\nu}{\tilde h_\nu \tilde h'_\nu}\sum_{\mu\leftarrow\nu}c_{\mu\nu}^{h_1^\perp}\frac{\tilde h_\mu \tilde h'_\mu}{\tilde h_\mu \tilde h'_\mu}T_\mu\Pi_\mu \qquad 2.22\\ \text{(by 2.17)}\qquad &= \sum_{\nu\vdash m-1}\frac{T_\nu}{\tilde h_\nu \tilde h'_\nu}\sum_{\mu\leftarrow\nu}d_{\mu\nu}^{e_1^*}(q,t)\,T_\mu\Pi_\mu\,.\end{aligned}$$

Now 2.21 gives

$$\sum_{\mu\leftarrow\nu}d_{\mu\nu}^{e_1^*}(q,t)\,T_\mu\Pi_\mu \;=\; T_\nu\Pi_\nu B_\nu(q,t)$$

which when substituted in 2.22 immediately yields 2.10.

For 2.11 we use 2.19 with

$$\omega g \;=\; e_2\big[(1-t)X\big]$$

and obtain

$$\frac1q\sum_{\nu\subseteq_2\mu}c_{\mu\nu}^{\omega g^\perp}T_\nu \;=\; t(1-t)\;\;T_\mu\;B_\mu\big(\tfrac1t,\tfrac1q\big)\;\;+\;\;T_\mu\;e_2\big[(1-t)B_\mu\big(\tfrac1t,\tfrac1q\big)\big]\;,$$

where the symbol "$\nu\subseteq_2\mu$" means that ν is obtained by removing two corners from μ. Substituting this in the left hand side of 2.11 gives

$$\begin{aligned}\mathit{left\ hand\ side\ of}\ (2.11) &= \sum_{\mu\vdash m}\frac{T_\mu\Pi_\mu}{\tilde h_\mu \tilde h'_\mu}\,\frac1q\sum_{\nu\subseteq_2\mu}c_{\mu\nu}^{\omega g^\perp}T_\nu\\ &= \frac1q\sum_{\nu\vdash m-2}\frac{T_\nu}{\tilde h_\nu \tilde h'_\nu}\sum_{\mu\supseteq_2\nu}d_{\mu\nu}^{f}\,T_\mu\Pi_\mu\end{aligned}$$

with

$$f \;=\; h_2\big[\tfrac{X}{1-q}\big]$$

and 2.21 gives

$$\frac1q\sum_{\mu\supseteq_2\nu}d_{\mu\nu}^{f}\,T_\mu\Pi_\mu \;=\; T_\nu\Pi_\nu\;h_2\big[(1-t)B_\nu(q,t)\big]$$

and proves 2.11.

The details of all these calculations and the complete proof of Theorem 1.2 will appear in the proceedings of the September 2000 Montreal Colloquium in Algebraic Combinatorics which is to be published by Discrete Mathematics.

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