ROOK THEORY FOR PERFECT MATCHINGS

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Abstract. In classical rook theory there is a fundamental relationship between the rook numbers and the hit numbers relative to any board. In that theory the k-th hit number of a board B can be interpreted as the number of permutations whose intersection with B is of size k. In the case of Ferrers boards there are q-analogues of the hit numbers and the rook numbers developed by Garsia and Remmel [GaRe], Dworkin [D1], [D2] and Haglund [H]. In this paper we develop a rook theory appropriate for shifted partitions, where hit numbers can be interpreted as the number of perfect matchings in the complete graph whose intersection with the board is of size k. We show there is also analogous q-theory for the rook and hit numbers for these shifted Ferrers boards.

Introduction. Perfect Matchings and Rook Boards

In classical rook theory there is a fundamental relationship between the rook numbers and the hit numbers relative to any board. A board B is a subset of the $n \times n$ board A_n pictured in Fig. 1.

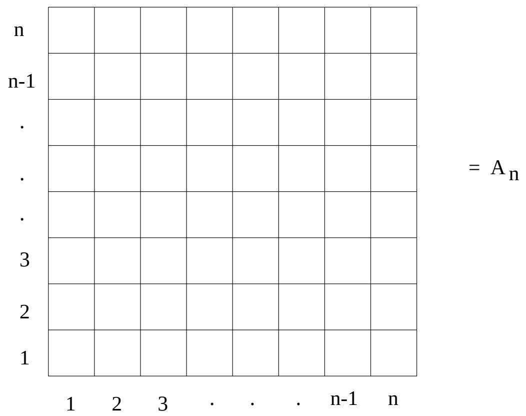

Figure 1

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Given a board $B \subseteq A_n$, we let $R_k(B)$ denote the set of all k element subsets p of B such that no two elements of p lie in the same row or column. Such a set p is called a rook placement of nonattacking rooks on B and $r_k(B) = |R_k(B)|$ is called the k-th rook number of B. For example, if $B \subseteq A_4$ is the board consisting of all shaded squares in Fig. 2, then $r_0(B) = 1$, $r_1(B) = 6$, $r_2(B) = 10$, $r_3(B) = 4$, and $r_4(B) = 0$.

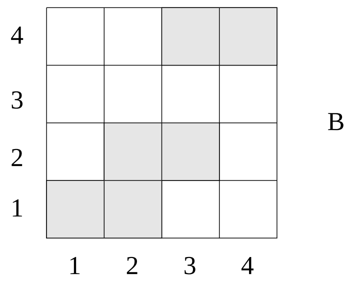

FIGURE 2

Given a permutation σ in the symmetric group S_n, we identify σ with the rook placement $p_\sigma = \{(i, j) : \sigma(i) = j\}$. We then define $H_{k,n}(B)$ to be the set of all $\sigma \in S_n$ such that $|p_\sigma \cap B| = k$ and we call $h_{k,n}(B) = |H_{k,n}(B)|$ the k-th hit number of B relative to A_n. One can easily prove the following formula which relates the rook numbers $r_k(B)$ to the hit numbers $h_{k,n}(B)$ for any board $B \subseteq A_n$.

$$\sum_{k=0}^{n} h_{k,n}(B)(z+1)^k = \sum_{k=0}^{n} r_k(B)(n-k)!z^k. \tag{1}$$

That is, it is easy to see that the left-hand side of (1) equals the sum

$$\sum_{\substack{(T,p_\sigma) \\ T \subseteq p_\sigma \cap B}} z^{|T|}. \tag{2}$$

However, the right-hand side of (1) also counts (2) since we can first pick $T \in R_k(B)$ and then extend it to a placement p_σ for some $\sigma \in S_n$ in $(n-k)!$ ways.

Replacing z by $z-1$ in (1) gives the following classical formula of Riordan and Kaplansky [KaRi]

$$\sum_{k=0}^{n} h_{k,n}(B)z^k = \sum_{k=0}^{n} r_k(B)(n-k)!(z-1)^k. \tag{3}$$

Garsia and Remmel [GaRe] gave a q-analogue of the rook numbers and hit numbers for a certain collection of boards $B \subseteq A_n$ called *Ferrers boards*. Let $A(a_1, a_2, \ldots, a_n)$ denote the board B contained in A_n consisting of all squares $\{(i, j) : j \le a_i\}$. For example, $A(1, 2, 2, 3)$ is pictured in Fig. 3.

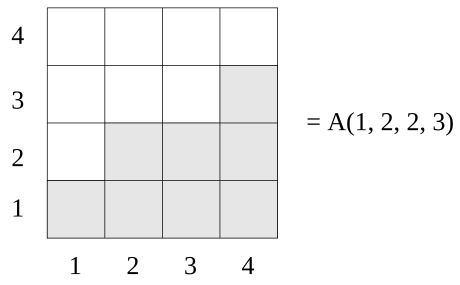

FIGURE 3

Thus $A(a_1, a_2, \ldots, a_n)$ denotes the board whose column heights reading from left to right are $a_1, a_2, \ldots, a_n$. We shall call a board $A(a_1, a_2, \ldots, a_n) \subseteq A_n$ a *skyline board.* $A(a_1, a_2, \ldots, a_n)$ is called a *Ferrers board* if $a_1 \leq a_2 \leq \cdots \leq a_n$.

Let $F = A(a_1, a_2, \ldots, a_n)$ be some fixed Ferrers board contained in A_n. Given a placement $p \in R_k(B)$, let each rook $\mathbf{r}$ cancel all squares to its right and all squares below $\mathbf{r}$. We let $u_F(p)$ denote the number of squares of F which are uncancelled, i.e. the number of squares which are neither in p nor cancelled by a rook in p. For example, if $F = A(1, 2, 2, 3, 4, 4)$ and p is the placement of $R_3(F)$ consisting of the squares containing an x in Fig. 4, then we put dots $\bullet$ in the squares which are cancelled by a rook in p. Then $u_F(p) = 5$ is the number of uncancelled squares, i.e. the squares which contain neither a dot nor an x.

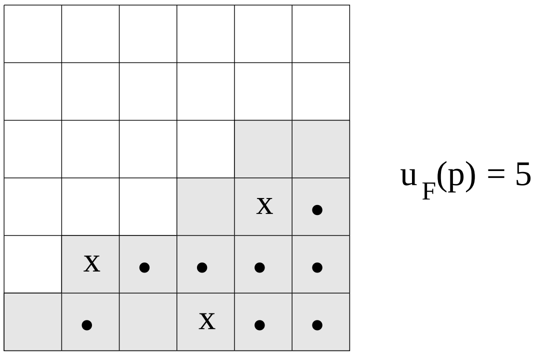

FIGURE 4

Garsia and Remmel [GaRe] then defined a q-analogue of $r_k(F)$ by setting

$$r_k(F, q) = \sum_{p \in R_k(F)} q^{u_F(p)}. \tag{4}$$

Garsia and Remmel proved [GaRe] that if $F = A(a_1, \ldots, a_n)$ where $0 \leq a_1 \leq \cdots \leq a_n \leq n$, then

$$\prod_{i=1}^{n} [x + a_i + i - 1] = \sum_{k=0}^{n} r_{n-k}(F, q)[x] \downarrow_k \tag{5}$$

where $[n] = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$ and $[n] \downarrow_k = [n][n-1]\cdots[n-k+1]$. We also define $[n]! = [n][n-1]\cdots[2][1]$ and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Garsia and Remmel also defined a q-analogue of the hit numbers, $h_{k,n}(F,q)$, for Ferrers boards by the formula

$$\sum_{k=0}^{n} h_{k,n}(F,q) z^k = \sum_{k=0}^{n} r_k(F,q)[n-k]! \prod_{i=n-k+1}^{n} (z - q^i).$$

Garsia and Remmel proved that $h_{k,n}(F,q)$ is a polynomial in q with nonnegative coefficients. In fact, they proved that there is a statistic $s_F(p)$ such that

$$h_{k,n}(F,q) = \sum_{p \in H_{k,n}(F)} q^{s_F(p)}. \tag{6}$$

However, Garsia and Remmel did not provide a direct description of $s_F(p)$ but only defined $s_F(p)$ indirectly via a recursive definition. Later Dworkin [D1], [D2] and Haglund [H] independently gave direct descriptions of statistics $s_{F,d}(p)$ and $s_{F,h}(p)$ on $p \in H_{k,n}(F)$ such that

$$h_{k,n}(F,q) = \sum_{p \in H_{k,n}(F)} q^{s_{F,d}(p)} = \sum_{p \in H_{k,n}(F)} q^{s_{F,h}(p)}.$$

The Dworkin statistic $s_{F,d}(p)$ and the Haglund statistic $s_{F,h}(p)$ have very similar descriptions. Given a placement $p \in H_{k,n}(F)$, first let each rook cancel all squares to its right. Then for each rook $\mathbf{r} = (i,j)$ which is not in F, $\mathbf{r}$ cancels all squares below $\mathbf{r}$ which are not in F. Finally for each rook $\mathbf{r} = (i,j)$ in F, in the Dworkin statistic the rook cancels all squares below $\mathbf{r}$, plus all squares off the board in its column, and $s_{F,d}(p)$ is the number of uncancelled squares. In the Haglund statistic, each rook $\mathbf{r}$ in F cancels all squares in F which lie above $\mathbf{r}$, plus all squares off the board in its column, and $s_{F,h}(p)$ is the number of uncancelled squares. For example, in Fig. 5, we picture the two types of cancellations for a placement $p \in H_{3,6}(F)$ where $F = A(1,4,4,4,4,4)$. Once again, we indicate the squares of the placement by placing an x in those squares and we indicate the cancelled squares by placing a dot in the cancelled squares.

We should note that the methods of proof employed by Dworkin [D] and Haglund [H] are very different and up until now there was no known weight preserving bijection which shows that both statistics give rise to the same q-analogue of the hit numbers for Ferrers boards. (As part of our research for this article we discovered such a bijection, which we describe in section 5). Indeed, it is easy to see that the definitions of $s_{F,d}(p)$ and $s_{F,h}(p)$ make sense for any skyline board $F = A(a_1, \ldots, a_n)$. However, Dworkin proved combinatorially that for any skyline board $F = A(a_1, \ldots, a_n)$ and any permutation $\sigma \in S_n$,

$$\sum_{p \in H_k(F)} q^{s_{F,d}(p)} = \sum_{p \in H_k(\sigma(F))} q^{s_{\sigma(F),d}(p)} \tag{7}$$

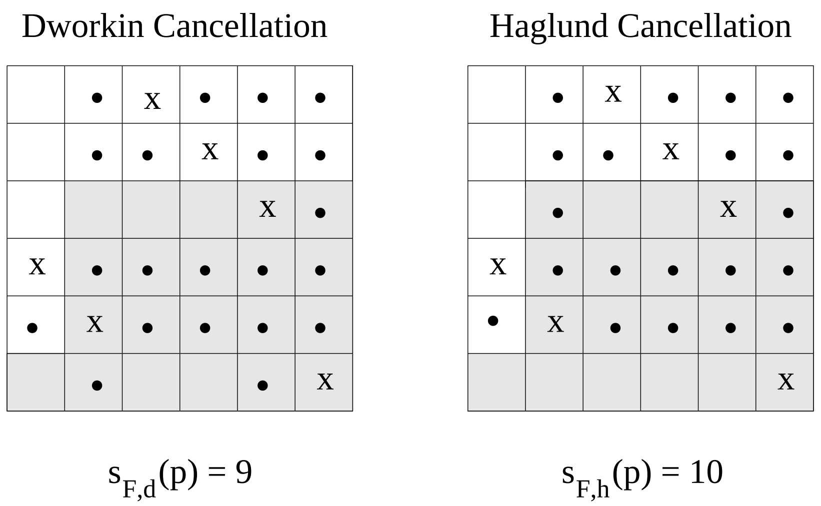

FIGURE 5

where $\sigma(F) = A(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. Haglund showed that (7) does not always hold if $s_{F,d}(p)$ and $s_{\sigma(F),d}(p)$ are replaced by $s_{F,h}(p)$ and $s_{\sigma(F),h}(p)$ respectively.

The main purpose of this paper is to prove analogues of the results described above where we replace permutations by perfect matchings. Our work was initially inspired by unpublished work of Reiner and White [ReWh], who suggested that one consider the board B_{2n} pictured in Fig. 6.

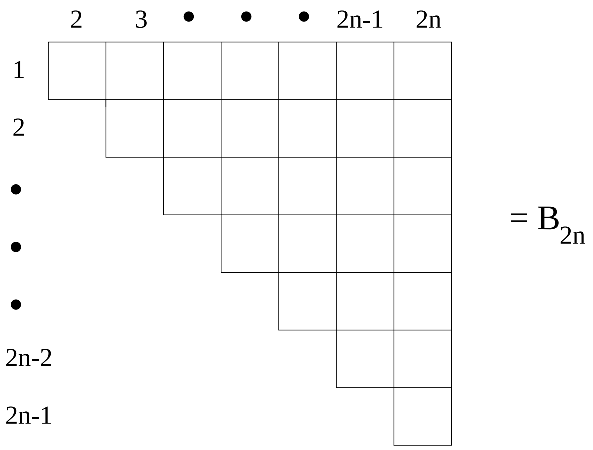

FIGURE 6

Note that for the board A_n, a rook placement p is just a partial permutation, i.e. a set of squares of A_n that can be extended to a permutation p_σ for some $\sigma \in S_n$. For the board B_{2n}, we replace permutations by perfect matchings of the complete graph K_{2n} on vertices $1, 2, \ldots, 2n$. That is, for each perfect matching m of K_{2n} consisting of n pairwise vertex disjoint edges in K_{2n}, we let $p_m = \{(i,j) : i < j \text{ and } \{i,j\} \in m\}$ where (i,j) denotes the square in row i and column j of B_{2n}

according to the labeling of rows and columns pictured in Fig. 6. For example, p_m is pictured in Fig. 7 where $m = \{\{1,4\},\{2,7\},\{3,5\},\{6,8\}\}$ is a perfect matching of K_8.

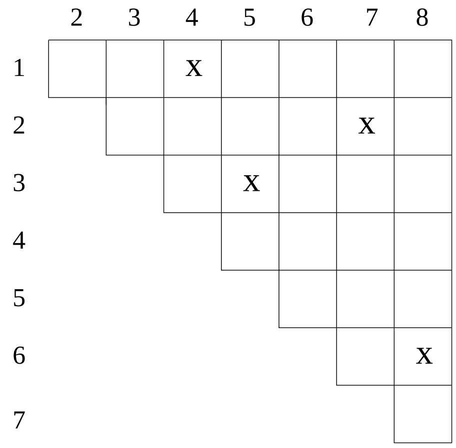

FIGURE 7

For the board B_{2n}, we thus define a rook placement to be a subset of some p_m for a perfect matching m of K_{2n}. Given a board $B \subseteq B_{2n}$, we let $M_k(B)$ denote the set of k element rook placements of B and let $m_k(B) = |M_k(B)|$. Similarly, we let $F_{k,2n}(B) = \{p_m : |p_m \cap B| = k \text{ and } m \text{ is a perfect matching of } K_{2n}\}$ and let $f_{k,2n}(B) = |F_{k,2n}(B)|$. We call $m_k(B)$ the k-th rook number of B and $f_{k,2n}(B)$ the k-th hit number of B. One can prove that

$$\sum_{k=0}^{n} f_{k,2n}(B) z^k = \sum_{k=0}^{n} m_k(B)(n-k)!!(z-1)^k \tag{8}$$

in much the same way that one proved (3). Here we let

$$n!! = \prod_{i=1}^{n}(2i-1) \qquad \text{and} \qquad [n]!! = \prod_{i=1}^{n}[2i-1].$$

The analogue of a skyline board in this setting is a board $B(a_1, a_2, \ldots, a_n) = \{(i, i+j) : 1 \leq j \leq a_i\}$. Thus $B(a_1, a_2, \ldots, a_{2n-1})$ is the board whose row lengths are $a_1, a_2, \ldots, a_{2n-1}$ respectively. We say that $B(a_1, \ldots, a_{2n-1})$ is a shifted Ferrers board if $2n-1 \geq a_1 \geq a_2 \geq \cdots \geq a_{2n-1} \geq 0$, and the non-zero entries of $a_1, \ldots, a_{2n-1}$ are strictly decreasing. For example, $B(5,3,2,1,0,0,0) \subseteq B_8$ is pictured in Fig. 8.

We note that if we identify a board $B \subseteq B_{2n}$ with the graph $G_B = (V, E_B)$ where $V = \{1, \ldots, 2n\}$ and $E_B = \{\{i,j\} : (i,j) \in B\}$, then the graph of a shifted Ferrers board is called a threshold graph in the graph theory literature.

Our investigation of rook numbers and hit numbers was, in part, motivated by trying to find a q-analogue of the following formula of Reiner and White which

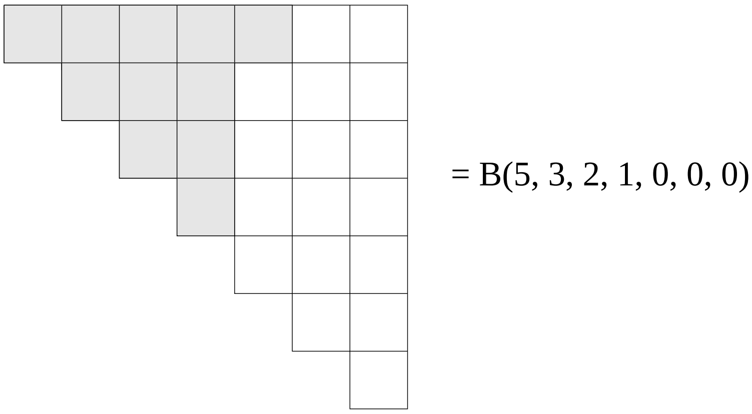

FIGURE 8

holds for any shifted Ferrers board $F = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$.

$$\prod_{i=1}^{2n-1} (x + a_{2n-i} - 2i + 2) = \sum_{k=0}^{2n-1} m_k(F)(x) \downarrow\downarrow_{2n-1-k} . \tag{9}$$

Here $(x) \downarrow\downarrow_k = x(x-2)(x-4)\cdots(x-2k+2)$. We can define q-rook numbers for which a q-analogue of Reiner and White's formula (9) holds as follows. We say that rook (i, j) with $i < j$ in a rook placement rook-cancels all cells (i, s) in B_n with $i < s < j$ and all cells (t, j) and (t, i) with $t < i$. For example the cells rook-cancelled by $(4, 7)$ in B_8 are pictured in Fig. 9.

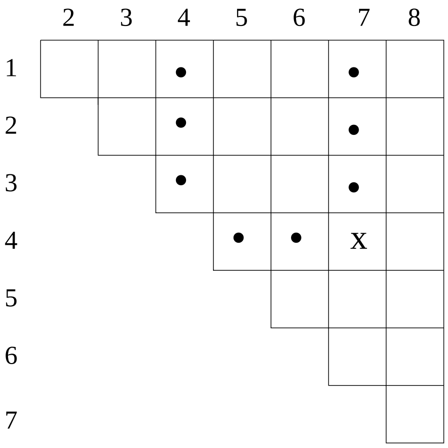

FIGURE 9

Given a shifted Ferrers board $F = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$ and a placement $p \in M_k(F)$, we let $u_F(p)$ denote the number of cells of F which are neither in p nor rook-cancelled by a rook in p. Then we define

$$m_k(F, q) = \sum_{p \in m_k(B)} q^{u_F(p)}. \tag{10}$$

This given, we shall prove the following q-analogue of (9).

$$\prod_{i=1}^{2n-1}[x+a_{2n-i}-2i+2]=\sum_{k=0}^{2n-1}m_k(F,q)[x]\downarrow\downarrow_{2n-1-k} \tag{11}$$

where $[x]\downarrow\downarrow_k=[x][x-2]\cdots[x-2k+2]$.

We define the q-analogue of the hit numbers for F by defining $f_{k,n}(F,q)$ via the formula

$$\sum_{j=0}^{n} f_{k,2n}(F,q)z^j=\sum_{k=0}^{n}m_k(F,q)[n-k]!!\prod_{i=n-k+1}^{n}(z-q^{2i-1}). \tag{12}$$

We shall show that one can define a Dworkin type statistic $t_F(p)$ for $p\in F_k(F)$ such that

$$f_{k,2n}(F,q)=\sum_{p\in F_k(F)}q^{t_F(p)} \tag{13}$$

so that (12) ensures that the $f_{k,2n}(F,q)$ are polynomials in q with nonnegative coefficients. We note that the q-rook numbers $m_k(F,q)$ appear as a special case of a more general rook placement model due to Remmel and Wachs [ReWa]. Our results suggest that there is a natural extension of q-hit numbers that can be defined in their model. However, there is no obvious way to define the analogue of our perfect matchings in the Remmel-Wachs model much less how one could find a statistic.

The outline of this paper is as follows. In section 1 we develop basic results for the $q=1$ case of rook numbers and hit numbers for shifted Ferrers boards. In section 2 we define natural q-analogues of the rook and hit numbers for shifted Ferrers boards and prove some basic identities that these numbers satisfy. Our basic definition of the q-hit numbers for shifted Ferrers boards is algebraic. However we also define a combinatorial interpretation of these numbers. In section 3 we prove the combinatorial interpretation and the algebraic definition of the q-hit numbers for shifted Ferrers boards are the same. Section 4 contains a number of algebraic identities satisfied by the q-rook and q-hit numbers for shifted Ferrers boards, which are used in the proofs of our theorems. In section 5 we introduce new families of statistics for the q-hit numbers in both the classical Ferrers board and shifted Ferrers board case so that q-counting permutations/perfect matchings with respect to these statistics generate the corresponding q-hit numbers. In the classical case this will give a direct proof that the statistics introduced by Dworkin [D1], [D2] and Haglund [H] give rise to the same q-hit numbers.

1 Basic results for rook numbers and hit numbers for boards in B_{2n}

In this section, we shall prove a number of basic results for the hit numbers and rook numbers for boards contained in B_{2n}. Let $PM(B_{2n})=\{p_m: m \text{ is a perfect matching of } K_{2n}\}$. It is easy to see that $|PM(B_{2n})|=n!!$. That is, there are $2n-1$ choices for an edge that contains vertex 1, i.e. $\{1,i\}$, $i=2,\ldots,2n$. If we pick an edge $\{1,j\}$, then the number of ways to complete $\{1,j\}$ to a perfect matching of K_{2n} is clearly just the number of perfect matchings on the complete graph on vertices $\{1,\ldots,2n\}-\{1,j\}$. Thus $|PM(B_{2n})|=(2n-1)|PM(B_{2n-2})|$ and hence

$|PM(B_{2n})| = 1 \times 3 \times \cdots \times (2n-1) = n!!$ by induction. More generally, it follows that if we are given k pairwise vertex disjoint edges $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ in K_{2n}, then the number of ways to extend $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ to a perfect matching of K_{2n} is equal to $|PM(B_{2n-2k})| = (n-k)!!$.

Now recall that given a board $B \subseteq B_{2n}$, $F_{k,2n}(B) = \{p_m \in PM(B_{2n}) : |p_m \cap B| = k\}$ and the k-th hit number of B is $f_{k,2n}(B) = |F_{k,2n}(B)|$. A set $p \subseteq B$ is a *rook placement* of B if $p \subseteq B \cap p_m$ for some $p_m \in PM(B_{2n})$. We let $M_k(B)$ denote the set of all k-element rook placements of B and we define $m_k(B) = |M_k(B)|$ to be the k-th rook number of B.

Our first result is the analogue of (1) for B_{2n}.

Theorem 1. *Let B be a board in B_{2n}. Then*

$$\sum_{k=0}^{n} f_{k,2n}(B)(z+1)^k = \sum_{k=0}^{n} m_k(B)(n-k)!!z^k. \tag{14}$$

Proof. It is easy to see that the left-hand side of (14) is just

$$\sum_{\substack{(T,p_m) \\ T \subseteq p_m \cap B \\ p_m \in PM(B_{2n})}} z^{|T|}. \tag{15}$$

On the other hand, for each rook placement $T \subseteq B$, there are $(n-k)!!$ ways to extend T to a perfect matching $p_m \in PM(B_{2n})$ if $|T| = k$ so that the right-hand side of (14) is also equal to (15). $\square$

Note if we replace z by $z-1$ in (14), we get the following analogue of the Riordan-Kaplansky formula (3) for any $B \subseteq B_{2n}$.

$$\sum_{k=0}^{n} f_{k,2n}(B)z^k = \sum_{k=0}^{n} m_k(B)(n-k)!!(z-1)^k. \tag{16}$$

Next we prove a number of simple recursions for the rook numbers and hit numbers of B_{2n}-boards. To this end, given a board $B \subseteq B_{2n}$ and a pair $(i,j) \in B$ with $i < j$, we define two boards, $B/(i,j)$ and $B/\overline{(i,j)}$. $B/(i,j)$ is just the board which is the result of removing the square (i,j) from B. $B/\overline{(i,j)}$ is the board contained in B_{2n-2} which is obtained as follows. First let $C^{2n}_{(i,j)}$ denote the set of all squares of B_n which have either i or j as a coordinate. It is easy to see that $B_{2n} - C^{2n}_{(i,j)}$ will be a copy of B_{2n-2} except that it will involve the coordinates $\{1, \ldots, 2n\} - \{i,j\}$ instead of $\{1, \ldots, 2n-2\}$. Thus we can isomorphically map the resulting board onto B_{2n-2} by sending a coordinate k to $\varphi_{i,j}(k)$ where

$$\varphi_{i,j}(k) = \begin{cases} k & \text{if } k < i \\ k-1 & \text{if } i < k < j \\ k-2 & \text{if } j < k. \end{cases}$$

Then

$$B/\overline{(i,j)} = \{(\varphi_{i,j}(s), \varphi_{i,j}(t)) : (s,t) \in B - C^{2n}_{(i,j)}\}.$$

This process is pictured in Fig. 10 for the board $B = B(6,4,3,2,0,0,0)$ and $(i,j) = (3,5)$. In Fig. 10, we construct $B/(3,5)$ and $B/\overline{(3,5)}$ and we indicate the cells in B_8 which have a coordinate equal to 3 or 5 by placing dots in those squares.

This given, we have the following.

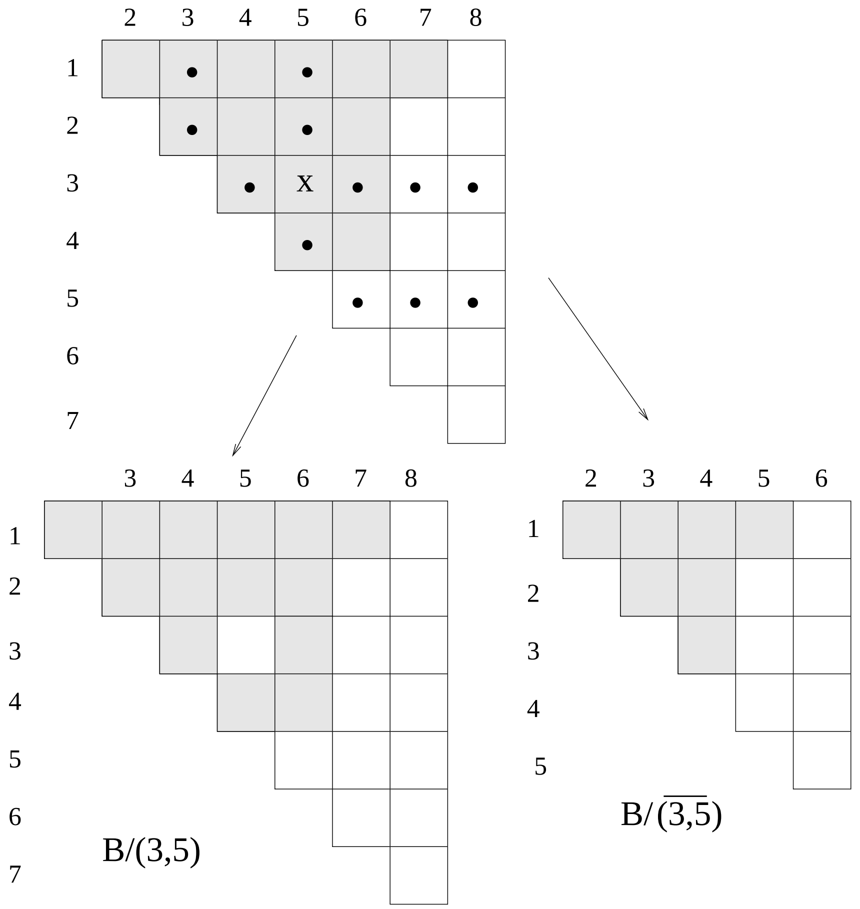

FIGURE 10

Theorem 2. *For any board* $B \subseteq B_{2n}$ *and* $(i,j) \in B$,

(i)
$$m_k(B) = m_k(B/(i,j)) + m_{k-1}(B/\overline{(i,j)}). \tag{17}$$

(ii)
$$f_{k,2n}(B) = f_{k,2n}(B/(i,j)) + f_{k-1,2n-2}(B/\overline{(i,j)}) - f_{k,2n-2}(B/\overline{(i,j)}). \tag{18}$$

Proof. For recursion (i), we simply classify the k-element rook placements p according to whether $(i,j) \in p$. That is, let $M_k^{(i,j)}(B) = \{p \in M_k(B) : (i,j) \in p\}$. Then it is easy to see that $M_k(B/(i,j)) = M_k(B) - M_k^{(i,j)}(B)$. Moreover $\varphi_{i,j}$ induces a $1:1$ correspondence between $M_k^{(i,j)}(B)$ and $M_{k-1}(B/\overline{(i,j)})$. That is, if $p \in M_k^{(i,j)}(B)$, then we let $\varphi_{i,j}(p) = \{(\varphi_{i,j}(s), \varphi_{i,j}(t)) : (s,t) \in p - \{(i,j)\}\}$. Recursion (i) easily follows.

To prove recursion (ii), we again partition the $p_m \in F_{k,2n}(B)$ into two sets according to whether $(i,j) \in p_m$. Let $F_{k,2n}^{(i,j)}(B) = \{p_m \in F_{k,2n}(B) : (i,j) \in p_m\}$. Again $\varphi_{i,j}$ induces a $1:1$ correspondence between $F_{k,2n}^{(i,j)}(B)$ and $F_{k-1,2n-2}(B/\overline{(i,j)})$ where if $p_m \in F_{k,2n}^{(i,j)}(B)$, then $\varphi_{i,j}(p_m) = \{(\varphi_{i,j}(s), \varphi_{i,j}(t)) : (s,t) \in p_m - \{(i,j)\}\}$. Next consider $F_{k,2n}(B/(i,j))$. Note that $F_{k,2n}(B/(i,j)) - F_{k,2n}^{(i,j)}(B/(i,j)) = F_{k,2n}(B) - F_{k,2n}^{(i,j)}(B)$. That is, if $(i,j) \notin p_m$, then $|p_m \cap B| = |p_m \cap B/(i,j)|$. By the same argument as above $\varphi_{i,j}$ induces a $1:1$ correspondence between $F^{(i,j)}_{k,2n}(B/(i,j))$ and $F_{k,2n-2}((B/(i,j))/\overline{(i,j)})$. However, it is easy to see that $(B/(i,j))/\overline{(i,j)} = B/\overline{(i,j)}$. Thus $|F_{k,2n}(B/(i,j))| = |F_{k,2n}(B) - F^{(i,j)}_{k,2n}(B)| + |F_{k,2n-2}(B/\overline{(i,j)})|$ or equivalently $|F_{k,2n}(B) - F^{(i,j)}_{k,2n}(B)| = f_{k,2n}(B/(i,j)) - f_{k,2n-2}(B/\overline{(i,j)})$. Since $|F^{(i,j)}_{k,2n}(B)| = |F_{k-1,2n-2}(B/\overline{(i,j)})| = f_{k-1,2n-2}(B/(i,j))$, recursion (ii) follows. $\square$

There is one other fundamental recursion for the hit numbers which we shall state since the q-analogue of this recursion will play a crucial role for our combinatorial interpretation of the q-hit numbers.

Theorem 3. *Suppose that B is a board contained in B_{2n} such that $B \cap \{(i,2n) : 1 \le i \le 2n-1\} = \emptyset$ (Thus B contains no elements in the last column of B_{2n}.) Then*

$$f_{k,2n}(B) = \sum_{i=1}^{2n-1} f_{k,2n-2}(B/\overline{(i,2n)}). \tag{19}$$

Proof: Note that every $p_m \in PM(B_{2n})$ must contain a square in the last column of B_{2n} since every perfect matching m of K_{2n} must contain one edge of the form $\{i,2n\}$ with $i \le 2n-1$. Thus $F_{k,2n}(B)$ can be partitioned into $\bigcup_{i=1}^{2n-1} F^{(i,2n)}_{k,2n}(B)$ since B contains no elements in the last column of B_{2n}. But $\varphi_{i,2n}$ induces a $1:1$ correspondence between $F^{(i,2n)}_{k,2n}(B)$ and $F_{k,2n-2}(B/\overline{(i,2n)})$ for $i = 1,\ldots,2n-1$. Hence (19) immediately follows. $\square$

We end this section with a proof of the factorization formula (9) for the rook polynomial $\sum_{k=0}^{n} m_k(B)(x)\downarrow\downarrow_{2n-1-k}$ for shifted Ferrers boards. Reiner and White's original proof of (9) was recursive. We will give a bijective proof of (9) for a slightly larger family of boards which we call *nearly Ferrers boards*. That is, we say a board $B \subseteq B_{2n}$ is nearly Ferrers if for all $(i,j) \in B$, the squares $\{(s,j) : s < i\} \cup \{(t,i) : t < i\}$ are also in B. It is easy to see that every shifted Ferrers board $F \subseteq B$ is nearly Ferrers. Moreover, you can construct a nearly Ferrers board by starting with a full triangle of squares $\Delta_i = \{(s,t) : s < t \le i\}$ and then adding any columns to the right of Δ_i of height $\le i$. See Fig. 11 for such an example.

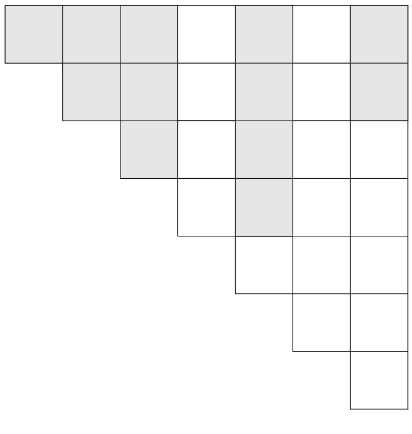

FIGURE 11

Theorem 4. *Let B be a nearly Ferrers board $\subseteq B_{2n}$ and let a_i denote the number of squares in B that lie in row i for $i = 1, \ldots, 2n-1$. Then*

$$\prod_{i=1}^{2n-1} (x + a_{2n-i} - 2i + 2) = \sum_{k=0}^{n} m_k(B)(x) \downarrow\downarrow_{2n-1-k} . \tag{20}$$

Proof. We let $B_{2n,x}$ denote the board B_{2n} with x columns of height $2n-1$ added to the right of B_{2n}; see Fig. 12.

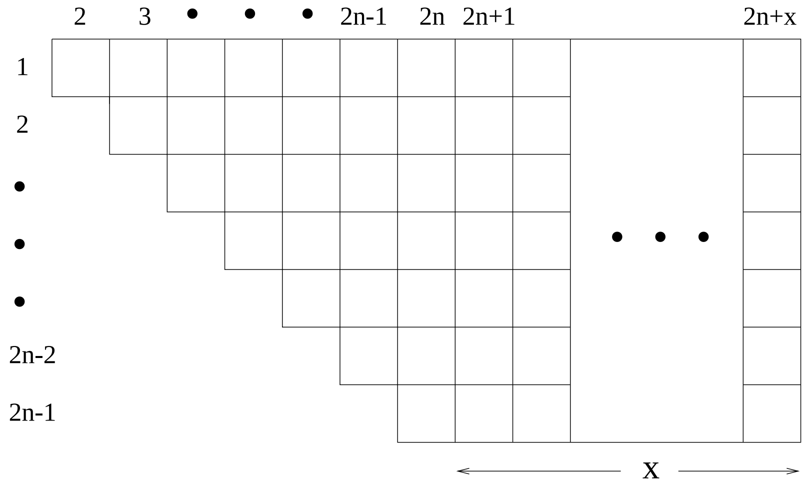

FIGURE 12: THE BOARD $B_{2n,x}$

We want to consider the set of all placements of $2n-1$ nonattacking rooks in $B_{2n,x}$ but we have to define the set of squares that a rook in a square (i,j) attacks. Now if $(i,j) \in B_{2n}$, then a rook $\mathbf{r}$ in (i,j) attacks all cells in row i and column j plus all cells in $\mathcal{A}^{2n}_{(i,j)} = \{(s,t) \in B_{2n} : |\{s,t\} \cap \{i,j\}| = 1\}$. However, if $(i,j) \in B_{2n,x} - B_{2n}$, then the cells that a rook in (i,j) attacks in a rook placement p depends on the other rooks in $p \cap (B_{2n,x} - B_{2n})$. That is, if (i,j) is the position of the lowest rook $\mathbf{r_1}$ in $p \cap (B_{2n,x} - B_{2n})$, then $\mathbf{r_1}$ attacks all cells in row i and column j other than (i,j) plus all cells in the column $j-1$ if $2n+1 < j$. If $j = 2n+1$ then $\mathbf{r_1}$ attacks all cells in row i and column j plus all cells in column $2n+x$. In general, if (i,j) is the position of the k-th lowest rook $\mathbf{r_k}$ in $p \cap (B_{2n,x} - B_{2n})$, then $\mathbf{r_k}$ attacks all cells in row i and column j other than (i,j) plus all cells in the first column in the following list of columns $j-1, j-2, \ldots, 2n, 2n+x, 2n+x-1, \ldots, j+1$ that contain a square which is not attacked by any of the $k-1$ lower rooks in $B_{2n,x} - B_{2n}$. Note that this means that each rook $\mathbf{r}$ in $p \cap (B_{2n,x} - B_{2n})$ will attack all cells in two columns of $B_{2n,x} - B_{2n}$. That is, if $\mathbf{r}$ is in cell (i,j), $\mathbf{r}$ attacks all cells in column j. It then looks for the first column $s > 2n$ to the left of column j which has a cell that is not attacked by a lower rook in $p \cap (B_{2n,x} - B_{2n})$. If there is no such column s, then $\mathbf{r}$ starts at column $2n+x$ and looks for the right most column s which has a square which is not attacked by any lower rook in $p \cap (B_{2n,x} - B_{2n})$. Note that we are guaranteed that such a column s exists if $x \geq 4n-2$. Then $\mathbf{r}$ attacks all cells in column s as well. Our definition of nearly Ferrers board also ensures that each rook $\mathbf{r} \in p$ that lies in B also attacks the squares in two columns of B which lie above $\mathbf{r}$, namely, the squares in column i and j. For example, consider the placement p pictured in Fig. 13 consisting of 3 rooks, $\mathbf{r_1}$ in $(7,10)$, $\mathbf{r_2}$ in $(5,11)$, and $\mathbf{r_3}$ in $(3,7)$. We have indicated all cells attacked by $\mathbf{r_i}$ by placing an i in the cell.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1		3				3		1	1	2						2
2		3				3		1	1	2						2
3			3	3	3	$\mathbf{r_3}$	3	1,3	1,3	2,3	3	3	3	3	3	2,3
4						3		1	1	2						2
5					2	2,3	2	1,2	1,2	$\mathbf{r_2}$	2	2	2	2	2	2
6						3		1	1	2						2
7							1,3	1	$\mathbf{r_1}$	1,2	1	1	1	1	1	1,2

FIGURE 13

Now let B be a board contained in B_{2n} and assume that $x \geq 4n-2$. Let $\mathcal{N}_{2n,x}(B)$ denote the set of all placements p of $2n-1$ rooks in $B_{2n,x}$ such that no cell which contains a rook in p is attacked by another rook in p and any rook $\mathbf{r}$ in $B_{2n} \cap p$ is an element of B. We claim that (20) arises from two different ways of counting $\mathcal{N}_{2n,x}(B)$. That is, the number of ways to place a rook $\mathbf{r_{2n-1}}$ in row $2n-1$ is just $x + a_{2n-1}$. Then $\mathbf{r_{2n-1}}$ attacks two cells in row $2n-2$ of $B \cup (B_{2n,x} - B_{2n})$ so that there will be $x + a_{2n-2} - 2$ ways to place a rook $\mathbf{r_{2n-2}}$ in row $2n-2$. Next in row $2n-3$, $\mathbf{r_{2n-1}}$ and $\mathbf{r_{2n-2}}$ together will attack four cells so that there will be $x + a_{2n-3} - 4$ ways to place a rook $\mathbf{r_{2n-3}}$ in row $2n-3$. Continuing on in this way, it is easy to see that

$$|\mathcal{N}_{2n,x}(B)| = \prod_{i=1}^{2n-1} (x + a_{2n-i} - 2i + 2).$$

On the other hand, suppose that we fix a placement p of k nonattacking rooks on B. Thus $p \in M_k(B)$. We claim that the number of ways to extend p to a placement $q \in \mathcal{N}_{2n,x}(B)$ such that $q \cap B_{2n} = p$ is $x(x-2)\cdots(x-2(2n-1-k)+2)$. That is, there are $2n-1-k$ rows in $B_{2n,x} - B_{2n}$, say $1 \leq d_1 < \cdots < d_{2n-1-k} \leq 2n-1$ which have no cells which are attacked by rooks in p. Now for the lowest such row d_{2n-1-k}, we have x choices of where to place a rook in $B_{2n,k} - B_{2n}$ that lies in

d_{2n-1-k}. The rook in row d_{2n-1-k} will attack exactly two cells in $B_{2n,x} - B_{2n}$ that lie in row $d_{2n-1-k-1}$ so that there will be $x-2$ choices of where to place a rook in row $d_{2n-1-k-1}$. The rooks in rows d_{2n-1-k} and d_{2n-k-2} will attack a total of four rooks in $(B_{2n,x} - B_{2n})$ that lie in row d_{2n-k-3} so that there will be a total of $x-4$ ways to place a rook in row d_{2n-k-3}. Continuing on in this way, it is easy to see that there are a total of $(x)\downarrow\downarrow_{2n-1-k}$ ways to extend p to a rook placement $q \in \mathcal{N}_{2n,x}(B)$ such that $q \cap B_{2n} = p$. Thus

$$|\mathcal{N}_{2n,x}(B)| = \sum_{k=0}^{n} m_k(B)(x)\downarrow\downarrow_{2n-1-k}. \quad \square$$

Now suppose that we set $x = 2n-2$ in (20). Then $(2n-2)\downarrow\downarrow_{2n-1-k} = 0$ for $k = 0, \ldots, n-1$. Thus the only term that survives on the right-hand side of (20) is $m_n(B)(2n-2)\downarrow\downarrow_{n-1}$. Note $(2n-2)\downarrow\downarrow_{n-1} = 2^{n-1}(n-1)!$. Thus the following is an immediate corollary of Theorem 4.

Corollary 1. *Let B be a nearly Ferrers board $\subseteq B_{2n}$ and for $i = 1, \ldots, 2n-1$, let a_i be the number of squares in row i that are in B. Then the number of perfect matchings of the graph $G_B = (\{1, \ldots, 2n\}, \{\{i,j\} : (i,j) \in B\})$ is*

$$\prod_{i=1}^{2n-1} (a_{2n-i} - 2(n-i))/2^{n-1}(n-1)!.$$

2. q-Rook numbers and q-hit numbers for boards in B_{2n}.

In this section we shall define q-rook numbers and q-hit numbers for boards in B_{2n} and prove some of their basic properties.

Let B be any board contained in B_{2n}. For any rook $\mathbf{r}$ in a square (i,j), we say that $\mathbf{r}$ *rook-cancels* squares $\{(r,i) : r < i\} \cup \{(i,s) : i+1 \leq s < j\} \cup \{(t,j) : t < i\}$. For example, the squares that are *rook-cancelled* by a rook in $(4,7)$ in B_8 are pictured in Fig. 9 with a dot in them. Thus the squares rook-cancelled by a rook $\mathbf{r}$ in cell (i,j) is just the squares (a,b) which are attacked by $\mathbf{r}$ such that $a+b < i+j$. Next for any rook placement $p \in M_k(B)$ for some k, we let $u_B(p)$ denote the number of squares in $B - p$ that are not rook-cancelled by any rook in p. We then define $m_k(B,q)$ for $k > 0$ by

$$m_k(B,q) = \sum_{p \in m_k(B)} q^{u_b(p)}. \tag{21}$$

We define $m_0(B,q) = q^{|B|}$.

We call $m_k(B,q)$ the *k-th q-rook number of B*. We shall define the k-th hit number of B, $f_{k,2n}(B,q)$, for any board $B \subseteq B_{2n}$ by the formula

$$\sum_{k=0}^{n} f_{k,2n}(B,q)z^k = \sum_{k=0}^{n} m_k(B,q)[n-k]!! \prod_{i=n-k+1}^{n} (z - q^{2i-1}). \tag{22}$$

Note for $k = 0$, the product $\prod_{i=n-k+1}^{n}(z - q^{2i-1})$ is equal to 1 by definition. We shall call $f_{k,2n}(B,q)$ the *k-th hit number of B relative to B_{2n}*. For example consider the shifted Ferrers board $B(2,1,0) \subseteq B_4$. Then $m_0(B,q) = q^3$ since B has 3 squares $m_1(B,q) = 1+q+q^2$ since there are three rook placements in $M_1(B)$ pictured in Fig. 14, and $m_2(B,q) = 0$ since $M_2(B) = \emptyset$. Thus

$$\begin{aligned}\sum_{k=0}^{2} f_{k,4}(B(2,1,0),q)z^k &= \sum_{k=0}^{2} m_k(B(2,1,0),q)[2-k]!! \prod_{i=2-k+1}^{2}(z-q^{2i-1}) \\ &= q^3[3][1] + (1+q+q^2)[1](z-q^3) \\ &= (1+q+q^2)z.\end{aligned}$$

Thus $f_{0,4}(B(2,1,0),q) = f_{2,4}(B(2,1,0),q) = 0$ and $f_{1,4}(B(2,1,0),q) = 1+q+q^2$.

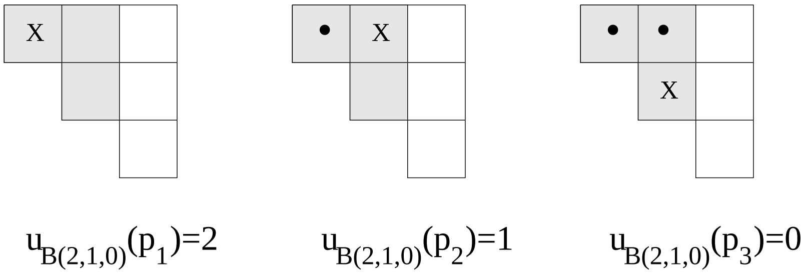

FIGURE 14

We should note that in general the $f_{k,2n}(B,q)$ are not polynomials in q with nonnegative coefficients. That is, consider the board B pictured in Fig. 15, which gives the 3 rook placements of $M_1(B)$, the 1 rook placement in $M_2(B)$, and the corresponding values of $u_B(p)$.

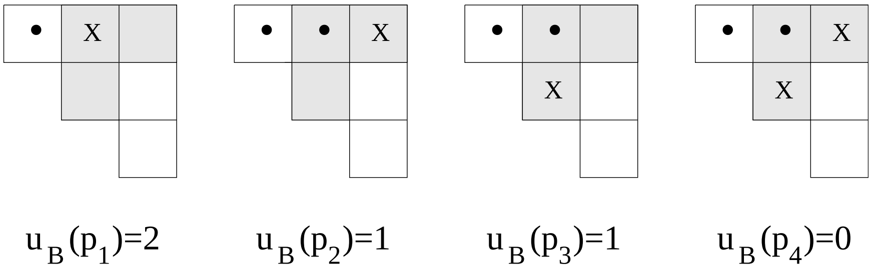

FIGURE 15

Thus $m_0(B,q) = q^3$, $m_1(B,q) = 2q+q^2$, and $m_2(B,q) = 1$. Hence

$$\begin{aligned}\sum_{k=0}^{2} f_{k,4}(B,q)z^k &= \sum_{k=0}^{2} m_k(B,q)[2-k]!! \prod_{i=2-k+1}^{2}(z-q^{2i-1}) \\ &= q^3[3] + (2q+q^2)(z-q^3) + (z-q^3)(z-q) \\ &= q^3 + (q+q^2-q^3)z + z^2\end{aligned}$$

so that $f_{0,4}(B,q) = q^3$, $f_{1,4}(B,q) = q + q^2 - q^3$, and $f_{2,4}(B,q) = 1$.

As mentioned in the introduction, the main result of this paper is to show that if F is a shifted Ferrers board of B_{2n}, then the q-hit numbers $f_{k,2n}(F,q)$ are polynomials in q with nonnegative coefficients. Indeed, we can define an analogue $t_F(p)$ of the Dworkin statistic $s_{F,d}(p)$ for boards contained in A_n such that

$$f_{k,2n}(F,q) = \sum_{p \in F_{k,2n}(F)} q^{t_F(p)}. \tag{23}$$

Let B be any board contained in B_{2n} and suppose that we are given a placement $p \in F_{k,2n}(B)$. If rook $\mathbf{r}$ is on cell $(i,j) \in p \cap B$, then $\mathbf{r}$ p_m-*cancels* all squares

$$\{(r,i) : r < i\} \cup \{(i,s) : i+1 \leq s < j\} \\ \cup \{(t,j) : t < i\} \cup \{(u,j) : u > j \text{ and } (u,j) \notin B\}.$$

That is, if $\mathbf{r}$ is on B, then it p_m-cancels all squares s to the left of $\mathbf{r}$ that it rook-cancels, and also all squares above $\mathbf{r}$ as in the definition of u_F, plus all squares in its column which are below $\mathbf{r}$ and not in B. However, if a rook $\mathbf{r}$ is on (i,j) and $(i,j) \notin B$, then $\mathbf{r}$ p_m-cancels all squares in

$$\{(r,i) : r < i\} \cup \{(i,s) : i+1 \leq s < j\} \cup \{(t,j) : t < i \text{ and } (t,j) \notin B\}.$$

That is, if $\mathbf{r}$ is off the board, $\mathbf{r}$ p_m-cancels the same squares to the left of $\mathbf{r}$ that it rook-cancels plus squares in its column which lie above $\mathbf{r}$ and are off the board. We then let $t_B(p)$ be the number of squares in $B_{2n} - p$ which are not p_m-cancelled. For example, for the placement $p \in F_{k,10}(B(9,7,5,4,2,0,0,0,0))$ pictured in Fig. 16, we have put dots in all the p_m-cancelled squares. There are a total of 13 uncancelled squares so that $t_F(p) = 13$.

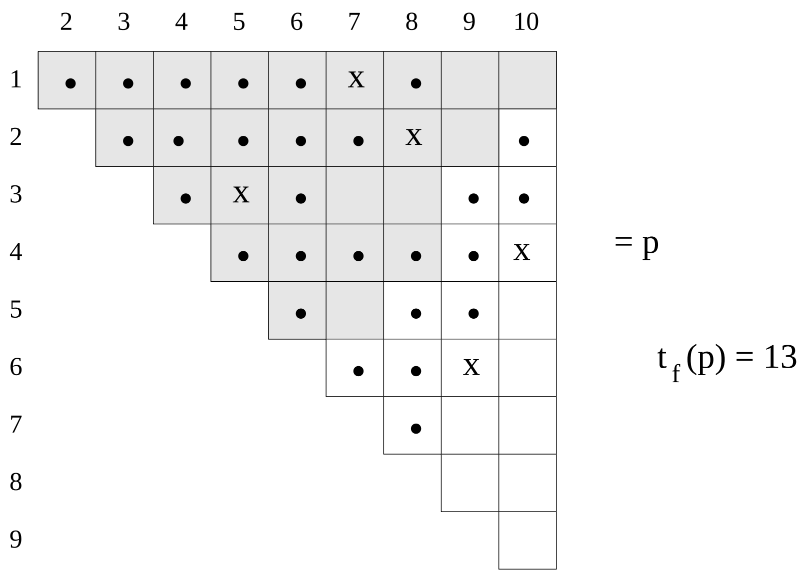

FIGURE 16

The main goal of this paper is to prove that if B is a shifted Ferrers board contained in B_{2n} and $f_{k,2n}(B,q)$ is defined via (22), then (23) holds. For any board $B \subseteq B_{2n}$, let

$$\tilde{f}_{k,2n}(B,q) = \sum_{p \in F_{k,2n}(B)} q^{t_B(p)}. \tag{24}$$

There are two simple recursions that are satisfied by the $\tilde{f}_{k,2n}(B,q)$ which are q-analogues of (18) and (19).

Theorem 5. *Suppose that B is a board contained in B_{2n} such that*

$$B \cap \{(j,2n) : 1 \leq j \leq 2n-1\} = \{(j,2n) : j \leq i\},$$

where $i \geq 1$. Thus in the last column of B_{2n}, B contains exactly the squares $(1,2n),(2,2n),\dots,(i,2n)$. Let $\alpha = (i,2n)$. Then

$$\tilde{f}_{k,2n}(B,q) = q\tilde{f}_{k,2n}(B/\alpha,q) + \tilde{f}_{k-1,2n-2}(B/\overline{\alpha},q) - q^{2n-1}\tilde{f}_{k,2n-2}(B/\overline{\alpha},q). \tag{25}$$

Proof. Just as in the proof of Theorem 2.*ii*, we partition $F_{k,2n}(B)$ into two sets, $F^{(i,2n)}_{k,2n}(B)$ and $F_{k,2n}(B) - F^{(i,2n)}_{k,2n}(B)$. Now if $p \in F^{(i,2n)}_{k,2n}(B)$, then the rook $\mathbf{r}$ on $(i,2n)$ p_m-cancels all cells $(j,2n)$ such that $j \neq i$ since $(i,2n)$ is the lowest cell of B in column $2n$. It follows that $\varphi_{i,2n}$ induces a weight preserving bijection between $F^{(i,2n)}_{k,2n}(B)$ and $F_{k-1,2n-2}(B/\overline{\alpha})$ so that

$$\sum_{p \in F^{(i,2n)}_{k,2n}(B)} q^{t_B(p)} = \sum_{p \in F_{k-1,2n-2}(B/\overline{\alpha})} q^{t_{B/\overline{\alpha}}(p)} = \tilde{f}_{k-1,2n-2}(B/\overline{\alpha},q). \tag{26}$$

Again it is the case that $F_{k,2n}(B/(i,2n)) - F^{(i,2n)}_{k,2n}(B/(i,2n)) = F_{k,2n}(B) - F^{(i,2n)}_{k,2n}(B)$ since if $(i,2n) \notin p_m$ for some $p_m \in PM(B_{2n})$, then $|p_m \cap B| = |p_m \cap B/(i,2n)|$. However, there is a difference between $t_{B/(i,2n)}(p_m)$ and $t_B(p_m)$ for such p_m. That is, p_m contains one rook $\mathbf{r}$ in the last column of B_{2n}. Say $\mathbf{r}$ is on square $(j,2n)$. Now if $j < i$, then $\mathbf{r}$ is on both B and $B/(i,2n)$. However relative to B, $\mathbf{r}$ p_m-cancels all cells $(s,2n)$ with $s < j$ or $s > i$. Relative to $B/(i,2n)$, $\mathbf{r}$ p_m-cancels all cells $(s,2n)$ with $s < j$ or $s \geq i$. That is, $(i,2n)$ is not p_m-cancelled relative to B but it is p_m-cancelled relative to $B/(i,2n)$. Similarly, if $j > i$ so that $(j,2n) \notin B$ and $(j,2n) \notin B/(i,2n)$, $(i,2n)$ is not p_m-cancelled relative to B but it is cancelled relative to $B/(i,2n)$. If $\mathbf{r}'$ is any rook in $p - \{\mathbf{r}\}$ then it is easy to see that $\mathbf{r}'$ p_m-cancels the same squares relative to B that it p_m-cancels relative to $B/(i,2n)$. It follows that for all

$$p \in F_{k,2n}(B) - F^{(i,2n)}_{k,2n}(B) = F_{k,2n}(B/\alpha) - F^{(i,2n)}_{k,2n}(B/\alpha),$$

$$t_B(p) = 1 + t_{B/\alpha}(p). \tag{27}$$

Next suppose that $p \in F^{(i,2n)}_{k,2n}(B/\alpha)$. Then the rook $\mathbf{r}$ on $(i,2n)$ in p does not p_m-cancel any squares in the last column relative to $B/(i,2n)$ so there are $2n-2$ uncancelled squares in the last column of B_{2n}. This given, it is easy to see that $\varphi_{i,2n}$

induces a $1:1$ correspondence between $F^{(i,2n)}_{k,2n}(B/\alpha)$ and $F_{k,2n-2}((B/\alpha))/\overline{\alpha}) = F_{k,2n-2}(B/\overline{\alpha})$ which shows that

$$\begin{aligned}\sum_{p\in F^{(i,2n)}_{k,2n}(B/\alpha)} q^{t_{B/\alpha}(p)} &= q^{2n-2}\sum_{p\in F_{k,2n-2}(B/\overline{\alpha})} q^{t_{B/\overline{\alpha}}(p)}\\ &= q^{2n-2}\tilde{f}_{k,2n-2}(B/\overline{\alpha},q).\end{aligned} \tag{28}$$

Thus by (27) and (28),

$$\begin{aligned} q\tilde{f}_{k,2n}(B/\alpha,q) &= q\sum_{p\in F^{(i,2n)}_{k,2n}(B/\alpha)} q^{t_{B/\alpha}(p)} + q\sum_{p\in F_{k,2n}(B/\alpha)-F^{(i,2n)}_{k,2n}(B/\alpha)} q^{t_{B/\alpha}(p)}\\ &= q^{2n-1}\tilde{f}_{k,2n-2}(B/\overline{\alpha},q) + \sum_{p\in F_{k,2n}(B)-F^{(i,2n)}_{k,2n}(B)} q^{t_B(p)}.\end{aligned} \tag{29}$$

Hence

$$\sum_{p\in F_{k,2n}(B)-F^{(i,2n)}_{k,2n}(B)} q^{t_B(p)} = q\tilde{f}_{k,2n}(B/\alpha,q) - q^{2n-1}\tilde{f}_{k,2n-2}(B/\overline{\alpha},q). \tag{30}$$

Clearly (25) follows immediately from (26) and (30). □

We have the following analogue of Theorem 3.

Theorem 6. *Suppose that B is any board contained in B_{2n} such that B has no cells in the last column, then for any k*

$$\tilde{f}_{k,2n}(B) = \sum_{i=1}^{2n-1} q^{2n-i-1}\tilde{f}_{k-1,2n-2}(B/\overline{(i,2n)}). \tag{31}$$

Proof. As in the proof of Theorem 3, we partition $F_{k,2n}(B)$ into $\bigcup_{i=1}^{2n-1} F^{(i,2n)}_{k,2n}(B)$. For a placement $p \in F^{(i,2n)}_{k,2n}(B)$, the rook $\mathbf{r}$ on $(i,2n)$ in p p_m-cancels all squares $(j,2n)$ with $j<i$ since there are no cells in B in the last column. Thus there are $2n-1-i$ uncancelled squares in the last column of B_{2n} relative to p. It is then easy to see that the $1:1$ correspondence that $\varphi_{i,2n}$ induces between $F^{(i,2n)}_{k,2n}(B)$ and $F_{k,2n-2}(B/\overline{(i,2n)})$ proves that for $i=1,\ldots,2n-1$,

$$\begin{aligned}\sum_{p\in F^{(i,2n)}_{k,2n}(B)} q^{t_B(p)} &= q^{2n-i-1}\sum_{p\in F_{k,2n-2}(B\overline{(i,2n)})} q^{t_{B/\overline{(i,2n)}}(p)}\\ &= q^{2n-i-1}\tilde{f}_{k,2n-2}(B\overline{(i,2n)}).\end{aligned} \tag{32}$$

Thus (31) holds. □

It is easy to check that for all boards $B \subseteq B_2$ and for all $k \in \{0,1\}$, $f_{k,2}(B,q) = \tilde{f}_{k,2}(B,q)$. Thus to prove that $f_{k,2n}(B,q) = \tilde{f}_{k,2n}(B,q)$ for all nearly Ferrers boards

$B \subseteq B_{2n}$ and all $k \in \{0, \ldots, n\}$, we only need show that the analogues of Theorems 5 and 6 hold for all shifted Ferrers boards B when $\tilde{f}_{k,2n}(B,q)$ is replaced by $f_{k,2n}(B,q)$.

First we shall show that the analogue of Theorem 5 follows from the following simple recursion for the $m_k(B,q)$'s. We shall say that a square (i,j) of a board $B \subseteq B_{2n}$ is a *corner square of* B if $B \cap \mathcal{A}_{i,j} = \emptyset$ where $\mathcal{A}_{i,j} = \{(s,t) \in B_{2n} : |\{s,t\} \cap \{i,j\}| = 1 \text{ and } s+t > i+j\}$. Note that $\mathcal{A}_{i,j}$ represents the squares $S \in B_{2n}$ such that a rook on S could rook-cancel the cell (i,j) relative to the $u_B(p)$ statistic. If B is a shifted Ferrers board, it is easy to see that (i,j) is a corner square if and only if there are no squares in B to the south-east of (i,j) in B_{2n}.

Theorem 7. *Let B be a board contained in B_{2n} and $\alpha = (i,j)$ be a corner square of B. Then for any k,*

$$m_k(B,q) = q m_k(B/\alpha, q) + m_{k-1}(B/\overline{\alpha}, q). \tag{33}$$

Proof. Set $M_k^{(i,j)}(B) = \{p \in M_k(B) : (i,j) \in p\}$. First we partition the rook placements of $M_k(B)$ into two sets, namely $M_k^{(i,j)}(B)$ and $M_k(B) - M_k^{(i,j)}(B)$. Now if $p \in M_k^{(i,j)}(B)$, then the rook $\mathbf{r}$ on (i,j) rook-cancels all squares in B in $C_{i,j}^{2n} \cap B$ since (i,j) is a corner square. Thus $\varphi_{i,j}$ induces a $1:1$ weight preserving correspondence between $M_k^{(i,j)}(B)$ and $M_{k-1}(B/\overline{\alpha})$. Hence it follows that

$$\sum_{p \in M_k^{(i,j)}(B)} q^{u_B(p)} = m_{k-1}(B/\overline{\alpha}, q). \tag{34}$$

If $p \in M_k(B) - M_k^{(i,j)}(B)$, then cell (i,j) is not rook-cancelled by any rook in p since (i,j) is a corner cell. Thus $u_B(p) = 1 + u_{B/\alpha}(p)$. Hence

$$\sum_{p \in M_k(B) - M_k^{(i,j)}(B)} q^{u_B(p)} = \sum_{p \in M_k(B/\alpha)} q^{1+u_{B/\alpha}(p)} = q\, m_k(B/\alpha, q). \qquad \Box \tag{35}$$

Corollary 2. *Let B be a board contained in B_{2n} and let $\alpha = (i,j)$ be a corner square of B. Then for any k,*

$$f_{k,2n}(B,q) = q f_{k,2n}(B/\alpha, q) + f_{k-1,2n-2}(B/\overline{\alpha}, q) - q^{2n-1} f_{k,2n-2}(B/\overline{\alpha}, q). \tag{36}$$

Proof. By (33),

$$\begin{aligned}
\sum_{k=0}^{n} f_{k,2n}(B,q) z^k &= \sum_{k=0}^{n} m_k(B,q)[n-k]!! \prod_{i=n-k+1}^{n} (z - q^{2i-1}) \\
&= \sum_{k=0}^{n} q m_k(B/\alpha, q)[n-k]!! \prod_{i=n-k+1}^{n} (z - q^{2i-1}) \\
&\quad + \sum_{k=0}^{n} m_{k-1}(B/\overline{\alpha}, q)[n-1-(k-1)]!!(z - q^{2n-1}) \prod_{i=n-1+1-(k-1)}^{n-1} (z - q^{2i-1}) \\
&= q \sum_{k=0}^{n} f_{k,2n}(B/\alpha, q) z^k + (z - q^{2n-1}) \sum_{j=0}^{n-1} m_j(B/\overline{\alpha}, q)[n-1-j]!! \prod_{i=n-1+1-j}^{n-1} (z - q^{2i-1}) \\
&= q \sum_{k=0}^{n} f_{k,2n}(B/\alpha, q) z^k + (z - q^{2n-1}) \sum_{k=0}^{n-1} f_{k,2n-2}(B/\overline{\alpha}, q) z^k.
\end{aligned} \tag{37}$$

Taking the coefficient of z^k on both sides of (37) yields (36). □

Note that the recursion (36) which holds for the $f_k(B,q)$'s represents a more general recursion than the recursion (25) which holds for the $\tilde{f}_{k,2n}(B,q)$'s. We could prove that $f_{k,2n}(B,q) = \tilde{f}_{k,2n}(B,q)$ for all shifted Ferrers boards B if we could give a direct combinatorial proof of the analogue of (36) for the $\tilde{f}_{k,2n}(B,q)$'s. However we have not been able to find such a direct combinatorial proof. The method of proof of Theorem 5 does not extend for arbitrary corner squares even for shifted Ferrers boards. For example consider the board $B = B(2,1,0) \subseteq B_4$. In our proof of Theorem 5, we showed that if B was a shifted Ferrers board and α is the corner square in the last column of B, then

$$\sum_{p \in F_k^{\alpha}(B)} q^{t_B(p)} = \tilde{f}_{k-1,2n-2}(B/\overline{\alpha}, q)$$

and

$$\sum_{p \in F_k(B) - F_k^{\alpha}(B)} q^{t_B(p)} = q\tilde{f}_{k,2n}(B/\alpha, q) - q^{2n-1}\tilde{f}_{k,2n-2}(B/\overline{\alpha}, q).$$

Now suppose $\alpha = (2,3)$. One can see from Fig. 17 that

$$\sum_{p \in F_1^{\alpha}(B)} q^{t_B(p)} = q^2 \text{ and } \sum_{p \in F_1(B) - F_1^{\alpha}(B)} q^{t_B(p)} = 1 + q.$$

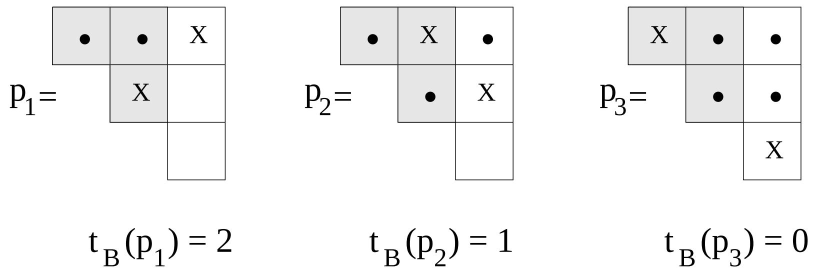

FIGURE 17

However one can easily calculate $\tilde{f}_{0,2}(B/\overline{\alpha}, q) = \tilde{f}_{0,2}(\emptyset, q) = 1$, $\tilde{f}_{1,2}(B/\overline{\alpha}, q) = 0$, and $\tilde{f}_{1,4}(B/\alpha, q) = 1 + q$. Thus

$$\sum_{p \in F_1^{\alpha}(B)} q^{t_B(p)} \neq \tilde{f}_{0,2}(B/\overline{\alpha}, q)$$

and

$$\sum_{p \in F_1(B) - F_1^{\alpha}(B)} q^{t_B(p)} \neq q\tilde{f}_{1,4}(B/\alpha, q) - q^3\tilde{f}_{1,2}(B/\overline{\alpha}, q).$$

Our inability to give a direct proof of the analogue of recursion (36) for the $\tilde{f}_{k,2n}(B,q)$'s forced us to take a different path of proof to establish the equality of the $f_{k,2n}(B,q)$

and $\tilde{f}_{k,2n}(B,q)$ for shifted Ferrers boards. Namely, we show that the $f_{k,2n}(B,q)$'s satisfy the analogue of the recursion (31) which holds for the $\tilde{f}_{k,2n}(B,q)$'s. Unfortunately it is not at all straightforward to show that the $f_{k,2n}(B,q)$'s satisfy the analogue of (31). Indeed most of section 3 will be devoted to proving such a recursion. In preparation for this proof, we shall end this section by proving a number of identities for the $m_k(B,q)$'s which will be used in section 3.

We start with a q-analogue of Theorem 4.

Theorem 8. *Let B be a nearly Ferrers board contained in B_{2n} such that B has a_i cells in row i. Then*

$$\prod_{i=1}^{2n-1}[x+a_{2n-i}-2i+2] = \sum_{k=0}^{n} m_k(B,q)[x]\downarrow\downarrow_{2n-1-k}. \tag{38}$$

Proof. As in the proof of Theorem 4, we shall consider rook placements in $\mathcal{N}_{2n,x}(B)$. Now suppose that $\mathbf{r}$ is a rook in p where $p \in \mathcal{N}_{2n,x}(B)$. Then if $\mathbf{r}$ is on $(i,j) \in B$, then we say $\mathbf{r}$ $\mathcal{N}$-cancels all cells in

$$\{(r,j) : r < i\} \cup \{(i,s) : i+1 \le s < j\} \cup \{(t,i) : t < i\} \\ \cup \{(i,u) : u > j \text{ and } (u,i) \notin B\}.$$

Note that the first three sets in this union are the same cells that $\mathbf{r}$ rook-cancels relative to the $u_B(p\cap B)$ statistic and the last set in the union is all the cells to the right of $\mathbf{r}$ which are not in B. If $\mathbf{r}$ is on $(i,j) \in B_{2n,x} - B_{2n}$, let $\mathcal{A}^{2n}_{(i,j)}$ denote the set of cells attacked by $\mathbf{r}$ as defined in Theorem 4. Then $\mathbf{r}$ $\mathcal{N}$-cancels all cells in $\mathcal{A}^{2n}_{(i,j)}$ that lie in a row s with $s < i$ plus all cells in row i that are either in $B_{2n} - B$ or to the right of (i,j). We let $u_{\mathcal{N}}(p)$ denote the number of squares in $B_{2n,x} - p$ which are not $\mathcal{N}$-cancelled by any rook in p. We claim that (38) results by computing the sum

$$\sum_{p\in\mathcal{N}_{2n,x}(B)} q^{u_{\mathcal{N}}(p)} \tag{39}$$

in two different ways.

Consider the ways to place a rook $\mathbf{r_{2n-1}}$ in row $2n-1$. If we place the rook in the rightmost position in B, then $\mathbf{r_{2n-1}}$ will $\mathcal{N}$-cancel all cells in row $2n-1$. If we place $\mathbf{r_{2n-1}}$ in the next to rightmost position in B, then $\mathbf{r_{2n-1}}$ will cancel all but one cell in row $2n-1$. As we continue to move $\mathbf{r_{2n-1}}$ to the left in B in row $2n-1$, we increase the number of uncancelled cells in row $2n-1$ by one until we reach the leftmost cell in row $2n-1$ of B where we would have $a_{2n-1}-1$ uncancelled cells. Next consider the placement of $\mathbf{r_{n-1}}$ in cell $(2n-1, 2n+1)$. In that case, we would have a total of a_{2n-1} uncancelled cells in row $2n-1$, namely the cells that lie in row $2n-1$ and in B. Then as we move $\mathbf{r_{2n-1}}$ successively to the right, we would increase the number of uncancelled cells by one until we reach the rightmost position, namely $(2n-1, 2n+x)$, where we would have a total of $x + a_{2n-1} - 1$ uncancelled cells. Thus the factor of (38) contributed by the possible placements of $\mathbf{r_{2n-1}}$ in row $2n-1$ is $1+q+\ldots+q^{x+a_{2n-1}-1} = [x+a_{2n-1}]$.

Note that if $\mathbf{r_{2n-1}}$ is placed in a cell in B, our definition of nearly Ferrers board ensures that it will $\mathcal{N}$-cancel exactly two cells in B in every row i with $i < 2n-1$. If

$\mathbf{r_{2n-1}}$ is placed in $B_{2n,x}-B_{2n}$, then it will $\mathcal{N}$-cancel exactly two cells in $B_{2n,x}-B_{2n}$ in every row i with $i < 2n-1$. Thus when we consider the placement of a rook $\mathbf{r_{2n-2}}$ in row $2n-2$, we can use the same argument to prove that the factor of (39) contributed by the possible placement of $\mathbf{r_{2n-2}}$ is $[x + a_{2n-2} - 2]$. Once again if $\mathbf{r_{2n-2}}$ is placed in a cell in B, it will $\mathcal{N}$-cancel an additional two cells in B in each row i with $i < 2n-2$ and if $\mathbf{r_{2n-2}}$ is placed in a cell in $B_{2n,x}-B_{2n}$, it will $\mathcal{N}$-cancel an additional two cells in $B_{2n,x} - B_{2n}$ in each row i with $i < 2n-2$. Hence the factor of (39) contributed by the possible placement of a rook $\mathbf{r_{2n-3}}$ in row $2n-3$ is $[x + a_{2n-3} - 4]$. Continuing on in this way, it is easy to see that

$$\sum_{p\in\mathcal{N}_{2n,x}(B)} q^{u_{\mathcal{N}}(p)} = \prod_{i=1}^{2n-1} [x + a_{2n-i} - 2i + 2]. \tag{40}$$

Next suppose that we fix a placement $p \in M_k(B)$ and we consider the sum

$$\sum_{\substack{p'\in\mathcal{N}_{2n,x}(B)\\ p'\cap B=p}} q^{u_{\mathcal{N}}(p)}.$$

It is easy to check that our definitions ensure that for any $p' \in \mathcal{N}_{2n,x}(B)$ such that $p' \cap B = p$, the number of squares of $B_{2n} - p$ that are not $\mathcal{N}$-cancelled by some rook in p' is just $u_B(p)$. Moreover, by the same type of argument that we used above, the factor of (4) that arises from the possible placements of $2n-1-k$ rooks in $B_{2n,x} - B_{2n}$ is just $[x][x-2]\cdots[x-2(2n-1-k)+2] = [x]\downarrow\downarrow_{2n-1-k}$. Thus it follows that

$$\begin{aligned}\sum_{p\in\mathcal{N}_{2n,x}(B)} q^{u_{\mathcal{N}}(p)} &= \sum_{k=0}^{n}\sum_{p\in M_k(B)} q^{u_B(p)}[x]\downarrow\downarrow_{2n-1-k}\\ &= \sum_{k=0}^{n} m_k(B,q)[x]\downarrow\downarrow_{2n-1-k}\ . \qquad \square\end{aligned}$$

We end this section by proving three recursions for the $m_k(B,q)$, where B is a shifted Ferrers board or nearly Ferrers board which has no cells in the last column of B_{2n}.

Theorem 9. *Suppose that B is a board contained in B_{2n} which has no cells in the last column of B_{2n}. Let $\alpha = (i,r)$ be the cell which is at the bottom of the rightmost column of B. Then*
(a) if B is a nearly Ferrers board,

$$m_k(B/\overline{(i,2n)},q) = q^{i-1}m_k(B/\overline{\alpha},q) + \sum_{j=1}^{i-1} q^{i-1-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q), \tag{41}$$

(b) if B is a shifted Ferrers board,

$$m_k(B/\overline{(r,2n)},q) = [r-2k]m_{k-1}(B/\overline{\alpha},q) + q^{r-2-2k}m_k(B/\overline{\alpha},q), \tag{42}$$

(c) if B is a shifted Ferrers board,

$$\sum_{j=1}^{2n-1} q^{2n-j-1} m_k(B/\overline{(j,2n)},q)$$
$$= [2n-1-2k] m_k(B,q) - (q^{2n-1} - q^{2n-3-2k}) m_{k+1}(B,q). \quad (43)$$

Proof. Before proceeding with the proof of these three recursions, it will be useful to see the relations between three boards mentioned in recursion (a) and (b). It is easy to see from Fig. 18 that $B/\overline{(i,2n)}$ is just the board $B/\overline{\alpha}$ with an extra column of height $i-1$ added in column $r-1$. The board $B/\overline{(r,2n)}$ is just the board B with the last column removed.

For recursion (a), we simply classify the rook placements p of $m_k(B/\overline{(i,2n)})$ according to whether or not p has a rook in the last column of $B/\overline{(i,2n)}$. That is, if $p \in M_k^{(j,r-1)}(B/\overline{(i,2n)})$ where $j \leq i-1$, then the rook on square $(j,r-1)$ in p will cancel all but $i-1-j$ squares in the last column. It follows that

$$q^{u_{B/\overline{(i,2n)}}(p)} = q^{i-1-j} q^{u_{(B/\overline{(i,2n)})/\overline{(j,r-1)}}(\varphi_{j,r-1}(p))}.$$

Clearly $(B/\overline{(i,2n)})/\overline{(j,r-1)}$ is the same board as $(B/\overline{(i,r)})/\overline{(j,2n-2)}$. Thus

$$\sum_{j=1}^{i-1} \sum_{p \in M_k^{(j,r-1)}(B/\overline{(i,2n)})} q^{u_{B/\overline{(i,2n)}}(p)}$$
$$= \sum_{j=1}^{i-1} q^{i-1-j} \sum_{p' \in M_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)})} q^{u_{(B/\overline{\alpha})/\overline{(j,2n-2)}}(p')}$$
$$= \sum_{j=1}^{i-1} q^{i-1-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q). \quad (44)$$

On the other hand given a $p \in M_k(B/\overline{(i,2n)})$ having no rook in column $r-1$, all the squares in column $r-1$ will not be rook-cancelled so that

$$u_{B/\overline{(i,2n)}}(p) = q^{i-1} u_{B/\overline{\alpha}}(p).$$

Thus

$$\sum_{p \in M_k(B/\overline{(i,2n)}) - \bigcup_{j=1}^{i-1} M_k^{(j,r-1)}(B/\overline{(i,2n)})} q^{u_{B/\overline{(i,2n)}}(p)} = q^{i-1} m_k(B/\overline{\alpha},q). \quad (45)$$

Combining (44) and (45) yields (41) as desired.

For recursion (b), note that the shifted Ferrers board $B/\overline{\alpha}$ is the board which results by removing the last column and the first row from B. Thus $B/\overline{\alpha}$ is the result of removing the first row from $B/\overline{(r,2n)}$. It follows that recursion (b) can be rephrased as follows. Suppose that D is a shifted Ferrers board with $r-2$ columns

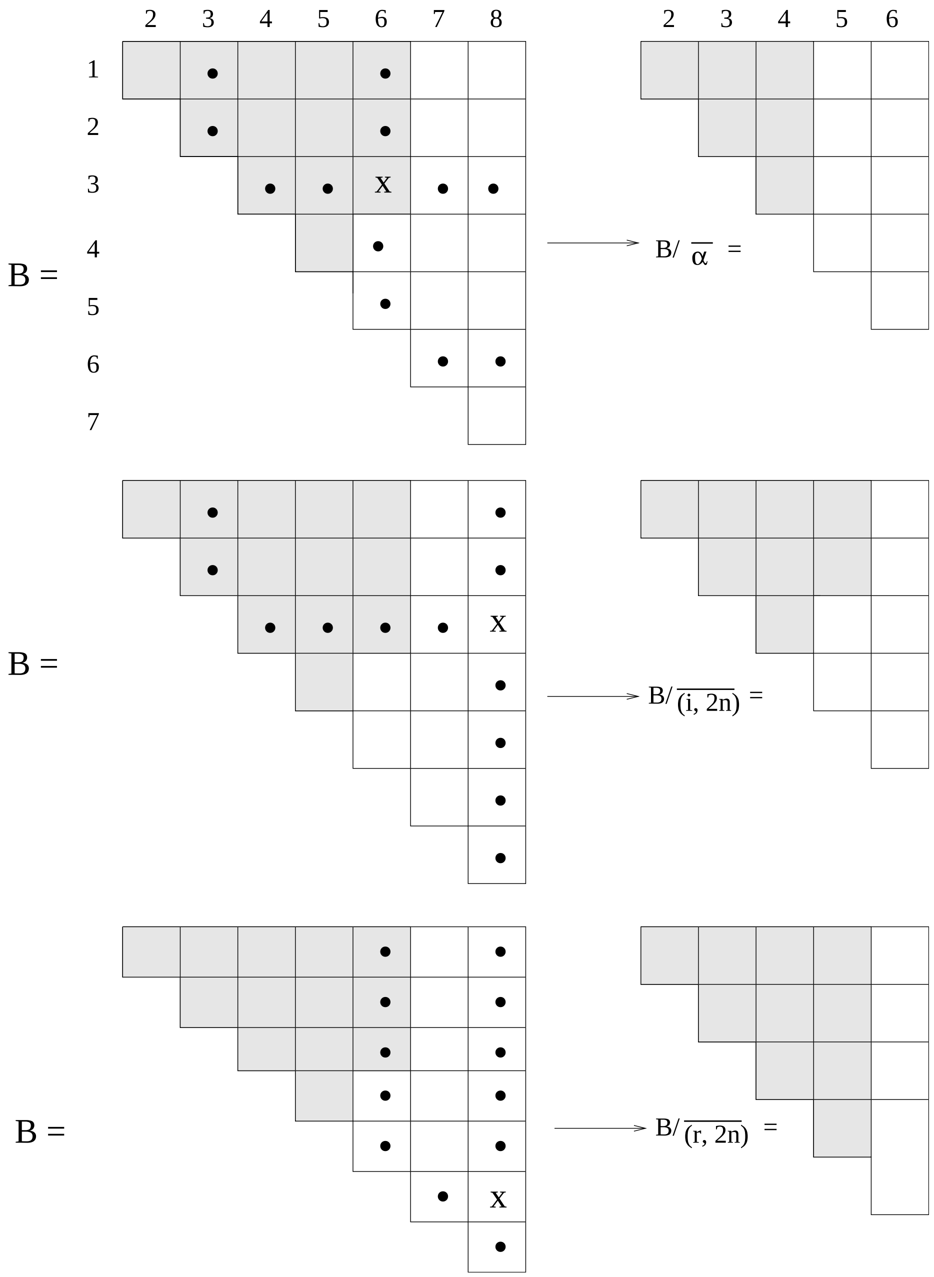

FIGURE 18

and C is the shifted Ferrers board that results from removing the first row of D. Then

$$m_k(D,q) = [r-2k]m_{k-1}(C,q) + q^{r-2-2k}m_k(C,q). \tag{46}$$

Once we have rephrased recursion (b) in this way, it is simple to prove. Namely we simply partition the elements p of $M_k(D)$ depending on whether or not p has a rook in the first row of D. That is, let $M_k^1(D) = \{p \in M_k(D) : \ p \text{ has a rook in the first row}\}$. Now if $p \in M_k(D) - M_k^1(D)$, then p has all k rooks below the first row. Since each of these rooks rook-cancel two squares in row 1, there will be $r-2-2k$ uncancelled squares in the first row. Of course, the board C is just the rows of D below row 1 so that

$$\sum_{p \in M_k(D) - M_k^1(D)} q^{u_D(p)} = q^{r-2-2k} \sum_{p \in M_k(C)} q^{u_C(p)}$$
$$= q^{r-2-2k} m_k(C,q). \tag{47}$$

Next suppose that $p' \in M_{k-1}(C)$. We can think of p' as a rook placement in D with no rooks in the first row. There will be $r-2-2(k-1) = r-2k$ uncancelled squares in the first row of D. Thus we can extend p' to a placement $p \in M_k^1(D)$ in $r-2k$ ways by placing a rook $\mathbf{r}$ in one of these $r-2k$ uncancelled squares in the first row of D. If we placed $\mathbf{r}$ in the i-th uncancelled square in row 1 starting from the right, $\mathbf{r}$ will rook-cancel all squares to its left and leave $i-1$ uncancelled squares in row 1. It follows that

$$\sum_{p \in M_k^1(D)} q^{u_D(p)} = (1 + q + \ldots + q^{r-k-1}) \sum_{p' \in M_{k-1}(C)} q^{u_C(p')}$$
$$= [r-2k] m_{k-1}(C,q). \tag{48}$$

Hence (46) holds.

We do not have a simple combinatorial proof of recursion (c). Instead we shall prove recursion (c) by induction, first on $2n$ and then on the number of squares in B. It is easy to verify that recursion (c) holds for all boards $B \subseteq B_2$. Thus assume that (c) holds for all boards $B' \subseteq B_{2n-2}$. Now if B is the empty board contained in B_{2n}, then it is easy to see that both sides of (43) are zero if $k \geq 1$. If $k = 0$, $B/\overline{(j,2n)}$ is the empty board for all j so that $m_0(B/\overline{(j,2n)},q) = m_0(B,q) = 1$ and $m_1(B,q) = 0$. Thus in that case (43) becomes

$$\sum_{j=1}^{2n-1} q^{2n-1-j} = [2n-1].$$

Thus (43) holds for the empty board for all n.

Finally by induction, assume that (43) holds for all shifted Ferrers boards with less than t squares and that $B \subseteq B_{2n}$ is a shifted Ferrers board with t squares which has no squares in the last column of B_{2n}. Let $\alpha = (i,r)$ denote the corner square in the rightmost column of B. Applying recursion (33) and then recursion

(c) to B/α and $B/\overline{\alpha}$ by induction, we find that

$$\begin{aligned}
&[2n-1-2k]m_k(B,q) - (q^{2n-1} - q^{2n-3-2k})m_{k+1}(B,q) = \\
&\qquad q([2n-1-2k]m_k(B/\alpha,q) - (q^{2n-1} - q^{2n-3-2k})m_{k+1}(B/\alpha,q)) \\
&\qquad + [2n-1-2k]m_{k-1}(B/\overline{\alpha},q) - (q^{2n-1} - q^{2n-3-2k})m_k(B/\overline{\alpha},q) = \\
&\qquad\qquad q\sum_{j=1}^{2n-1} q^{2n-1-j} m_k((B/\alpha)/\overline{(j,2n)},q) \\
&+ [2(n-1)-1-2(k-1)]m_{k-1}(B/\overline{\alpha},q) - (q^{2(n-1)-1} - q^{2(n-1)-3-2(k-1)})m_k(B/\overline{\alpha},q) \\
&\qquad + (q^{2(n-1)-1} - q^{2(n-1)-3-2(k-1)} - q^{2n-1} + q^{2n-3-2k})m_k(B/\overline{\alpha},q) = \\
&\qquad\qquad q\sum_{j=1}^{2n-1} q^{2n-1-j} m_k((B/\alpha)/\overline{(j,2n)},q) \\
&+ \sum_{j=1}^{2n-3} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q) - (q^{2n-1} - q^{2n-3})m_k(B/\overline{\alpha},q). \quad (49)
\end{aligned}$$

We would also like to apply recursion (33) to the left-hand side of (43) but this requires some care. That is, if $j < i$, then the image of $\alpha = (i,r)$ under $\varphi_{j,2n}$ is $\beta = (i-1, r-1)$ which will still be the rightmost corner square of $B/\overline{(j,2n)}$. Similarly if $i < j < r$, then the image of α under $\varphi_{j,2n}$ is $\gamma = (i, r-1)$ will also be the rightmost corner square of $B/\overline{(j,2n)}$. If $j > r$, then $(j,2n)$ only attacks empty squares so that α is the rightmost corner cell of $B/\overline{(j,2n)}$. See Fig. 19.

It is easy to see that if $j < i$, then $B/\overline{(j,2n)}/\beta = (B/\alpha)/\overline{(j,2n)}$ and $B/\overline{(j,2n)}/\overline{\beta} = (B/\overline{\alpha})/\overline{(j,2n-2)}$. If $i < j < r$, then $B/\overline{(j,2n)}/\gamma = (B/\alpha)/\overline{(j,2n)}$ and $B/\overline{(j,2n)}/\overline{\gamma} = (B/\overline{\alpha})/\overline{(j-1,2n-2)}$. Finally if $j > r$, then $B/\overline{(j,2n)}/\alpha = (B/\alpha)/\overline{(j,2n)}$ and $B/\overline{(j,2n)}/\overline{\alpha} = (B/\overline{\alpha})/\overline{(j-2,2n-2)}$. This given, we can apply recursion (a) to obtain the following

$$\begin{aligned}
&\sum_{j=1}^{2n-1} q^{2n-1-j} m_k(B/\overline{(j,2n)},q) \\
&\qquad = q^{2n-1-i} m_k(B/\overline{(i,2n)},q) + q^{2n-1-r} m_k(B/\overline{(r,2n)},q) \\
&\qquad + \sum_{j=1}^{i-1} q^{2n-1-j}(q m_k((B/\alpha)/\overline{(j,2n)},q) + m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q)) \\
&\quad + \sum_{j=i+1}^{r-1} q^{2n-1-j}(q m_k((B/\alpha)/\overline{(j,2n)},q) + m_{k-1}((B/\overline{\alpha})/\overline{(j-1,2n-2)},q)) \\
&+ \sum_{j=r+1}^{2n-1} q^{2n-1-j}(q m_k((B/\alpha)/\overline{(j,2n)},q) + m_{k-1}((B/\overline{\alpha})/\overline{(j-2,2n-2)},q)). \quad (50)
\end{aligned}$$

Comparing the right-hand sides of (49) and (50), we can prove (43) if we can show

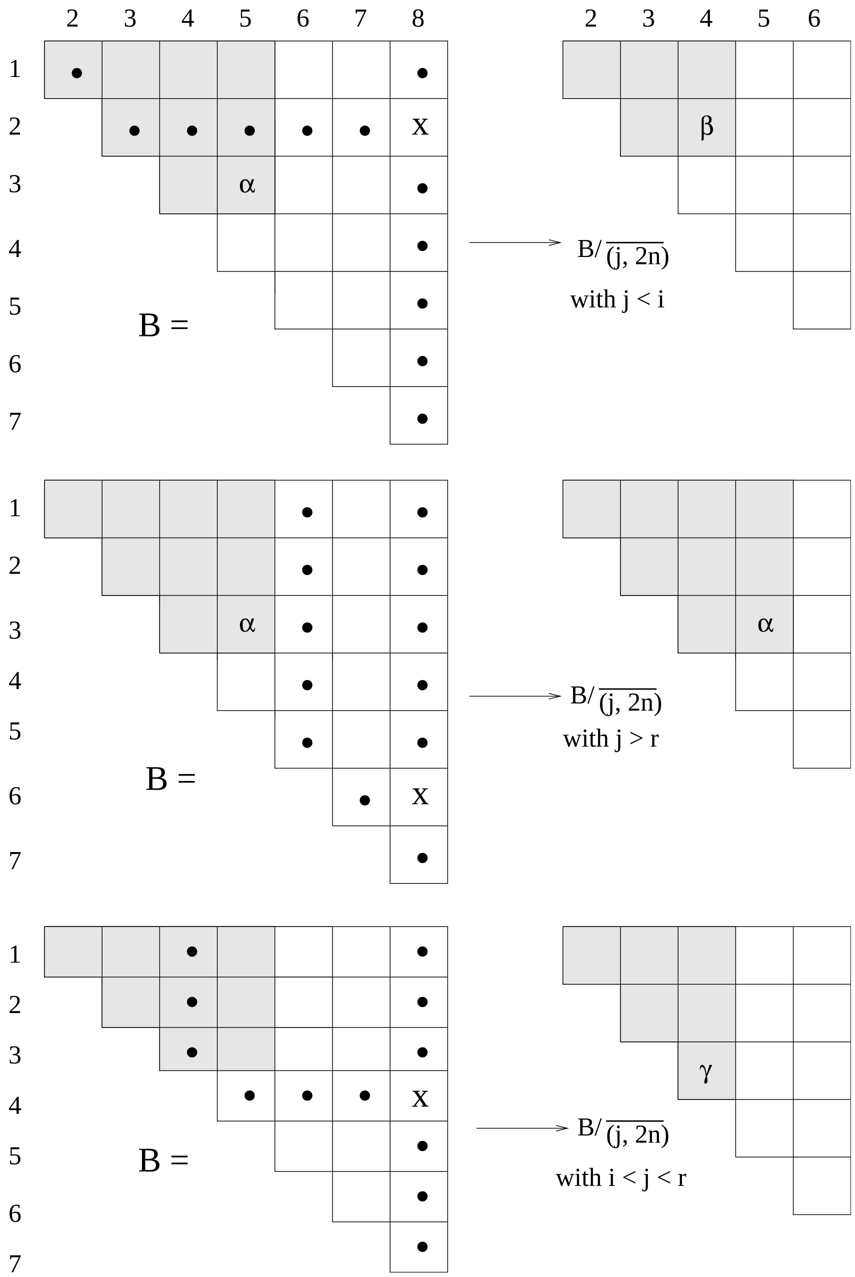
B =
B/$\overline{(j, 2n)}$
with j < i
B =
B/$\overline{(j, 2n)}$
with j > r
B =
B/$\overline{(j, 2n)}$
with i < j < r

FIGURE 19

that

$$
\begin{aligned}
q^{2n-1-i} m_k(B/\overline{(i,2n)}, q) + q^{2n-1-r} m_k(B/\overline{(r,2n)}, q) \\
+ \sum_{j=1}^{i-1} q^{2n-1-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
+ \sum_{j=i}^{r-2} q^{2n-2-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
+ \sum_{j=r-1}^{2n-3} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
= q^{2n-i} m_k((B/\alpha)\overline{(i,2n)}, q) + q^{2n-r} m_k((B/\alpha)\overline{(r,2n)}, q) \\
+ \sum_{j=1}^{2n-3} q^{2n-3-j} m_k((B/\overline{\alpha})/\overline{(j,2n-2)}, q) - (q^{2n-1} - q^{2n-2}) m_k(B/\overline{\alpha}, q). \quad (51)
\end{aligned}
$$

It is easy to see that $(B/\alpha)/\overline{(i,2n)} = B/\overline{(i,2n)}$ and $(B/\alpha)/\overline{(r,2n)} = B/\overline{(r,2n)}$ since both $(i,2n)$ and $(j,2n)$ attack α. Thus (51) is equivalent to

$$
\begin{aligned}
\sum_{j=1}^{i-1} (q^{2n-1-j} - q^{2n-3-j}) m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
+ \sum_{j=i}^{r-2} (q^{2n-2-j} - q^{2n-3-j}) m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-1)}, q) \\
= (q^{2n-i} - q^{2n-i-1}) m_k(B/\overline{(i,2n)}, q) + (q^{2n-r} - q^{2n-r-1}) m_k(B/\overline{(r,2n)}, q) \\
- (q^{2n-1} - q^{2n-3}) m_k(B/\overline{\alpha}, q). \quad (52)
\end{aligned}
$$

Dividing both sides by $q-1$ gives

$$
\begin{aligned}
(q+1) \sum_{j=1}^{i-1} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
+ \sum_{j=i}^{r-2} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
= q^{2n-i-1} m_k(B/\overline{(i,2n)}, q) + q^{2n-r-1} m_k(B/\overline{(r,2n)}, q) - (1+q) q^{2n-3} m_k(B/\overline{\alpha}, q). \quad (53)
\end{aligned}
$$

But we can now apply recursions (a) and (b) to the first two terms on the right-hand side of (53) to show that the right-hand side of (53) is

$$
\begin{aligned}
q^{2n-2} m_k(B/\overline{\alpha}, q) + \sum_{j=1}^{i-1} q^{2n-2-j} m_k((B/\overline{\alpha})/\overline{(j,2n-2)}, q) \\
+ q^{2n-r-1} [r-2k] m_{k-1}(B/\overline{\alpha}, q) + q^{2n-3-2k} m_k(B/\overline{\alpha}, q) - (1+q) q^{2n-3} m_k(B/\overline{\alpha}, q). \quad (54)
\end{aligned}
$$

Now replacing the right-hand side of (53) by (54) and collecting terms we get that (43) is equivalent to proving

$$\sum_{j=1}^{r-2} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q)$$
$$= q^{2n-r-1}[r-2k]m_{k-1}(B/\overline{\alpha},q) - (q^{2n-3} - q^{2n-3-2k})m_k(B/\overline{\alpha},q). \quad (55)$$

Note however that by induction

$$\sum_{j=1}^{2n-3} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q)$$
$$= [2n-3-2(k-1)]m_{k-1}(B/\overline{\alpha},q) - (q^{2n-3} - q^{2n-5-2(k-1)})m_k(B/\overline{\alpha},q)$$
$$= [2n-1-2k]m_{k-1}(B/\overline{\alpha},q) - (q^{2n-3} - q^{2n-3-2k})m_k(B/\overline{\alpha},q). \quad (56)$$

Moreover since B had only $r-1$ columns then $B/\overline{\alpha}$ has at most $r-3$ columns. Thus $(B/\overline{\alpha})/\overline{(j,2n-2))} = B/\overline{\alpha}$ for $j \geq r-2$ since $(j,2n-2)$ will only attack cells in empty columns. Hence

$$\sum_{j=r-1}^{2n-3} q^{2n-3-j} m_{k-1}((B/\overline{\alpha})/\overline{(j,2n-2)},q)$$
$$= \sum_{j=r-1}^{2n-3} q^{2n-3-j} m_{k-1}(B/\overline{\alpha},q) = [2n-1-r]m_{k-1}(B/\overline{\alpha},q).$$

Thus subtracting $[2n-1-r]m_{k-1}(B/\overline{\alpha},q)$ from both sides of (56) yields (55) as desired. □

3. Main Theorem

In this section we prove our main result, namely that $f_j(B,q) = \tilde{f}_j(B,q)$ for all shifted Ferrers boards B. We start by proving two identities which hold for any board.

Theorem 10. *If B is a board, $B \subseteq B_{2n}$, $0 \leq j \leq n$, and $\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$ is the q-binomial coefficient base q^2, then*

$$f_{j,2n}(B,q) = \sum_{k \geq j} m_k(B,q)[n-k]!!(-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{(k-j)(2n-k-j)}.$$

Proof. Recall the q-binomial theorem [A]:

$$\prod_{i=0}^{m-1}(1+xq^i) = \sum_{k=0}^{m} x^k q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}.$$

Theorem 10 follows by applying this to the product on the right-hand side of (22). □

Theorem 11. *If B is a board, $B \subseteq B_{2n}$, then for $0 \leq k \leq n$,*

$$\sum_{j\geq k} f_{j,2n}(B,q) \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} z^{j-k} q^{(j-k)(2n-1-2k)}$$
$$= \sum_{p \geq k} m_p(B,q)[n-p]!! \begin{bmatrix} p \\ k \end{bmatrix}_{q^2} q^{(p-k)(2n-k-p)} (z;q^2)_{p-k}(-1)^{p-k}, \quad (57)$$

where $(z;q)_k = (1-z)(1-zq)\cdots(1-zq^{k-1})$.

Proof. Using Theorem 10, the left-hand side of (57) equals

$$\sum_{j\geq k} \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} z^{j-k} q^{(j-k)(2n-1-2k)}$$
$$\sum_{p\geq j} m_p(B,q)[n-p]!!(-1)^{p-j} \begin{bmatrix} p \\ j \end{bmatrix}_{q^2} q^{(p-j)(2n-p-j)}$$

$$= \sum_{p\geq k} m_p(B,q)[n-p]!!(-1)^{p-k}$$
$$\sum_{k\leq j\leq p} \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} \begin{bmatrix} p \\ j \end{bmatrix}_{q^2} q^{(j-k)(2n-1-2k)+(p-j)(2n-p-j)} (-1)^{j-k} z^{j-k}$$

$$= \sum_{p\geq k} m_p(B,q)[n-p]!!(-1)^{p-k}$$
$$\begin{bmatrix} p \\ k \end{bmatrix}_{q^2} \sum_{u=0}^{p-k} \begin{bmatrix} p-k \\ u \end{bmatrix}_{q^2} q^{u(2n-1-2k)+(p-k-u)(2n-p-k-u)}(-1)^u z^u$$

$$= \sum_{p\geq k} m_p(B,q)[n-p]!!(-1)^{p-k} \begin{bmatrix} p \\ k \end{bmatrix}_{q^2} q^{(p-k)(2n-p-k)} \sum_{u=0}^{p-k} \begin{bmatrix} p-k \\ u \end{bmatrix}_{q^2} q^{u^2-u}(-1)^u z^u$$
$$= \sum_{p\geq k} m_p(B,q)[n-p]!!(-1)^{p-k} \begin{bmatrix} p \\ k \end{bmatrix}_{q^2} q^{(p-k)(2n-p-k)} (z;q^2)_{p-k}$$

using the q-binomial theorem. □

Theorem 12. *If B is a shifted Ferrers board, $B \subseteq B_{2n}$, then*

$$f_{j,2n}(B,q) = \sum_{i=1}^{2n-1} q^{2n-i-1} f_{j,2n-2}(B/\overline{(i,2n)},q).$$

Proof. We start by setting $z = q^{-2}$ in eq. (57) to get

$$\sum_{j\geq k} f_{j,2n}(B,q) \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-3-2k)}$$
$$= m_k(B,q)[n-k]!! - m_{k+1}(B,q)[n-1-k]!![k+1]_{q^2} q^{2n-2k-1}(1-q^{-2})$$
$$= m_k(B,q)[n-k]!! - m_{k+1}(B,q)[n-1-k]!![k+1]_{q^2} q^{2n-3-2k}(q^2-1). \quad (58)$$

On the other hand if B has less than $2n-1$ columns in B_{2n} then

$$\sum_{j\geq k}(\sum_{i=1}^{2n-1} q^{2n-i-1} f_{j,2n-2}(B/\overline{(i,2n)},q))z^{j-k}\begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-1-2k)}$$
$$= \sum_{i=1}^{2n-1} q^{2n-i-1}\sum_{j\geq k} f_{j,2n-2}(B/\overline{(i,2n)},q)\begin{bmatrix} j \\ k \end{bmatrix}_{q^2} z^{j-k}q^{(j-k)(2n-1-2k)}. \quad (59)$$

Setting $z=q^{-2}$ in (59) we get

$$\sum_{j\geq k}(\sum_{i=1}^{2n-1} q^{2n-i-1} f_{j,2n-2}(B/\overline{(i,2n)},q))\begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-3-2k)}$$
$$= \sum_{i=1}^{2n-1} q^{2n-i-1}\sum_{j\geq k} f_{j,2n-2}(B/\overline{(i,2n)},q)\begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-3-2k)}$$
$$= \sum_{i=1}^{2n-1} q^{2n-i-1} m_k(B/\overline{(i,2n)},q)[n-1-k]!! \quad (60)$$

where the last equality follows by using the special case of (57) with $z=1$ for the boards $B/\overline{(i,2n)}$. Comparing (59) and (60), we get that if B has less than $2n-1$ columns,

$$\sum_{j\geq k} f_{j,2n}(B,q)\begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-3-2k)}$$
$$= \sum_{j\geq k}(\sum_{i=1}^{2n-1} q^{2n-i-1} f_{j,2n-2}(B/\overline{(i,2n)},q))\begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-3-2k)} \quad (61)$$

if we can show that

$$\sum_{i=1}^{2n-1} q^{2n-i-1} m_k(B/\overline{(i,2n)},q)$$
$$= [2n-1-2k]m_k(B,q) - \frac{q^{2k+2}-1}{q^2-1}q^{2n-3-2k}(q^2-1)m_{k+1}(B,q)$$
$$= [2n-1-2k]m_k(B,q) - (q^{2n-1}-q^{2n-3-2k})m_{k+1}(B,q).$$

Since this is equation (43), (61) holds for all k assuming B is contained in the first $2n-2$ columns of B_{2n}. If $k=n$ this reduces to

$$f_{n,2n}(B,q) = \sum_{i=1}^{2n-1} q^{2n-i-1} f_{n,2n-2}(B/\overline{(i,2n)},q)). \quad (62)$$

If $k = n-1$ (61) reduces to

$$\begin{aligned} f_{n,2n}(B,q)\begin{bmatrix} n \\ n-1 \end{bmatrix}_{q^2} & q^{2n-3-2(n-1)} + f_{n-1,2n}(B,q) \\ &= \sum_{i=1}^{2n-1} q^{2n-i-1} f_{n,2n-2}(B/\overline{(i,2n)},q)\begin{bmatrix} n \\ n-1 \end{bmatrix}_{q^2} q^{2n-3-2(n-1)} \\ &\quad + \sum_{i=1}^{2n-1} q^{2n-i-1} f_{n-1,2n-2}(B/\overline{(i,2n)},q). \end{aligned} \tag{63}$$

Thus we can use (62) to cancel the first terms on both sides of (63) to get

$$f_{n-1,2n}(B,q) = \sum_{i=1}^{2n-1} q^{2n-i-1} f_{n-1,2n-2}(B/\overline{(i,2n)},q).$$

Continuing in this manner we get

$$f_{j,2n}(B,q) = \sum_{i=1}^{2n-1} q^{2n-i-1} f_{j,2n-2}(B/\overline{(i,2n)},q)$$

for all j. $\square$

Corollary 3. *If B is a shifted Ferrers board, $B \subseteq B_{2n}$, and $0 \le k \le n$, then (23) holds, i.e.*

$$f_{k,2n}(B,q) = \tilde{f}_{k,2n}(B,q).$$

Proof. If B has no cells in the last column of B_{2n}, then $f_{k,2n}(B,q)$ and $\tilde{f}_{k,2n}(B,q)$ satisfy the same recursion by Theorem's 6 and 10. If B has at least one cell in the last column of B_{2n}, then they both satisfy the same recursion by Theorem 5 and Corollary 2. If B is the empty board, then $\tilde{f}_{k,2n}(B,q) = \chi(k=0)[n]!!$. For this board, $m_k(B,q) = \chi(k=0)$ and so by (22), $f_{k,2n}(B,q) = \chi(k=0)[n]!!$. Since the $f_{k,2n}$ and the $\tilde{f}_{k,2n}$ satisfy the same recursion with the same initial conditions, they are equal for all B. $\square$

4. Algebraic Identities

In this section we prove a number of algebraic identities for the m_k and the f_j. In many cases these are analogues for nearly Ferrers boards of known identities for q-rook and q-hit numbers. We use the notation $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$.

Theorem 13. *If $B \subseteq B_{2n}$ is a nearly Ferrers board with b_i squares in row i for $i = 1,\ldots,2n-1$, and $0 \le k \le 2n-1$, then*

$$[2][4]\cdots[2k]m_{2n-1-k}(B,q) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{2\binom{k-j}{2}}(-1)^{(k-j)} \prod_{i=1}^{2n-1}[2j + b_i - 2i + 2].$$

Proof. By Theorem 8, the right-hand side above equals

$$\sum_{j\geq 0} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{2\binom{k-j}{2}}(-1)^{(k-j)} \sum_{s\geq 0}[2j][2j-2]\cdots[2j-2s+2]m_{2n-1-s}(B,q)$$

$$= \sum_{s\geq 0} m_{2n-1-s}(B,q) \sum_{s\leq j\leq k} [2j][2j-2]\cdots[2j-2s+2] \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{2\binom{k-j}{2}}(-1)^{(k-j)}$$

$$= \sum_{s\geq 0} m_{2n-1-s}(B,q) \sum_{u=0}^{k-s}[2(u+s)]\cdots[2u+2] \begin{bmatrix} k \\ u+s \end{bmatrix}_{q^2} q^{2\binom{k-s-u}{2}}(-1)^{(k-s-u)}$$

$$= \sum_{s\geq 0} m_{2n-1-s}(B,q) \begin{bmatrix} k \\ s \end{bmatrix}_{q^2} (-1)^{(k-s)}[2][4]\cdots[2s]$$

$$\times \sum_{u=0}^{k-s} \frac{[2k-2s][2k-2s-2]\cdots[2k-2s-2u+2]}{[2][4]\cdots[2u]} q^{2\binom{k-s-u}{2}}(-1)^{u}$$

$$= \sum_{s\geq 0} m_{2n-1-s}(B,q)[2][4]\cdots[2s] \begin{bmatrix} k \\ s \end{bmatrix}_{q^2} (-1)^{(k-s)} \sum_{u=0}^{k-s} \begin{bmatrix} k-s \\ u \end{bmatrix}_{q^2} q^{2\binom{k-s-u}{2}}(-1)^{u}$$

$$= \sum_{s\geq 0} m_{2n-1-s}(B,q)[2][4]\cdots[2s] \begin{bmatrix} k \\ s \end{bmatrix}_{q^2} (-1)^{(k-s)}(-1)^{(k-s)}$$

$$\sum_{u=0}^{k-s} \begin{bmatrix} k-s \\ u \end{bmatrix}_{q^2} q^{2\binom{u}{2}}(-1)^{u}$$

$$= [2][4]\cdots[2k]m_{2n-1-k}(B,q)$$

by the q-binomial theorem. □

Theorem 14. *If $B \subseteq B_{2n}$ is a nearly Ferrers board with b_i squares in row i for $i = 1, \ldots, 2n-1$, then*

$$\frac{1}{[n]!!} \sum_{j\geq 0} f_{j,2n}(B,q)[x] \downdownarrows_{2n-1-j} [x-2n+2j+1] \downdownarrows_{j} = \prod_{i=1}^{2n-1} [x+b_i-2i+2]. \quad (64)$$

Proof. By Theorem 10,

$$f_{j,2n}(B,q) = \sum_{k\geq j} m_k(B,q)[n-k]!!(-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{(k-j)(2n-k-j)},$$

so the left-hand side of (64) equals

$$\frac{1}{[n]!!} \sum_{j\geq 0} [x] \downdownarrows_{2n-1-j} [x-2n+2j+1] \downdownarrows_{j}$$

$$\times \sum_{k\geq j} m_k(B,q)[n-k]!!(-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{(k-j)(2n-k-j)}$$

$$= \sum_{k \geq 0} m_k(B,q) \frac{[n-k]!!}{[n]!!}$$
$$\times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} (-1)^{k-j} q^{(k-j)(2n-k-j)} [x] \downdownarrows_{2n-1-j} [x-2n+2j+1] \downdownarrows_{j}$$

$$= \sum_{k \geq 0} m_k(B,q) \frac{[n-k]!!}{[n]!!}$$
$$\times \sum_{j=0}^{k} \frac{(q^{-2k};q^2)_j}{(q^2;q^2)_j} (-q^{2k})^j q^{-2\binom{j}{2}} (-1)^{k-j} q^{(k-j)(2n-k-j)}$$
$$[x][x-2]\cdots[x-2(2n-2)] \frac{(q^{x-2n+3};q^2)_j}{(q^{x-4n+4};q^2)_j}$$

$$= [x][x-2]\cdots[x-4n+4] \sum_{k \geq 0} m_k(B,q) \frac{[n-k]!!}{[n]!!} (-1)^k q^{k(2n-k)}$$
$${}_2\phi_1 \left(\begin{matrix} q^{-2k}, & q^{x-2n+3}; & q^{2k+1-2n}, & q^2 \\ q^{x-4n+4} & & & \end{matrix} \right)$$

where in the last equality we have used the fact that $2kj-(j^2-j)+k(2n-k)-j2n+jk-jk+j^2 = j(2k+1-2n)+k(2n-k)$. Using the q-Vandermonde convolution for the sum of a terminating ${}_2\phi_1$ [GR, p.236], the equation above equals

$$[x][x-2]\cdots[x-4n+4] \sum_{k \geq 0} m_k(B,q) \frac{[n-k]!!}{[n]!!} (-1)^k q^{k(2n-k)} \frac{(q^{1-2n};q^2)_k}{(q^{x-4n+4};q^2)_k}$$

$$= \sum_{k \geq 0} m_k(B,q) [x][x-2]\cdots[x-2(2n-1-k)+2] \times C,$$

where

$$C = \frac{[n-k]!!}{[n]!!} (-1)^k \frac{q^{k(2n-k)}}{(1-q)^k} (1-q^{1-2n})(1-q^{3-2n})\cdots(1-q^{2k-1-2n})$$

$$= \frac{[n-k]!!}{[n]!!} \frac{q^{k(2n-k)}}{(1-q)^k}$$
$$q^{1-2n+3-2n+\ldots+2k-1-2n} (q^{2n-1}-1)(q^{2n-3}-1)\cdots(q^{2n-2k+1}-1)(-1)^k$$

$$= \frac{[n-k]!!}{[n]!!} q^{k(2n-k)+k^2-2nk} [2n-1][2n-3]\cdots[2n-2k+1] = 1.$$

Theorem 14 now follows from Theorem 8. $\square$

Corollary 4. *If B is a nearly Ferrers board, $B \subseteq B_{2n}$, then*

$$\sum_{j=0}^{n} f_{j,2n}(B,q) = [n]!!.$$

Proof. Letting $x \to \infty$ in the left-hand side of Theorem 14 we get

$$\frac{1}{[n]!!}\sum_{j=0}^{n} f_{j,2n}(B,q)\frac{1}{(1-q)^{2n-1}} = \frac{1}{(1-q)^{2n-1}}. \qquad \square$$

Remark: Corollaries 3 and 4 together show that for any fixed shifted Ferrers board $B \subseteq B_{2n}$, the statistic $t_f(B)$ has what could be called the "Mahonian" property for perfect matchings, i.e. its distribution is $[n]!!$.

Theorem 15. *If $B \subseteq B_{2n}$ is a nearly Ferrers board with b_i squares in row i for $i = 1, \dots, 2n-1$, then*

$$f_{j,2n}(B,q) = \sum_{s=0}^{n-j} \begin{bmatrix} n+1/2 \\ n-j-s \end{bmatrix}_{q^2} q^{2\binom{n-j-s}{2}}(-1)^{n-j-s} \times \frac{[s]!!}{[2][4]\cdots[2n-2+2s]} \prod_{i=1}^{2n-1} [2n-2+2s+b_i-2i+2].$$

Proof. Using Theorem 8, the right-hand side above equals

$$\sum_{s=0}^{n-j} \begin{bmatrix} n+1/2 \\ n-j-s \end{bmatrix}_{q^2} q^{2\binom{n-j-s}{2}}(-1)^{n-j-s} \times \frac{[s]!!}{[2][4]\cdots[2n-2+2s]} \sum_{k\geq 0} m_k(B,q)[2n-2+2s] \downdownarrows_{2n-1-k} .$$

The coefficient of $m_k(B,q)$ above is clearly zero unless $2n-2+2s \geq 2(2n-1-k)$, or $2k \geq 2n-2s$, or $k \geq n-s$ and since $s \leq n-j$ we have $k \geq n-(n-j) = j$. Thus the right-hand side of Theorem 15 equals

$$\sum_{k\geq j} m_k(B,q) \sum_{\substack{u=0 \\ (s=n-k+u)}}^{k-j} \begin{bmatrix} n+1/2 \\ k-j-u \end{bmatrix}_{q^2} q^{2\binom{n-j-(n-k+u)}{2}}(-1)^{n-j-(n-k+u)} \frac{[n-k+u]!!}{[2][4]\cdots[2u]}$$

$$= \sum_{k\geq j} m_k(B,q)[n-k]!! \begin{bmatrix} n+1/2 \\ k-j \end{bmatrix}_{q^2} \sum_{u=0}^{k-j} q^{2\binom{u}{2}} \frac{(q^{-2(k-j)};q^2)_u}{(q^{2n+1-2k+2j+2};q^2)_u} \times q^{2\binom{k-j+u}{2}}(-1)^{(k-j+u)} \frac{(q^{2n-2k+1};q^2)_u}{(q^2;q^2)_u}(-q^{2(k-j)})^u.$$

Now

$$-2\binom{u}{2}+2\binom{k-j-u}{2}+u(2k-2j) = -u^2+u+(k-j-u)(k-j-1-u)+2uk-2uj$$

$$= 2\binom{k-j}{2} - u^2 + u + u^2 + u - uk + uj - uk + uj + 2uk + 2uj = 2u + 2\binom{k-j}{2}$$

so the right-hand side of Theorem 15 now equals

$$\sum_{k\geq j} m_k(B,q)[n-k]!! \begin{bmatrix} n+1/2 \\ k-j \end{bmatrix}_{q^2} q^{2\binom{k-j}{2}}$$
$$\times (-1)^{k-j}{}_2\phi_1\left(\begin{matrix} q^{-2(k-j)}, & q^{2n+1-2k}; & q^2, & q^2 \\ q^{2n-2k+2j+3} & & & \end{matrix} \right)$$

$$= \sum_{k\geq j} m_k(B,q)[n-k]!! \begin{bmatrix} n+1/2 \\ k-j \end{bmatrix}_{q^2} q^{2\binom{k-j}{2}}$$
$$\times (-1)^{k-j} \frac{(q^{2j+2};q^2)_{k-j}}{(q^{2n-2k+2j+3};q^2)_{k-j}} q^{(2n+1-2k)(k-j)}$$

$$= \sum_{k\geq j} m_k(B,q)[n-k]!!(-1)^{k-j} q^{(2n+1-2k+k-j-1)(k-j)}$$
$$\times \frac{[2n+1][2n-1]\cdots[2n+1-2k+2j+2]}{[2][4]\cdots[2k-2j]}$$
$$\frac{[2j+2][2j+4]\cdots[2j+2+2(k-j)-2]}{[2n-2k+2j+3][2n-2k+2j+5]\cdots[2n-2k+2j+3+2k-2j-2]}$$

$$= \sum_{k\geq j} m_k(B,q)[n-k]!!(-1)^{k-j} q^{(2n-k-j)(k-j)} \begin{bmatrix} k \\ k-j \end{bmatrix}_{q^2}$$

$= f_{j,2n}(B,q)$ by Theorem 10. $\square$

Corollary 5. *For $B \subseteq B_{2n}$ a nearly Ferrers board with b_i squares in row i for $i = 1, \ldots, 2n-1$,*

$$\sum_{j=0}^{n} z^j f_{j,2n}(B,q) = \frac{(z;q^2)_\infty}{(zq^{2n+1};q^2)_\infty} \sum_{k=0}^{\infty} \frac{z^k [k]!! \prod_{i=1}^{2n-1}[2n+2k+b_i-2i]}{[2][4]\cdots[2n-2+2k]}.$$

Proof. Using the q-binomial theorem, the coefficient of z^{n-j} in the right-hand side above is

$$\sum_k \frac{[k]!! \prod_{i=1}^{2n-1}[2n+2k+b_i-2i]}{[2][4]\cdots[2n-2+2k]} \frac{(q^{-(2n+1)};q^2)_{n-j-k}}{(q^2;q^2)_{n-j-k}} q^{(2n+1)(n-j-k)}.$$

Since

$$\frac{(q^{-(2n+1)};q^2)_{n-j-k}}{(q^2;q^2)_{n-j-k}} q^{(2n+1)(n-j-k)} =$$
$$\frac{(1-q^{2n+1})(1-q^{2n-1})\cdots(1-q^{2n+1-(2n-2j-2k)+2})}{(q^2;q^2)_{n-j-k}}(-1)^{n-j-k}q^{2\binom{n-j-k}{2}}$$
$$= \begin{bmatrix} (2n+1)/2 \\ n-j-k \end{bmatrix}_{q^2} (-1)^{n-j-k} q^{2\binom{n-j-k}{2}},$$

the corollary follows from Theorem 15. □

Theorem 16. *If B is a nearly Ferrers board, $B \subseteq B_{2n}$, and $0 \leq k \leq n$,*

$$m_k(B,q)[n-k]!! = \sum_{j \geq k} \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} f_{j,2n}(B,q) q^{(j-k)(2n-1-2k)}.$$

Proof. By Theorem 10,

$$f_{j,2n}(B,q) = \sum_{k \geq j} m_k(B,q)[n-k]!!(-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} q^{(k-j)(2n-k-j)}.$$

Plugging this into the right-hand side of Theorem 16 yields

$$\sum_{j \geq k} \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} q^{(j-k)(2n-1-2k)}$$
$$\sum_{m \geq j} m_m(B,q)[n-m]!!(-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} q^{(m-j)(2n-m-j)}$$

$$= \sum_{m \geq k} m_m(B,q)[n-m]!!(-1)^m$$
$$\sum_{j=k}^{m} \begin{bmatrix} j \\ k \end{bmatrix}_{q^2} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} (-1)^j q^{(j-k)(2n-1-2k)+(m-j)(2n-m-j)}$$

$$= \sum_{m \geq k} m_m(B,q)[n-m]!!(-1)^m \sum_{\substack{u \geq 0 \\ (j=k+u)}} \begin{bmatrix} k+u \\ u \end{bmatrix}_{q^2} \begin{bmatrix} m \\ k+u \end{bmatrix}_{q^2}$$
$$(-1)^{k+u} q^{u(2n-1-2k)+(m-k-u)(2n-m-k-u)}$$

$$= \sum_{m \geq k} m_m(B,q)[n-m]!!(-1)^{m+k} \sum_{u \geq 0} \frac{(q^{2k+2};q^2)_u}{(q^2;q^2)_u} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}$$
$$q^{-(u^2-u)} q^{2(m-k)u} q^{u(2n-1-2k)+(m-k-u)(2n-m-k-u)} \frac{(q^{-2(m-k)};q^2)_u}{(q^{2k+2};q^2)_u}$$

$$= \sum_{m\geq k} m_m(B,q)[n-m]!!(-1)^{m+k} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{(m-k)(2n-m-k)}$$

$$\sum_{u\geq 0} \frac{(q^{-2(m-k)};q^2)_u}{(q^2;q^2)_u} q^{u(1+2(m-k)+2n-1-2k-2n+m+k-m+k)}$$

$$= \sum_{m\geq k} m_m(B,q)[n-m]!!(-1)^{m+k} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{(m-k)(2n-m-k)}$$

$$\times {}_1\phi_0 \left(q^{-2(m-k)}; \quad q^{2(m-k)}; \quad q^2 \right)$$

$$= \sum_{m\geq k} m_m(B,q)[n-m]!!(-1)^{m+k} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{(m-k)(2n-m-k)} \frac{(1;q^2)_\infty}{(q^{2(m-k)};q^2)_\infty}$$

$$= m_k(B,q)[n-k]!!. \qquad \square$$

5. Some Related Statistics for the q-Hit Numbers

We begin this section by giving the first direct combinatorial proof that the statistics $s_{F,d}(p)$ and $s_{F,h}(p)$ discussed in the introduction generate the same q-hit numbers for Ferrers boards contained in A_n. We then derive an analogous result for shifted Ferrers boards contained in B_{2n}.

Theorem 17. *Let $F = A(a_1, \dots, a_n)$ be a Ferrers board. Then for $0 \leq k \leq n$,*

$$\sum_{p\in H_{k,n}(F)} q^{s_{F,d}(p)} = \sum_{p\in H_{k,n}(F)} q^{s_{F,h}(p)}.$$

Proof. Let γ be a fixed placement of $n-k$ nonattacking rooks on A_n, all of which are off F, and consider the set $\Lambda(\gamma, F)$ of all placements λ which extend γ to a placement of n nonattacking rooks on A_n, with k rooks on F. We first show that

$$\sum_{\lambda\in\Lambda(\gamma,F)} q^{s_{F,d}(p)} = \sum_{\lambda\in\Lambda(\gamma,F)} q^{s_{F,h}(p)}. \tag{65}$$

It follows from the definition of $s_{F,d}(p)$ and $s_{F,h}(p)$ that the set of uncancelled squares in either Dworkin or Haglund cancellation that occur either off F, or in columns of F which contain a rook from γ, is the same for all λ. Let $F' \subseteq A_k$ be the Ferrers board obtained from F by deleting all the rows and columns in A_n containing a rook in γ, and collapsing the remaining rows to form a smaller Ferrers board. To complete the proof of (65), we need to show that

$$\sum_{p\in H_{0,k}(F')} q^{s_{F',d}(p)} = \sum_{p\in H_{0,k}(F')} q^{s_{F',h}(p)}. \tag{66}$$

If $F' = A(b_1, \dots, b_k)$, it is easy to see by induction that both sides of (66) equal $[b_1][b_2-1]\cdots[b_k-k+1]$. For if we place rooks in columns 1 thru $k-1$ of F', there will be $b_k - k + 1$ open squares in column k of F', and whether we use Dworkin or Haglund cancellation, we will generate a factor of $[b_k - k + 1]$ when placing a rook in the last column of F' in the $b_k - k + 1$ open squares. This proves (65), and Theorem 17 follows by summing over all γ. $\square$

The proof of Theorem 17 shows that one could also define other "hybrid" statistics to generate $h_{k,n}(F, q)$, by changing the cancellation scheme for the squares of F in columns with rooks on F to any scheme which gives the same value for (66). For example, one could use Dworkin cancellation in some columns and Haglund cancellation in others.

In [H] it was shown that to any statistic for the q-hit numbers there is an associated pair of "Euler-Mahonian" permutation statistics which are equi-distributed with the number of descents and the major index. The pair associated to $s_{F,d}(p)$ is the number of excedances and Denert's statistic, while associated to $s_{F,h}(p)$ was a new Euler-Mahonian pair. This new pair has been analyzed and placed within a general classification scheme of Mahonian statistics by Babson and Steingrímson [BaSt]. The proof of Theorem 17 shows that these pairs are part of a general family of related pairs.

The proof of Theorem 17 also carries over to shifted Ferrers boards $F \subseteq B_{2n}$. To construct other statistics for $f_{k,2n}(F, q)$, we could use the same cancellation as in $t_F(p)$ for those rooks off F, and modify the cancellation for rooks on F appropriately. In particular, we could count squares of F which are above rooks on F instead of below rooks on F (and not to the left of any rook). All that we need for the cancellation scheme for the rooks on the board is that when we sum over all perfect matchings with all rooks on a shifted Ferrers board $F = B(a_1, \ldots, a_{2n-1})$ we get $\prod_{i=1}^{2n-1}[a_{2n-i} - 2i + 2]$ (the $x = 0$ case of eq. (11)).

In [H] the proof that $s_{F,h}(p)$ generates $h_{k,n}(F, q)$ grew out of a relationship between q-rook numbers and matrices over finite fields. Theorem 18 shows there is a corresponding connection for rook placements on shifted Ferrers boards, although we have been unable to prove Corollary 3 by exploiting this relationship.

Given a skew-symmetric matrix S, let S' denote the upper-triangular portion of S.

Theorem 18. *Let $B \subseteq B_{2n}$ be a shifted Ferrers board. Let $P_{2k}(B, q)$ denote the number of $2n \times 2n$ skew-symmetric matrices S of rank $2k$ with entries in the finite field $\mathbb{F}_q$, where the entries in S' are zero outside of the squares of B. Then for $0 \le k \le n$,*

$$P_{2k}(B, q) = (q - 1)^k q^{|B|-k} m_k(B, q^{-1}).$$

Sketch of Proof. We perform a modified form of Gaussian elimination on such a matrix S. Find the lowest nonzero entry in the rightmost nonzero column of S', occurring say in square (i, j). By adding appropriate multiples of row i and column j, zero out the entries of S in row i and column j above and to the left of (i, j), and leave a 1 in square (i, j). Do similar operations to square (j, i). The resulting matrix S_1 is also skew-symmetric. Call squares (i, j) and (j, i) "pivot spots". Now iterate; find the lowest nonzero entry in the first nonzero column of S_1' to the left of column j and pivot as before. We eventually end up with k pivots above the main diagonal and k below, where if we placed rooks on the pivot spots in S', they would form a set of k nonattacking rooks on F. How many matrices S give rise to the same set of pivot spots? Theorem 18 follows after noting that the pivots spots of S' could originally have held any of $q-1$ entries, and the entries above and to their left which they attack could have been any of q entries. $\square$

In section 2 we pointed out that our algebraic definition of $f_{k,2n}(B,q)$ (eq. (12)) did not always result in a polynomial with nonnegative coefficients for boards which are not shifted Ferrers boards. However, there are larger classes of boards than shifted Ferrers boards for which we can show the $f_{k,2n}(B,q) \in \mathbb{N}[q]$ and give a combinatorial interpretation of these polynomials. For example, we can start with a shifted Ferrers board and shift the rightmost nonzero column all the way to the right. However, it is not clear what the most general class is and we'll pursue this question in subsequent work.

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