A Proof of the q,t-Schröder Conjecture

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Abstract

We prove a recent conjecture of Egge, Haglund, Killpatrick and Kremer (Elec. J. Combin. **10** (2003), $\#R16$), which gives a combinatorial formula for the coefficients of a hook shape in the Schur function expansion of the symmetric function ∇e_n, which Haiman (Invent. Math. **149** (2002), 371 − 407) has shown has a representation-theoretic interpretation. More precisely, we show that $\langle \nabla e_n, e_{n-d}h_d \rangle$ can be expressed as $\sum q^{\text{area}} t^{\text{bounce}}$, where the sum is over all Schröder lattice paths and area, bounce are simple statistics on these paths. For $d = 0$ this reduces to Garsia and Haglund's formula for the q,t-Catalan sequence (PNAS **98** (2001), 4313-4316). Our results build on symmetric function identities for sums of generalized Pieri coefficients and Macdonald polynomials due to Bergeron, Garsia, Haiman and Tesler (Asian J. Math. **6** (1999), 363−420) and Garsia and Haglund (Discrete Math. **256** (2002), 677-717). We also derive several transformation identities for sums of rational functions occurring in the theory of Macdonald polynomials and Diagonal Harmonics, and apply these to obtain a combinatorial formula for $\langle \nabla e_n, h_{n-d}h_d \rangle$. We discuss how our formulas for $\langle \nabla e_n, e_{n-d}h_d \rangle$ and $\langle \nabla e_n, h_{n-d}h_d \rangle$ prove two special cases of a recent conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov.

1 Introduction

Let

$$\mathcal{R}_n = \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]/ < \sum_{i=1}^{n} x_i^h y_i^k, \forall h+k>0 > . \tag{1}$$

$\mathcal{R}_n$ is known to be isomorphic to

$$\mathcal{H}_n = \{f : \sum_{i=1}^{n} \partial x_i^h \partial y_i^k f = 0, \forall h+k>0\}, \tag{2}$$

the so-called "space of Diagonal Harmonics" [Hai94]. The symmetric group S_n acts on $\mathcal{H}_n$ via $\sigma f = f(x_{\sigma_1},\ldots,x_{\sigma_n},y_{\sigma_1},\ldots,y_{\sigma_n})$. If we let $\mathcal{H}_n^{i,j}$ denote the portion of $\mathcal{H}_n$ of bi-homogeneous (x,y) degree (i,j), then the S_n-action respects the bigrading. We define the Frobenius Series $\mathcal{F}_n(q,t)$ to be the sum

$$\sum_{\lambda\vdash n}\sum_{i,j\ge 0} q^i t^j \mathrm{mult}(\chi^\lambda, \mathcal{H}_n^{i,j}) s_\lambda, \tag{3}$$

where $\lambda \vdash n$ means λ is a partition of n, s_λ is the Schur function, and $\mathrm{mult}(\chi^\lambda, \mathcal{H}_n^{i,j})$ is the multiplicity of the irreducible S_n-character χ^λ in the character of $\mathcal{H}_n^{i,j}$ induced by the S_n-action.

Let e_j be the jth elementary symmetric function and h_j be the jth elementary symmetric function, with generating functions

$$\prod_i (1+zx_i) = \sum_j z^j e_j[X], \quad \prod_i \frac{1}{(1-zx_i)} = \sum_j z^j h_j[X]. \tag{4}$$

We adopt the convention that e_n and h_n are zero if $n<0$. Both $e_n = s_{1^n}$ and $h_n = s_n$ are special cases of Schur functions.

Given partitions λ, μ, let $K_{\lambda,\mu}(q,t)$ be Macdonald's (q,t)-Kostka polynomial [Mac95]. Furthermore set $\eta(\mu) = \sum_i (i-1)\mu_i$, $\tilde{K}_{\lambda,\mu} = \tilde{K}_{\lambda,\mu}(q,t) = t^{\eta(\mu)} K_{\lambda,\mu}(q,1/t)$, and let

$$\tilde{H}_\mu = \tilde{H}_\mu[X;q,t] = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t) s_\lambda \tag{5}$$

be the modified Macdonald polynomial. The study of $\mathcal{F}_n(q,t)$ is closely connected to these polynomials via the following theorem, which was conjectured by Garsia and Haiman in the early 1990's.

Theorem 1 *(Haiman, [Hai02])*

$$\mathcal{F}_n(q,t) = \nabla e_n, \tag{6}$$

where ∇ is a linear operator defined on the $\tilde{H}_\mu$ basis via

$$\nabla \tilde{H}_\mu = t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_\mu, \tag{7}$$

with μ' denoting the conjugate of μ.

Let $C_n(q,t) = \langle \nabla e_n, s_{1^n} \rangle$, where $\langle , \rangle$ is the usual Hall scalar product defined by $\langle s_\beta, s_\mu \rangle = 1$ if $\beta = \mu$ and 0 otherwise. A few years ago Garsia and Haglund [GH01], [GH02] proved that

$$C_n(q,t) = \sum_\pi q^{\mathrm{area}(\pi)} t^{\mathrm{bounce}(\pi)}, \tag{8}$$

where the sum is over all Catalan lattice paths β from $(0,0)$ to (n,n), which are paths consisting of N $(0,1)$ and E $(1,0)$ steps which never go below the diagonal $y = x$. Here $\mathrm{area}(\beta)$ is the number of "lower triangles" (triangles whose vertex set is of the form (i,j), $(i+1,j)$ and $(i+1,j+1)$) between β and the diagonal $y = x$. The statistic bounce was first introduced in [Hag03] (although called differently there). To calculate it, first form the "bounce path" for β (if β is the path on the right in Figure 1, the bounce path for β is the dotted line) by starting at (n,n), going left until we reach the top of an N step of β, then "bouncing" down to the line $y = x$, then iterating: left to the path, down to the line $y = x$, and so on until we reach $(0,0)$. As we travel from (n,n) to $(0,0)$ our bounce path hits the line $y = x$ at various points, say at $(j_1,j_1),(j_2,j_2),\ldots,(j_k,j_k)$ $((3,3),(1,1),(0,0)$ in Figure 1) with $n > j_1 > \cdots > j_k = 0$. We then set

$$\mathrm{bounce}(\beta) = j_1 + j_2 + \ldots + j_{k-1}. \tag{9}$$

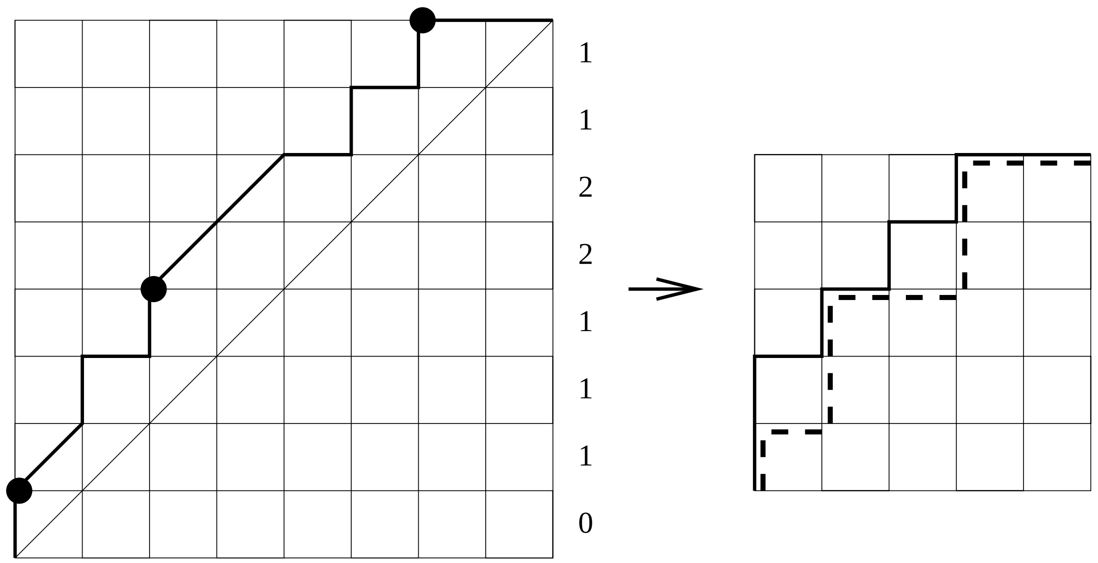

Figure 1: On the left, a Schröder path Π, with the top of each peak marked by a dot. To the right of each row is the length of the row. On the right is the Catalan path $C(\Pi)$ and its bounce path (the dotted path).

Recently Egge, Haglund, Killpatrick and Kremer [EHKK03] introduced extensions of (area, bounce) to Schröder paths, which are paths that are identical to Catalan paths except diagonal D $(1,1)$ steps are also allowed. Specifically, given a Schröder path Π, we let $\text{area}(\Pi)$ be the number of lower triangles between the path and the line $y = x$ as before. To calculate $\text{bounce}(\Pi)$, first remove all the D steps from Π and collapse in the obvious way to form a Catalan path $C(\Pi)$. Next construct the bounce path for $C(\Pi)$; note that N steps of this bounce path occurring just before E steps are also N steps of $C(\Pi)$. The corresponding N steps of Π are called the "peaks" of Π. For each D step α of Π let $b(\alpha)$ denote the number of peaks above it, and define

$$\text{bounce}(\Pi) = \text{bounce}(C(\Pi)) + \sum_{\alpha} b(\alpha), \tag{10}$$

where the sum is over all D steps of Π. For the path of Figure 1, bounce $= (3+1)+(1+1+2) = 8$.

Let $\mathcal{S}_{n,d}$ denote the set of all Schröder paths from $(0,0)$ to (n,n) with exactly d D steps, and let

$$S_{n,d}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{area}(\Pi)} t^{\text{bounce}(\Pi)}. \tag{11}$$

Egge et. al conjecture that

$$S_{n,d}(q,t) = \langle \nabla e_n, e_{n-d} h_d \rangle. \tag{12}$$

By the the Pieri rule for multiplying Schur functions [Mac95],

$$e_{n-d} h_d = s_{d+1,1^{n-d-1}} + s_{d,1^{n-d}}, \tag{13}$$

so (12) gives a combinatorial expression for the sum of two consecutive hook shapes in ∇e_n. One of the main results of this article is a proof of this conjecture. Our proof makes heavy use of extensions of results of Bergeron, Garsia, Haiman and Tesler [BGHT99], and Garsia and Haglund [GH02], involving summation formulas for generalized Pieri coefficients and Macdonald polynomials.

Garsia and Haiman [GH96] obtained an explicit expression for $\langle \nabla e_n, s_\lambda \rangle$ as a sum of rational functions in q,t. We use our Pieri coefficient summation formulas to obtain some transformation identities for these and other related sums of rational functions, which allow us to also find a combinatorial expression for $\langle \nabla e_n, h_d h_{n-d} \rangle$. We then show how this result and (12) are special cases of a general conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov [HHL$^+$].

2 Transformation Formulas

We begin by reviewing some notation and basic results in the theory of symmetric functions. By convention we say there is one partition of 0, the partition $\emptyset$. For $n > 0$, we label the squares of the Ferrers diagram of a partition of n with (row, column) coordinates so that the upper left-hand square has coordinates $(0,0)$. For a given square (i,j), we let

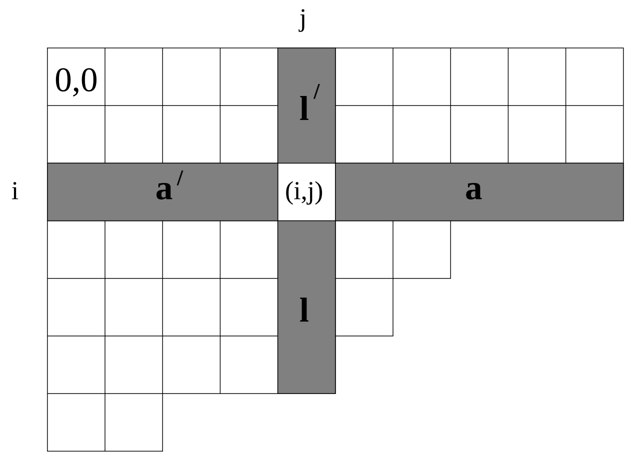

Figure 2: Arm a, leg l, co-arm a' and co-leg l' of (i,j)

the arm a, leg l, coarm a' and coleg l' be the number of squares in the Ferrers diagram of the partition in question to the right, below, left, and above (i,j), respectively. For example, for the cell labeled (i,j) in Figure 2 we have $a=5$, $a'=4$, $l=3$ and $l'=2$.

In [GH96] Garsia and Haiman proved that

$$e_n = \sum_{\mu \vdash n} \frac{M\tilde{H}_\mu \Pi_\mu B_\mu(q,t)}{w_\mu}, \tag{14}$$

where

$$\Pi_\mu = \prod_{(i,j)\in\mu,(i,j)\neq(0,0)} (1-q^{a'}t^{l'}), \quad B_\mu(q,t) = \sum_{(i,j)\in\mu} q^{a'}t^{l'}, \tag{15}$$

$$w_\mu = \prod_{(i,j)\in\mu} (q^a - t^{l+1})(t^l - q^{a+1}) \quad \text{and} \quad M = (1-q)(1-t). \tag{16}$$

Thus

$$\langle \nabla e_n, s_\lambda \rangle = \sum_{\mu \vdash n} \frac{M\tilde{K}_{\lambda,\mu}\Pi_\mu B_\mu(q,t) T_\mu}{w_\mu}, \tag{17}$$

where

$$T_\mu = t^{\eta(\mu)} q^{\eta(\mu')}. \tag{18}$$

It is known that

$$\tilde{K}_{1^n,\mu} = T_\mu, \tilde{K}_{n,\mu} = 1, \tag{19}$$

and more generally that [Mac95, p.362, ex. 2]

$$\left\langle \tilde{H}_\mu, e_{n-d}h_d \right\rangle = e_{n-d}[B_\mu], \quad 0 \le d \le n. \tag{20}$$

We let $\mathbb{N}$ stand for the nonnegative integers and $\delta_{n,k}$ denote the function which is one if $n = k$ and zero otherwise. If P is a symmetric function we write $P \in \Lambda$, and if in addition P is of homogeneous degree n we write $P \in \Lambda^n$. Unless otherwise stated, any $P \in \Lambda$ will depend on a (possibly infinite) set of variables X, which we supress, so for example P will stand for $P(X) = P(x_1, x_2, \ldots)$. As usual ω denotes the homomorphism on symmetric functions defined on the "power sum" symmetric functions $p_k = \sum_i x_i^k$ by $\omega p_k = (-1)^{k-1} p_k$, or, equivalently, on the Schur basis by $\omega s_\lambda = s_{\lambda'}$. Also, given symmetric functions P and Q, $P[Q(X)]$ will denote the "plethystic" substitution of $Q(X)$ into $P(X)$. To define this, first express $P(X)$ as a polynomial in the $p_k(X)$, then replace each $p_k(X)$ by $Q(x_1^k, x_2^k, \ldots)$. For example, to evaluate $P[\frac{X(1-z)}{1-q}]$ we would first express $P(X)$ as a polynomial in the $p_k(X)$, then replace $p_k(X)$ by $p_k(X)(1 - z^k)/(1 - q^k)$. Note that $P[X] = P(X)$, and that inside the plethystic brackets a minus sign has special meaning, so for example $p_k(-x_1, -x_2, \ldots) = (-1)^k p_k(X)$ while $p_k[-X] = -p_k[X]$. When we need to discriminate between the two types of minus signs we use ϵX to indicate the set of variables $(-x_1, -x_2, \ldots)$. Thus $p_k[\epsilon X] = (-1)^k p_x$ and for any $P \in \Lambda$, $P[-\epsilon X] = \omega P$. Recalling that M stands for $(1 - q)(1 - t)$, we will often abbreviate $P[X/M]$ by P^*.

Two identities which will be very useful to us are the "addition" formulas

$$e_n[X+Y] = \sum_{k=0}^{n} e_k[X] e_{n-k}[Y], \qquad h_n[X+Y] = \sum_{k=0}^{n} h_k[X] h_{n-k}[Y], \tag{21}$$

and the Cauchy identities

$$e_n[XY] = \sum_{\lambda \vdash n} s_\lambda[X] s_{\lambda'}[Y], \qquad h_n[XY] = \sum_{\lambda \vdash n} s_\lambda[X] s_\lambda[Y]. \tag{22}$$

We will also make use of Cauchy's q-binomial series [GR90, p.7];

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{23}$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$. Two special cases of (23) are

$$(z;q)_n = \sum_{k=0}^{n} z^k q^{\binom{k}{2}} (-1)^k e_k[1, q, \ldots, q^{n-1}] \tag{24}$$

$$= \sum_{k=0}^{n} z^k q^{\binom{k}{2}} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{25}$$

and

$$\frac{1}{(z;q)_n} = \sum_{k=0}^{\infty} z^k h_k[1, q, \ldots, q^{n-1}] \tag{26}$$

$$= \sum_{k=0}^{\infty} z^k \begin{bmatrix} k+n-1 \\ k \end{bmatrix}_q. \tag{27}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient, defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}, \tag{28}$$

with $[n]! = (q;q)_n/(1-q)^n = \prod_{i=2}^n (1+q+\ldots+q^{i-1})$.

In [GH01] Garsia and Haglund define symmetric functions $E_{n,k}$ via

$$e_n\left[X\frac{1-z}{1-q}\right] = \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}. \tag{29}$$

They prove that

$$\langle \nabla E_{n,k}, e_n \rangle = q^{\binom{k}{2}} t^{n-k} \sum_{r=0}^{n-k} \begin{bmatrix} r+k-1 \\ r \end{bmatrix}_q \langle \nabla E_{n-k,r}, e_{n-r} \rangle, \tag{30}$$

and then show that (8) follows easily from this recurrence. Based on Maple calculations, Garsia and the author conjecture the following.

Conjecture 1 *For all n, k, λ,*

$$\langle \nabla E_{n,k}, s_\lambda \rangle \in \mathbb{N}[q,t]. \tag{31}$$

In this section we will obtain formulas for $E_{n,k}$ which will allow us in later sections to prove some significant special cases of this conjecture.

Haiman [Hai02, Thm. 3.3] has proven that for any partitions λ, β,

$$\left\langle \Delta_{s_\beta} \nabla e_n, s_\lambda \right\rangle \in \mathbb{N}[q,t], \tag{32}$$

where for any symmetric function P, the operator Δ_P is a linear operator defined on the modified Macdonald basis via

$$\Delta_P \tilde{H}_\mu = P[B_\mu]\tilde{H}_\mu. \tag{33}$$

Note that if $\mu \vdash n$, then by (19) and (20)

$$\Delta_{e_n} \tilde{H}_\mu = \nabla \tilde{H}_\mu. \tag{34}$$

We now advance a more general form of Conjecture 1, which has also been tested using Maple, namely

Conjecture 2 *For all n, k, λ, β,*

$$\left\langle \Delta_{s_\beta} \nabla E_{n,k}, s_\lambda \right\rangle \in \mathbb{N}[q,t]. \tag{35}$$

(In [BGHT99] it is also conjectured that $\langle \Delta_{s_\lambda} e_n, s_\lambda \rangle \in \mathbb{N}[q,t]$, although this stronger conjecture doesn't hold with e_n replaced by $E_{n,k}$. In particular $\langle E_{n,k}, s_\lambda \rangle$ is generally not in $\mathbb{N}[q,t]$.)

The following result shows Conjecture 2 is true if $k = n$. We will also prove it holds for $k = n-1$, although we defer a discussion of this until Section 5 in order to quickly arrive at more significant results.

Proposition 1 *For any partition β and $\lambda \vdash n$,*

$$\left\langle \Delta_{s_\beta} \nabla E_{n,n}, s_\lambda \right\rangle = s_\beta[1, q, \ldots, q^{n-1}] \sum_{T \in SYT(\lambda)} q^{maj(T)}, \tag{36}$$

where the sum is over all standard Young tableaux T of shape λ, and

$$maj(T) = \sum_{i\,:\, i+1 \text{ is below } i \text{ in } T} i. \tag{37}$$

Proof. From (29) we see that

$$\Delta_{s_\beta} \nabla e_n[X \frac{1-z}{1-q}]|_{z^n} = (-1)^n q^{\binom{n}{2}} \frac{1}{(q;q)_n} \Delta_{s_\beta} \nabla E_{n,n}, \tag{38}$$

where $f(z)|_{z^n}$ stands for the coefficient of z^n in the Maclaurin series expansion of $f(z)$. On the other hand, (22) implies

$$e_n[X \frac{1-z}{1-q}] = \sum_{\lambda \vdash n} s_{\lambda'}[\frac{X}{1-q}] s_\lambda[1-z]. \tag{39}$$

This together with the well-known fact that

$$s_\lambda[1-z]|_{z^n} = \begin{cases} (-1)^n & \text{if } \lambda = 1^n \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

gives

$$\Delta_{s_\beta} \nabla h_n[\frac{X}{1-q}] = q^{\binom{n}{2}} \frac{1}{(q;q)_n} \Delta_{s_\beta} \nabla E_{n,n}. \tag{41}$$

Now, as noted in [GH02, (2.16)],

$$\tilde{H}_n = (q;q)_n h_n[\frac{X}{1-q}], \tag{42}$$

so

$$\Delta_{s_\beta} \nabla h_n[\frac{X}{1-q}] = s_\beta[B_n] q^{\binom{n}{2}} h_n[\frac{X}{1-q}] \tag{43}$$

$$= s_\beta[1, q, \ldots, q^{n-1}] q^{\binom{n}{2}} \sum_\lambda s_\lambda[X] s_\lambda[\frac{1}{1-q}] \quad \text{(by (22))} \tag{44}$$

$$= s_\beta[1, q, \ldots, q^{n-1}] q^{\binom{n}{2}} \sum_\lambda s_\lambda[X] \frac{1}{(q;q)_n} \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} \tag{45}$$

by [Sta99, p.363]. □

Given $A \in \Lambda$, define generalized "Pieri" coefficients $d^A_{\mu\nu}$ via

$$A\tilde{H}_\nu = \sum_{\mu \supseteq \nu} \tilde{H}_\mu d^A_{\mu\nu}. \tag{46}$$

Let $A^\perp$ be the "skewing" operator which is adjoint to multiplication by A with respect to the Hall scalar product; i.e., for any symmetric functions A, P, Q,

$$\langle A\, P, Q\rangle = \left\langle P, A^\perp Q\right\rangle. \tag{47}$$

Define skew coefficients $c^{A\perp}_{\mu\nu}$ via

$$A^\perp \tilde{H}_\mu = \sum_{\nu \subseteq \mu} \tilde{H}_\nu c^{A\perp}_{\mu\nu}. \tag{48}$$

The $c^{f\perp}_{\mu\nu}$ and $d^f_{\mu\nu}$ satisfy [GH02, (3.5)].

$$c^{f\perp}_{\mu\nu} w_\nu = d^{\omega f^*}_{\mu\nu} w_\mu. \tag{49}$$

One of the important tools used in the proof of (8) is the following.

Theorem 2 *[GH02, pp.698-701] Let $m \geq d \geq 0$. Then for any symmetric function g of degree at most d, and $\mu \vdash m$,*

$$\sum_{\substack{\nu \subseteq \mu \\ m-d \leq |\nu| \leq m}} c^{(\omega g)^\perp}_{\mu\nu} T_\nu = T_\mu G[D_\mu(1/q, 1/t)], \tag{50}$$

and

$$\sum_{\substack{\nu \subseteq \mu \\ m-d \leq |\nu| \leq m}} c^{g^\perp}_{\mu\nu} = F[D_\mu(q,t)], \tag{51}$$

where

$$G[X] = \omega \nabla \left(g \left[\frac{X+1}{(1-1/q)(1-1/t)} \right] \right), \tag{52}$$

$$F[X] = \nabla^{-1} \left((\omega g) \left[\frac{X - \epsilon}{M} \right] \right), \tag{53}$$

and

$$D_\mu(q,t) = M B_\mu(q,t) - 1. \tag{54}$$

We will use Theorem 2 to obtain the following formula for $\nabla E_{n,k}$.

Theorem 3 *For $k, n \in \mathbb{N}$ with $1 \leq k < n$,*

$$\nabla E_{n,k} = t^{n-k}(1-q^k) \sum_{\nu \vdash n-k} \frac{T_\nu}{w_\nu} \sum_{\substack{\mu \supseteq \nu \\ \mu \vdash n}} \tilde{H}_\mu \Pi_\mu d_{\mu\nu}^{h_k[\frac{X}{1-q}]} \tag{55}$$

Proof. From (1.6), (1.11) and (1.20) of [GH02] we have

$$\nabla E_{n,k} = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^{n-i} q^{\binom{i}{2}+k-in} \nabla h_n \left[X \frac{1-q^i}{1-q} \right]. \tag{56}$$

Using the following formula [GH02, p. 693]

$$\nabla h_n \left[X \frac{1-q^i}{1-q} \right] = (-t)^{n-i} q^{i(n-1)} (1-q^i) \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu \Pi_\mu e_i[(1-t)B_\mu(1/q, 1/t)]}{w_\mu} \tag{57}$$

we then have

$$\nabla E_{n,k} = \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i}{2}+k-i} (1-q^i) t^{n-i} \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu \Pi_\mu}{w_\mu} e_i[(1-t)B_\mu(1/q, 1/t)]. \tag{58}$$

On the other hand,

$$t^{n-k}(1-q^k) \sum_{\nu \vdash n-k} \frac{T_\nu}{w_\nu} \sum_{\substack{\mu \supseteq \nu \\ \mu \vdash n}} \tilde{H}_\mu \Pi_\mu d_{\mu\nu}^{h_k[\frac{X}{1-q}]} \tag{59}$$

$$= t^{n-k}(1-q^k) \sum_{\mu \vdash n} \frac{\tilde{H}_\mu \Pi_\mu}{w_\mu} \sum_{\substack{\nu \subseteq \mu \\ \nu \vdash n-k}} \frac{d_{\mu\nu}^{h_k[\frac{X}{1-q}]} T_\nu w_\mu}{w_\nu} \tag{60}$$

$$= t^{n-k}(1-q^k) \sum_{\mu \vdash n} \frac{\tilde{H}_\mu \Pi_\mu}{w_\mu} \sum_{\substack{\nu \subseteq \mu \\ \nu \vdash n-k}} c_{\mu\nu}^{e_k[(1-t)X]^\perp} T_\nu \qquad \text{(by (49))} \tag{61}$$

$$= t^{n-k}(1-q^k) \sum_{\mu \vdash n} \frac{\tilde{H}_\mu \Pi_\mu T_\mu}{w_\mu} \omega \nabla \left(h_k \left[\frac{qt(X+1)}{1-q} \right] \right) |_{X=D_\mu(1/q,1/t)} \qquad \text{(by Theorem 2)} \tag{62}$$

$$= t^n q^k (1-q^k) \sum_{\mu \vdash n} \frac{\tilde{H}_\mu \Pi_\mu T_\mu}{w_\mu} \omega \nabla \sum_{j=0}^{k} h_j \left[\frac{X}{1-q} \right] h_{k-j} \left[\frac{1}{1-q} \right] |_{X=D_\mu(1/q,1/t)} \tag{63}$$

$$= t^n q^k (1-q^k) \sum_{\mu \vdash n} \frac{\tilde{H}_\mu \Pi_\mu T_\mu}{w_\mu} \sum_{j=0}^{k} q^{\binom{j}{2}} e_j \left[\frac{(1-1/q)(1-1/t)B_\mu(1/q,1/t) - 1}{1-q} \right] \frac{1}{(q;q)_{k-j}} \tag{64}$$

using the $n=\infty$ case of (27) and (42). After using (21) again, (64) equals

$$t^n q^k(1-q^k)\sum_{\mu\vdash n}\frac{\tilde{H}_\mu \Pi_\mu T_\mu}{w_\mu}\sum_{j=0}^{k} q^{\binom{j}{2}-j}t^{-j}\sum_{i=0}^{j} e_i[(1-t)B_\mu(1/q,1/t)]e_{j-i}\left[\frac{-qt}{1-q}\right]\frac{1}{(q;q)_{k-j}} \tag{65}$$

$$= t^n q^k(1-q^k)\sum_{\mu\vdash n}\frac{\tilde{H}_\mu \Pi_\mu T_\mu}{w_\mu}\sum_{i=0}^{k} e_i[(1-t)B_\mu(1/q,1/t)]\sum_{j=i}^{k}\frac{q^{\binom{j}{2}-i}t^{-i}(-1)^{j-i}}{(q;q)_{j-i}(q;q)_{k-j}}, \tag{66}$$

since $e_{j-i}[-1/1-q]=(-1)^{j-i}h_{j-i}[1/1-q]=(-1)^{j-i}/(q;q)_{j-i}$. The inner sum in (66) equals

$$t^{-i}q^{-i}\sum_{s=0}^{k-i}\frac{q^{\binom{s+i}{2}}(-1)^s}{(q;q)_s(q;q)_{k-i-s}} = \frac{t^{-i}q^{-i}}{(q;q)_{k-i}}\sum_{s=0}^{k-i}\frac{q^{\binom{s}{2}+\binom{i}{2}+is}(-1)^s(1-q^{k-i})\cdots(1-q^{k-i-s+1})}{(q;q)_s} \tag{67}$$

$$= \frac{t^{-i}q^{\binom{i}{2}-i}}{(q;q)_{k-i}}\sum_{s=0}^{k-i}\frac{q^{\binom{s}{2}+is+(k-i)s-\binom{s}{2}}(1-q^{-k+i})\cdots(1-q^{-k+i+s-1})}{(q;q)_s} \tag{68}$$

$$= \frac{t^{-i}q^{\binom{i}{2}-i}}{(q;q)_{k-i}}\sum_{s=0}^{\infty}\frac{(q^{i-k};q)_s}{(q;q)_s}q^{ks} \tag{69}$$

$$= \frac{t^{-i}q^{\binom{i}{2}-i}}{(q;q)_{k-i}}(1-q^i)\cdots(1-q^{k-1}) \quad \text{(by (23))}, \tag{70}$$

and plugging this into (66) and comparing with (58) completes the proof. □

Given a partition ν we let $|\nu|=\sum_i \nu_i$, and for any $P\in\Lambda$ we let $P^*=P[\frac{X}{M}]=P[\frac{X}{(1-q)(1-t)}]$. The following summation theorem for the $d^A_{\mu\nu}$ will be crucial in reducing the inner sum in (55) to more useful forms. The case $\lambda=1^{|\nu|}$ is essentially equivalent to Theorem 0.3 of [GH02]. The statement involves a new linear operator $\mathcal{K}_P$, where $P\in\Lambda$, which we define for $P=s_\lambda$ on the modified Macdonald basis as follows.

$$\mathcal{K}_{s_\lambda}\tilde{H}_\mu=\tilde{K}_{\lambda,\mu}\tilde{H}_\mu. \tag{71}$$

Theorem 4 *Let $A\in\Lambda^b$, λ,ν partitions with $|\nu|>0, |\lambda|=m\in\mathbb{N}$. Then*

$$\sum_{\substack{\mu\supseteq\nu\\|\mu|=|\nu|+b}} d^A_{\mu,\nu}s_\lambda[B_\mu]\Pi_\mu=\Pi_\nu(\Delta_{A[MX]}s_\lambda\left[\frac{X}{M}\right])[MB_\nu] \tag{72}$$

and

$$\sum_{\substack{\mu\supseteq\nu\\|\mu|=|\nu|+b}} d^A_{\mu,\nu}s_\lambda[B_\mu]\Pi_\mu=\Pi_\nu\mathcal{K}_{s_{\lambda'}}\left(\sum_{r=0}^{\min(b,m)}(f)_r e_{m-r}\left[\frac{X}{M}\right]\right)[MB_\nu]. \tag{73}$$

Here

$$f = \tau_\epsilon \nabla \tau_1 A \tag{74}$$

with τ a linear operator defined by $\tau_a A = A[X+a]$, and for any $g \in \Lambda$, $(g)_r$ denotes the portion of g which is of homogeneous degree r (so $(g)_r \in \Lambda^r$).

Proof. Our proof follows the proof of Theorem 0.3 of [GH02] closely. By definition,

$$\sum_{\mu \supseteq \nu} d^A_{\mu,\nu} \tilde{H}_\mu = A\tilde{H}_\nu. \tag{75}$$

We now apply the following result, known as the Koornwinder-Macdonald reciprocity formula (see [GHT99]), which holds for any partitions μ, λ with $|\mu|, |\lambda| \geq 0$

$$\frac{\tilde{H}_\mu[1+z(MB_\lambda(q,t)-1)]}{\prod_{(i,j)\in\mu}(1-zq^{a'}t^{l'})} = \frac{\tilde{H}_\lambda[1+z(MB_\mu(q,t)-1)]}{\prod_{(i,j)\in\lambda}(1-zq^{a'}t^{l'})}. \tag{76}$$

Evaluating both sides of (75) at $X = 1+z(MB_\lambda - 1)$ and then applying (76) to both sides gives

$$\sum_{\mu\supseteq\nu} d^A_{\mu,\nu} \frac{\prod_{(i,j)\in\mu}(1-zq^{a'}t^{l'})\tilde{H}_\lambda[1+z(MB_\mu-1)]}{\prod_{(i,j)\in\lambda}(1-zq^{a'}t^{l'})} = \\ A[1+z(MB_\lambda-1)]\frac{\prod_{(i,j)\in\nu}(1-zq^{a'}t^{l'})\tilde{H}_\lambda[1+z(MB_\nu-1)]}{\prod_{(i,j)\in\lambda}(1-zq^{a'}t^{l'})}. \tag{77}$$

Since $|\nu| > 0$ there is a common factor of $1-z$ (corresponding to $(i,j) = (0,0)$) in the numerator of both sides of (77). Cancelling this as well as the denominators on both sides and then setting $z = 1$, (77) becomes

$$\sum_{\mu\supseteq\nu} d^A_{\mu,\nu} \Pi_\mu \tilde{H}_\lambda[MB_\mu] = A[MB_\lambda]\Pi_\nu \tilde{H}_\lambda[MB_\nu]. \tag{78}$$

Since the $\tilde{H}$ form a basis this implies that for any symmetric function G,

$$\sum_{\mu\supseteq\nu} d^A_{\mu,\nu} \Pi_\mu G[MB_\mu] = \Pi_\nu(\Delta_{A[MX]}G)[MB_\nu]. \tag{79}$$

Letting $G = s^*_\lambda$ proves (72).

Given $G \in \Lambda^m$, $m \geq 0$, by definition

$$G[X] = \sum_{\beta\vdash m} d^G_{\beta,\emptyset}\tilde{H}_\beta. \tag{80}$$

Thus

$$\sum_{\beta\vdash m} A[MB_\beta] d^G_{\beta,\emptyset} \tilde{H}_\beta = \Delta_{A[MX]} G. \tag{81}$$

Now by Theorem 2 we have

$$A[MB_\beta] = \sum_{\alpha\subseteq\beta} c^{f^\perp}_{\beta\alpha}, \tag{82}$$

where

$$A = \tau_{-1}\nabla^{-1}\left((\omega f)[\frac{X-\epsilon}{M}]\right), \tag{83}$$

or equivalently

$$\omega f^* = \tau_\epsilon \nabla \tau_1 A. \tag{84}$$

Hence using (49) we have

$$\Delta_{A[MX]} G = \sum_{\beta\vdash m} \tilde{H}_\beta d^G_{\beta\emptyset} \sum_{\alpha\subseteq\beta} c^{f^\perp}_{\beta\alpha} \tag{85}$$

$$= \sum_{r=0}^{m} \sum_{\alpha\vdash m-r} \frac{1}{w_\alpha} \sum_{\substack{\beta\supseteq\alpha \\ \beta\vdash m}} \tilde{H}_\beta d^G_{\beta\emptyset} w_\beta d^{\omega f^*}_{\beta\alpha}. \tag{86}$$

One of Macdonald's basic identities implies [GHT99, (1.18)])

$$e_n[\frac{XY}{M}] = \sum_{\beta\vdash n} \frac{\tilde{H}_\beta[X;q,t]\tilde{H}_\beta[Y;q,t]}{w_\beta}. \tag{87}$$

Combining this with (22) yields

$$s^*_\lambda = \sum_{\beta\vdash|\lambda|} \frac{\tilde{H}_\beta}{w_\beta} \tilde{K}_{\lambda',\beta}, \tag{88}$$

or equivalently

$$d^{s_\lambda *}_{\beta\emptyset} = \frac{\tilde{K}_{\lambda',\beta}}{w_\beta}. \tag{89}$$

Thus setting $G = s^*_\lambda$ in (86) we have

$$\Delta_{A[MX]} s^*_\lambda = \sum_{r=0}^{m} \sum_{\alpha \vdash m-r} \frac{1}{w_\alpha} \sum_{\substack{\beta \supseteq \alpha \\ \beta \vdash m}} \tilde{H}_\beta \tilde{K}_{\lambda',\beta} d^{\omega f^*}_{\beta\alpha} \tag{90}$$

$$= \mathcal{K}_{s_{\lambda'}}(\sum_{r=0}^{m} \sum_{\alpha \vdash m-r} \frac{1}{w_\alpha} \sum_{\substack{\beta \supseteq \alpha \\ \beta \vdash m}} \tilde{H}_\beta d^{\omega f^*}_{\beta\alpha} \tag{91}$$

$$= \mathcal{K}_{s_{\lambda'}}(\sum_{r=0}^{m} \sum_{\alpha \vdash m-r} \frac{1}{w_\alpha} \tilde{H}_\alpha (\omega f^*)_r) \tag{92}$$

$$= \mathcal{K}_{s_{\lambda'}}(\sum_{r=0}^{\min(b,m)} e^*_{m-r} (\omega f^*)_r) \tag{93}$$

using (88), and the fact that the degree of ωf^* is at most the degree of A, so $(\omega f^*)_r = 0$ for $r > b$. Plugging (93) into (81) we see the right-hand sides of (72) and (73) are equal. $\square$

Corollary 1 *Let $1 \le k < n$ and $m \in \mathbb{N}$ with $\lambda \vdash m$. Then*

$$\langle \Delta_{s_\lambda} \nabla E_{n,k}, s_n \rangle$$
$$= t^{n-k} \sum_{\nu \vdash n-k} \frac{T_\nu \Pi_\nu}{w_\nu} \mathcal{K}_{s_{\lambda'}} \left(\sum_{p=1}^{\min(k,m)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} (1-q^p) e_{m-p} \left[\frac{X}{M} \right] h_p \left[\frac{X}{1-q} \right] \right) [MB_\nu]. \tag{94}$$

Proof. Since $1 \le n-k$, for $\nu \vdash n-k$ we can apply Theorem 4 to the inner sum on the right-hand side of (55) to get

$$\langle \Delta_{s_\lambda} \nabla E_{n,k}, s_n \rangle = t^{n-k}(1-q^k) \sum_{\nu \vdash n-k} \frac{T_\nu}{w_\nu} \sum_{\substack{\mu \supseteq \nu \\ \mu \vdash n}} d^{h_k[\frac{X}{1-q}]}_{\mu\nu} \Pi_\mu s_\lambda [B_\mu] \tag{95}$$

$$= t^{n-k}(1-q^k) \sum_{\nu \vdash n-k} \frac{T_\nu \Pi_\nu}{w_\nu} \mathcal{K}_{s_{\lambda'}}(\sum_{p=0}^{\min(k,m)} (f)_p e^*_{m-p})[MB_\nu], \tag{96}$$

where

$$f = \tau_\epsilon \nabla \tau_1 h_k[\frac{X}{1-q}] = \tau_\epsilon \nabla h_k[\frac{X+1}{1-q}] \tag{97}$$

$$= \tau_\epsilon \nabla \sum_{j=0}^{k} h_j[\frac{X}{1-q}] h_{k-j}[\frac{1}{1-q}] \quad \text{(by (21))} \tag{98}$$

$$= \tau_\epsilon \sum_{j=0}^{k} q^{\binom{j}{2}} h_j[\frac{X}{1-q}] h_{k-j}[\frac{1}{1-q}] \quad \text{(by (42))} \tag{99}$$

$$= \sum_{j=0}^{k} q^{\binom{j}{2}} h_j[\frac{X+\epsilon}{1-q}] h_{k-j}[\frac{1}{1-q}] \tag{100}$$

$$= \sum_{j=0}^{k} q^{\binom{j}{2}} \sum_{p=0}^{j} h_p[\frac{X}{1-q}] h_{j-p}[\frac{\epsilon}{1-q}] h_{k-j}[\frac{1}{1-q}] \tag{101}$$

$$= \sum_{p=0}^{k} h_p[\frac{X}{1-q}] \sum_{j=p}^{k} q^{\binom{j}{2}} (-1)^{j-p} h_{j-p}[\frac{1}{1-q}] h_{k-j}[\frac{1}{1-q}]. \tag{102}$$

The inner sum in (102) equals

$$\sum_{b=0}^{k-p} q^{\binom{b+p}{2}} (-1)^b h_b[\frac{1}{1-q}] h_{k-p-b}[\frac{1}{1-q}] = \sum_{b=0}^{k-p} q^{\binom{b}{2}+\binom{p}{2}+pb} (-1)^b q^{-\binom{b}{2}} e_b[\frac{1}{1-q}] h_{k-p-b}[\frac{1}{1-q}] \tag{103}$$

$$= q^{\binom{p}{2}} \sum_{b=0}^{k-p} q^{pb} (-1)^b e_b[\frac{1}{1-q}] h_{k-p-b}[\frac{1}{1-q}] \tag{104}$$

$$= q^{\binom{p}{2}} \frac{(zq^p;q)_\infty}{(z;q)_\infty}|_{z^{k-p}} \tag{105}$$

$$= q^{\binom{p}{2}} \frac{1}{(z;q)_p}|_{z^{k-p}} \tag{106}$$

$$= q^{\binom{p}{2}} \begin{bmatrix} k-1 \\ k-p \end{bmatrix}_q. \tag{107}$$

Plugging this into (102) we see

$$(f)_p = h_p[\frac{X}{1-q}] q^{\binom{p}{2}} \begin{bmatrix} k-1 \\ k-p \end{bmatrix}_q. \tag{108}$$

Using this and

$$(1-q^k) \begin{bmatrix} k-1 \\ k-p \end{bmatrix}_q = (1-q^p) \begin{bmatrix} k \\ p \end{bmatrix}_q \tag{109}$$

in (96) completes the proof. □

The following lemma will allow us to obtain a formula for $\langle \Delta_{s_\lambda} \nabla E_{n,k}, s_n \rangle$ without the $\mathcal{K}$ operator.

Lemma 1 *Given positive integers m, n, k, a partition $\lambda \vdash m$ and a symmetric function P of homogeneous degree n,*

$$\sum_{\nu \vdash n} \frac{\Pi_\nu \left\langle \tilde{H}_\nu, P \right\rangle}{w_\nu} \mathcal{K}_{s_\lambda}\left(\sum_{p=1}^{k} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} (1-q^p) e^*_{m-p} h_p[\frac{X}{1-q}]\right)[MB_\nu]$$
$$= \sum_{\mu \vdash m} \frac{(1-q^k) h_k[(1-t)B_\mu](\omega P)[B_\mu] \Pi_\mu \tilde{K}_{\lambda,\mu}}{w_\mu}. \tag{110}$$

Proof. By linearity, it suffices to prove the theorem if $P = s_\beta$. By Theorem 4 with $A = h_k[(1-t)X]$ and the proof of Corollary 1, for $\nu \vdash n$ we have

$$\mathcal{K}_{s_\lambda}\left(\sum_{p=1}^{k} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} (1-q^p) e^*_{m-p} h_p[\frac{X}{1-q}]\right)[MB_\nu]$$
$$= (1-q^k)(\Delta_{h_k[(1-t)X]} s^*_{\lambda'})[MB_\nu] \tag{111}$$
$$= (1-q^k) \sum_{\mu \vdash m} \frac{\tilde{H}_\mu[MB_\nu]}{w_\mu} \tilde{K}_{\lambda,\mu} h_k[(1-t)B_\mu] \quad \text{(using (88))}. \tag{112}$$

Thus

$$\sum_{\nu \vdash n} \frac{\Pi_\nu \tilde{K}_{\beta,\nu}}{w_\nu} \mathcal{K}_{s_\lambda}\left(\sum_{p=1}^{k} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} (1-q^p) e^*_{m-p} h_p[\frac{X}{1-q}]\right)[MB_\nu] \tag{113}$$
$$= \sum_{\mu \vdash m} \frac{(1-q^k) h_k[(1-t)B_\mu] \tilde{K}_{\lambda,\mu}}{w_\mu} \sum_{\nu \vdash n} \frac{\Pi_\nu \tilde{H}_\mu[MB_\nu]}{w_\nu} \tilde{K}_{\beta,\nu} \tag{114}$$
$$= \sum_{\mu \vdash m} \frac{(1-q^k) h_k[(1-t)B_\mu] \tilde{K}_{\lambda,\mu}}{w_\mu} \sum_{\nu \vdash n} \frac{\Pi_\mu \tilde{H}_\nu[MB_\mu]}{w_\nu} \tilde{K}_{\beta,\nu} \quad \text{(using (76))} \tag{115}$$
$$= \sum_{\mu \vdash m} \frac{(1-q^k) h_k[(1-t)B_\mu] \tilde{K}_{\lambda,\mu} \Pi_\mu}{w_\mu} s_{\beta'}[B_\mu] \quad \text{(by (88))}. \tag{116}$$

□

Since $\mathcal{K}_{e_m}$ is the identity operator, if we let $k = 1$ and $\lambda = 1^m$ then the left-hand side of (110) becomes

$$\sum_{\nu \vdash n} \frac{\Pi_\nu M B_\nu e_{m-1}[B_\nu] \left\langle \tilde{H}_\nu, P \right\rangle}{w_\nu}, \tag{117}$$

and we get the following.

Corollary 2 *For n, m positive integers and $P \in \Lambda^n$,*

$$\left\langle \Delta_{e_{m-1}} e_n, P \right\rangle = \left\langle \Delta_{\omega P} e_m, s_m \right\rangle. \tag{118}$$

Example: Let $H_n(q,t)$ be the bigraded *Hilbert Series* of $\mathcal{H}_n$, i.e.

$$H_n(q,t) = \sum_{i,j \geq 0} q^i t^j \dim \mathcal{H}_n^{i,j}. \tag{119}$$

Letting $m = n+1$ and $P = h_1^n$ in Corollary 2 we get

$$H_n(q,t) = \sum_{\mu \vdash n+1} \frac{M(B_\mu)^{n+1} \Pi_\mu}{w_\mu}. \tag{120}$$

As another consequence of Lemma 1 we get

Theorem 5 *For $n, k \in \mathbb{N}$ with $1 \leq k \leq n$*

$$\langle \nabla E_{n,k}, s_n \rangle = \delta_{n,k}. \tag{121}$$

In addition if $m > 0$ and $\lambda \vdash m$

$$\langle \Delta s_\lambda \nabla E_{n,k}, s_n \rangle = t^{n-k} \left\langle \Delta_{h_{n-k}} e_m [X \frac{1-q^k}{1-q}], s_{\lambda'} \right\rangle, \tag{122}$$

or, equivalently,

$$\langle \Delta s_\lambda \nabla E_{n,k}, s_n \rangle = t^{n-k} \sum_{\mu \vdash m} \frac{(1-q^k) h_k[(1-t)B_\mu] h_{n-k}[B_\mu] \Pi_\mu \tilde{K}_{\lambda',\mu}}{w_\mu}. \tag{123}$$

Proof. Eq. (121) follows for $k < n$ from the $\lambda = \emptyset$ case of Corollary 1 and for $k = n$ by the $\beta = \emptyset$ case of Proposition 1. Next we note that for $m > 0$, (122) and (123) are equivalent due to the following result [GH02, p.692]

$$e_m [X \frac{1-q^k}{1-q}] = \sum_{\mu \vdash m} \frac{(1-q^k) h_k[(1-t)B_\mu] \Pi_\mu \tilde{H}_\mu}{w_\mu}. \tag{124}$$

If $k = n$, (122) follows from Proposition 1, (22) and (45). For $k < n$, to obtain (123) apply Lemma 1 with $n = n-k$, $P = s_{1^{n-k}}$ and λ replaced by λ', then use Corollary 1. □

3 The Hook Case

In this section we use the results in Section 2 and combinatorial reasoning to show both sides of (12) can be subdivided into components which satisfy the same recurrence and inital conditions.

Theorem 6 *Let $n, k, d \in \mathbb{N}$ with $1 \leq k \leq n$. Set*

$$F_{n,d,k} = \langle \nabla E_{n,k}, e_{n-d} h_d \rangle . \tag{125}$$

Then

$$F_{n,n,k} = \delta_{n,k}, \tag{126}$$

and if $d < n$,

$$F_{n,d,k} = t^{n-k} \sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} \sum_{b=0}^{n-k} \begin{bmatrix} p+b-1 \\ b \end{bmatrix}_q F_{n-k,p-k+d,b} \tag{127}$$

with the initial conditions

$$F_{0,0,k} = \delta_{k,0} \quad \textit{and} \quad F_{n,d,0} = \delta_{n,0}\delta_{d,0}. \tag{128}$$

Proof. Eq. (126) is equivalent to (121). Now assume $d < n$. If $k = n$,

$$\begin{align} F_{n,d,k} &= \langle \nabla E_{n,n}, e_{n-d} h_d \rangle \tag{129}\\ &= \left\langle \Delta_{e_{n-d}} \nabla E_{n,n}, s_n \right\rangle \quad \text{(by (20))} \tag{130}\\ &= e_{n-d}[1, q, \ldots, q^{n-1}] \quad \text{(by Proposition (1))} \tag{131}\\ &= \begin{bmatrix} n \\ n-d \end{bmatrix}_q q^{\binom{n-d}{2}} \quad \text{(by (25))}. \tag{132} \end{align}$$

Using the initial conditions this agrees with the right-hand side of (127).

If $k < n$,

$$\begin{align} F_{n,d,k} = \langle \nabla E_{n,k}, e_{n-d} h_d \rangle &= \left\langle \Delta_{e_{n-d}} \nabla E_{n,k}, s_n \right\rangle \quad \text{(by (20))} \tag{133}\\ = t^{n-k} \sum_{\nu \vdash n-k} \frac{T_\nu \Pi_\nu}{w_\nu} &\sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} (1-q^p) e_{n-d-p}[B_\nu] h_p[(1-t)B_\nu] \tag{134} \end{align}$$

by Corollary 1 with $\lambda = (n-d)$. (In the inner sum in (134), $e_{n-d-p}[B_\nu] = 0$ if $n-d-p > n-k$ since B_ν has $n-k$ terms). Reversing summation (134) equals

$$\begin{align} & t^{n-k} \sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} \sum_{\nu \vdash n-k} \frac{(1-q^p) h_p[(1-t)B_\nu] T_\nu \Pi_\nu e_{n-d-p}[B_\nu]}{w_\nu} \tag{135}\\ &= t^{n-k} \sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} \left\langle \nabla e_{n-k}[X \frac{1-q^p}{1-q}], e_{n-d-p} h_{d+p-k} \right\rangle \quad \text{(by (124) and (20))} \tag{136}\\ &= t^{n-k} \sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} \sum_{b=1}^{n-k} \begin{bmatrix} p+b-1 \\ b \end{bmatrix}_q \langle \nabla E_{n-k,b}, e_{n-d-p} h_{d+p-k} \rangle \tag{137} \end{align}$$

by letting $z = q^p$ in (29). The inner sum in (137) equals 0 if $b = 0$, so by the initial conditions we can write (137) as

$$t^{n-k} \sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} \sum_{b=0}^{n-k} \begin{bmatrix} p+b-1 \\ b \end{bmatrix}_q F_{n-k,p-k+d,b}. \tag{138}$$

$\square$

Recall that $\mathcal{S}_{n,d}$ is the set of Schröder paths from $(0,0)$ to (n,n) with d D steps. Let $\mathcal{S}_{n,d,k}$ be the subset of these which have k total $D+E$ steps after the highest N step, and define

$$S_{n,d,k}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d,k}} q^{\text{area}(\Pi)} t^{\text{bounce}(\Pi)}. \tag{139}$$

If Π has no peaks, then $d = n$ and we set $S_{n,n,k}(q,t) = \delta_{n,k}$. We let pword($\Pi$) denote the word of 0's, 1's and 2's obtained by starting at $(0,0)$ and travelling along Π, adding a 0, 1 or 2 to pword(Π) depending on whether the corresponding step of Π is an E, D or N step, respectively. For example, if Π is the path on the left in Figure 1 then pword(Π) = 2120211020200.

We now derive a recurrence for $S_{n,d,k}(q,t)$. This is similar, but not identical to, recurrences for stratified versions of $S_{n,d}(q,t)$ obtained in [EHKK03]. In fact, it was the discovery of (127) and a search for a corresponding way to stratify $S_{n,d}(q,t)$ which led to the following.

Theorem 7 *Let n, k, d be positive integers satisfying $1 \le k \le n$ and $0 \le d < n$. Then*

$$S_{n,d,k}(q,t) = t^{n-k} \sum_{p=\max(1,k-d)}^{\min(k,n-d)} \begin{bmatrix} k \\ p \end{bmatrix}_q q^{\binom{p}{2}} \sum_{b=0}^{n-k} \begin{bmatrix} p+b-1 \\ b \end{bmatrix}_q S_{n-k,d+p-k,b}(q,t), \tag{140}$$

with the initial conditions

$$S_{0,0,k} = \delta_{k,0} \quad \textit{and} \quad S_{n,d,0} = \delta_{n,0}\delta_{d,0}. \tag{141}$$

Proof. Let the highest peak of Π be denoted peak 1 of Π, the next highest peak 2 of Π, etc. Lets say Π has p E steps and $k-p$ D steps above peak 1. We first asume $p < n-d$, which means Π has at least two peaks. We now describe an operation we call truncation, which takes a Schröder path $\Pi \in \mathcal{S}_{n,d,k}$ and maps it to a Schröder path Π' with one less peak. Given such a Π, to create Π' start with pword(Π) and remove the last k letters. Also remove all the 2's which correspond to N steps above peak 2 of Π. The result is pword(Π'). For the path on the left in Figure 1, we start with pword(Π) = 2120211020200 and $k = 2$. We then remove the last two letters to form 21202110202, then finally remove the last two 2's (which correspond to N steps between peaks 1 and 2) to get pword(Π') = 212021100.

Lets say that Π' has b total $D+E$ steps after its highest peak (peak 2 of Π). How does bounce(Π) compare to bounce(Π')? First of all, by construction the bounce path for $C(\Pi')$ will be identical to the bounce path for $C(\Pi)$ except the first (top) bounce of $C(\Pi)$ is truncated. This bounce step hits the diagonal at $(n-d-p, n-d-p)$, and so the contribution to bounce(Π') from the bounce path will be $n-d-p$ less than to bounce(Π). Furthermore, for each D step of Π below peak 1 of Π, the number of peaks of Π' above it will be one less than the number of peaks of Π above it. Thus the contribution to bounce(Π') from the D steps will be $d-(k-p)$ less than that of bounce(Π). In summary we have

$$\mathrm{bounce}(\Pi) = \mathrm{bounce}(\Pi') + n - d - p + d - (k-p) = \mathrm{bounce}(\Pi') + n - k. \tag{142}$$

Note the factor of t^{n-k} in Theorem 7.

Next we will consider how area(Π) relates to area(Π'). We will use Figure 3 as a visual aid in our argument. Since Π has p E steps above peak 1, there is a $p \times p$ triangle (region 0) of area $\binom{p}{2}$. It is not hard to show (see the proof of Lemma 1 in [EHKK03]) that the area of region 1 equals the number of inversions in the last k letters of pword(Π), where an inversion in a word is a pair of letters (a, b) with $a > b$ and a occurring before b in the word. When we sum over all Π which get mapped to a given Π' under truncation, the contribution of region 1 to area will thus generate $\begin{bmatrix} k \\ p \end{bmatrix}_q$, which is MacMahon's formula for q to the number of inversions, summed over all words of p 0's and $k-p$ 1's.

Again by [EHKK03], the area of region 2 equals the number of inversions of the section of pword(Π) corresponding to the steps of Π between peaks 1 and 2. After truncation, the inversions of this subword involving 0 and 1 letters becomes part of area(Π'), so we wish to count only the inversions between 2's and 1's or between 2's and 0's. Since Π' is fixed, to count these we might as well assume we are summing q to the number of inversion over all words of $p-1$ 2's and b 0's (Although there are p N steps of Π above peak 2, the last one, peak 1, is fixed and contributes no inversions). Again by MacMahon's formula, this sum is $\begin{bmatrix} p-1+b \\ b \end{bmatrix}_q$. Eq. (140) now follows (since $p < n-d$, we must have $b > 0$, which is compatible with (140) since if $b = 0$ the initial conditions force $S_{n-k,d+p-k,b}(q,t) = 0$).

We now consider the case where $p = n-d$, so Π has only one peak. Clearly bounce$(\Pi) = n-k$, since there are $d - (k - (n-d)) = n-k$ D steps below peak 1. By the above analysis, the case $p = n-d$ contributes

$$t^{n-k} q^{\binom{n-d}{2}} \begin{bmatrix} k \\ n-d \end{bmatrix}_q \begin{bmatrix} n-k+n-d-1 \\ n-k \end{bmatrix}_q \tag{143}$$

to $S_{n,d,k}$. This also agrees with (7), since by definition and the initial conditions we have $S_{n-k,n-k,b} = \delta_{b,n-k}$ for $n-k \geq 0$. □

Since $S_{n,d,k}(q,t) = F_{n,d,k}$ when $d = n$, and for $d < n$ they satisfy the same recurrence relation and initial conditions, we have the following.

Theorem 8 *Let n, k be positive integers satisfying $1 \leq k \leq n$, and let $d \in \mathbb{N}$. Then*

$$\langle \nabla E_{n,k}, e_{n-d} h_d \rangle = S_{n,d,k}(q,t). \tag{144}$$

By summing Theorem 8 from $k = 1$ to n we obtain (12).

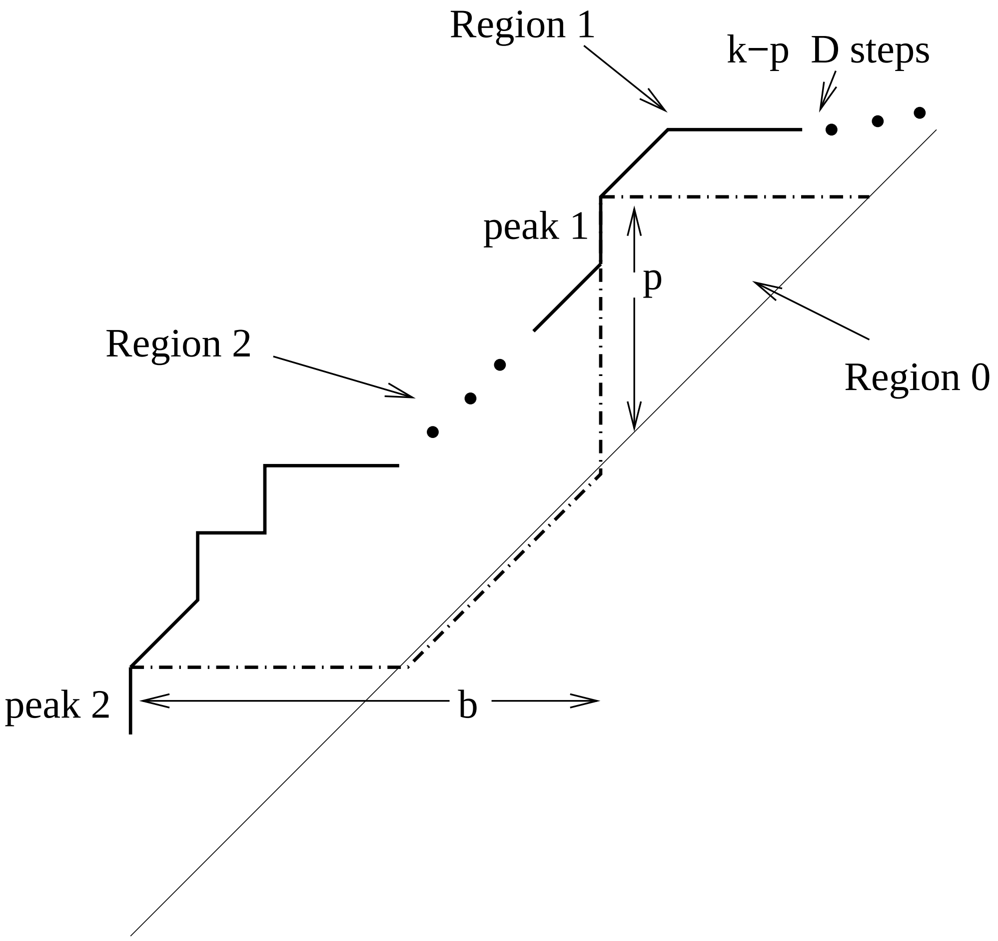

Figure 3: A Schröder path with p E and $k-p$ D steps above peak 1, and b $E+D$ steps between peaks 1 and 2

Corollary 3

$$\langle \nabla e_n, e_{n-d} h_d \rangle = S_{n,d}(q,t). \tag{145}$$

In [EHKK03] it is shown that (12) is equivalent to the following result, which now follows by Corollary 3.

Corollary 4 *For* $0 \le d \le n-1$,

$$\left\langle \nabla e_n, s_{d+1,1^{n-d-1}} \right\rangle = \sum_{\substack{\Pi \\ no\ D\ before\ first\ E\ in\ pword(\Pi)}} q^{area(\Pi)} t^{bounce(\Pi)}, \tag{146}$$

where the sum is over all Π *which have no* D *step below the lowest* E *step.*

Combining Theorem 8 and (122) we obtain an alternate formula for $S_{n,d,k}(q,t)$.

Corollary 5 *Let n, k be positive integers satisfying $1 \le k \le n$, and let $d \in \mathbb{N}$. Then*

$$S_{n,d,k}(q,t) = t^{n-k} \left\langle \Delta h_{n-k} e_{n-d}[X \frac{1-q^k}{1-q}], s_{n-d} \right\rangle. \tag{147}$$

It is easy to see combinatorially that $S_{n+1,d,1}(q,t) = t^n S_{n,d}(q,t)$. Thus Corollary 5 also implies

Corollary 6

$$S_{n,d}(q,t) = \langle \Delta h_n e_{n+1-d}, s_{n+1-d} \rangle. \tag{148}$$

We should mention that Corollaries 5 and 6 are new even in the $d = 0$ case, where they give new formulas for the q,t-Catalan.

Corollary 6 allows us to obtain the following result, which was an unpublished conjecture of the author (based on hueristics and Maple calculations) dating back to 2001.

Theorem 9 *For $n \in \mathbb{N}$,*

$$\lim_{d \to \infty} S_{n+d,d}(q,t) = \prod_{i,j \ge 0} (1 + q^i t^j z)|_{z^n}. \tag{149}$$

Proof. By Corollary 6 we have

$$S_{n+d,d}(q,t) = \sum_{\mu \vdash n+1} \frac{MB_\mu h_{n+d}[B_\mu] \Pi_\mu}{w_\mu}. \tag{150}$$

Now

$$\lim_{d \to \infty} h_{n+d}[B_\mu] = \lim_{d \to \infty} \prod_{(i,j) \in \mu} \frac{1}{1 - q^{a'} t^{l'} z}|_{z^{n+d}} \tag{151}$$

$$= \lim_{d \to \infty} \frac{1}{1-z} \prod_{\substack{(i,j) \in \mu \\ (i,j) \ne (0,0)}} \frac{1}{1 - q^{a'} t^{l'} z}|_{z^{n+d}} \tag{152}$$

$$= \prod_{(i,j) \in \mu, (i,j) \ne (0,0)} \frac{1}{1 - q^{a'} t^{l'}} = \frac{1}{\Pi_\mu}. \tag{153}$$

Thus

$$\lim_{d \to \infty} S_{n+d,d}(q,t) = \sum_{\mu \vdash n+1} \frac{MB_\mu}{w_\mu} \tag{154}$$

$$= M \left\langle e_{n+1}[\frac{X}{M}], e_1 h_n \right\rangle \quad \text{(by (88) and (20))} \tag{155}$$

$$= M \sum_{\lambda \vdash n+1} \left\langle s_\lambda[X] s_{\lambda'}[\frac{1}{M}], e_1 h_n \right\rangle \quad \text{(by (22))} \tag{156}$$

$$= M h_1[\frac{1}{M}] e_n[\frac{1}{M}] \tag{157}$$

$$= e_n[\frac{1}{M}]. \tag{158}$$

By (4)

$$\prod_{i,j\geq 0}(1+q^it^jz)|_{z^n} = e_n[\frac{1}{M}], \tag{159}$$

and the result follows. $\square$

Corollary 7 *For* $|q|, |t| < 1$,

$$\prod_{i,j\geq 0}(1+q^it^jz)|_{z^n} = \sum_{d=0}^{\infty} \sum_{\substack{\Pi\in\mathcal{S}_{n+d,d} \\ \Pi \text{ ends in an } E \text{ step}}} q^{area(\Pi)}t^{bounce(\Pi)} \tag{160}$$

$$= \sum_{\substack{\Pi \text{ has } n\ N,\ E \text{ steps, arbitrary } \#\text{ of } D \text{ steps} \\ \Pi \text{ ends in an } E \text{ step}}} q^{area(\Pi)}t^{bounce(\Pi)}. \tag{161}$$

Proof. Note that if we add a D step to the end of a path Π, area(Π) and bounce(Π) remain the same. We fix the number of N and E steps to be n, and require d D steps, and break up our paths according to how many D steps they end in. After truncating these D steps at the end, we get

$$S_{n+d,d}(q,t) = \sum_{p=0}^{d} \sum_{\substack{\Pi\in\mathcal{S}_{n+p,p} \\ \Pi \text{ ends in an } E \text{ step}}} q^{\mathrm{area}(\Pi)}t^{\mathrm{bounce}(\Pi)}. \tag{162}$$

Assuming $|q|, |t| < 1$ (159) implies (162) converges absolutely as $d \to \infty$, so it can be rearranged to the form (161). $\square$

4 Shuffles and $\langle \nabla e_n, h_{n-d}h_d\rangle$

We now show how the results of Section 2 yield a combinatorial formula for the polynomials $\langle \nabla e_n, h_{n-d}h_d\rangle$.
Defnition For $1 \leq k \leq n$ and $j \in \mathbb{N}$, let

$$H_{n,k,j} = \left\langle \Delta_{h_j e_n} E_{n,k}, s_n \right\rangle. \tag{163}$$

Theorem 10

$$H_{n,k,0} = \delta_{n,k}, \tag{164}$$

and if $j > 0$,

$$H_{n,k,j} = t^{n-k} \sum_{b=1}^{j} \begin{bmatrix} k+b-1 \\ b \end{bmatrix}_q H_{j,b,n-k}. \tag{165}$$

Proof. The fact that $H_{n,k,0} = \delta_{n,k}$ is equivalent to (121). If $j > 0$, by (122) we have

$$H_{n,k,j} = t^{n-k} \left\langle \Delta_{h_{n-k}} e_j[X\frac{1-q^k}{1-q}], e_j \right\rangle \tag{166}$$

$$= t^{n-k} \sum_{b=1}^{j} \begin{bmatrix} k+b-1 \\ b \end{bmatrix}_q \left\langle \Delta_{h_{n-k}e_j} E_{j,b}, s_j \right\rangle \quad \text{(by (29))} \tag{167}$$

$$= t^{n-k} \sum_{b=1}^{j} \begin{bmatrix} k+b-1 \\ b \end{bmatrix}_q H_{j,b,n-k}. \tag{168}$$

□

Corollary 8 *For $n, d \in N$, $\langle \nabla e_n, h_{n-d}h_d \rangle \in \mathbb{N}[q,t]$.*

Proof. By Corollary 2 with $P = h_{n-d}h_d$,

$$\langle \Delta_{e_n} e_n, h_{n-d}h_d \rangle = \left\langle \Delta_{e_{n-d}e_d} e_{n+1}, h_{n+1} \right\rangle \tag{169}$$

$$= \left\langle \Delta_{e_{n-d}} e_{n+1}, h_{n-d+1}e_d \right\rangle \quad \text{(using (20))} \tag{170}$$

$$= \left\langle \Delta_{h_d e_{n-d+1}} e_{n-d+1}, h_{n-d+1} \right\rangle, \tag{171}$$

by Corollary 2 again, this time with $P = e_d h_{n-d+1}$ and $m = n-d+1$. Since

$$\left\langle \Delta_{h_d e_{n-d+1}} e_{n-d+1}, s_{n-d+1} \right\rangle = \sum_{k=1}^{n-d+1} \left\langle \Delta_{h_d e_{n-d+1}} E_{n-d+1,k}, s_{n-d+1} \right\rangle, \tag{172}$$

and since Theorem 10 implies

$$\left\langle \Delta_{h_d e_{n-d+1}} E_{n-d+1,k}, s_{n-d+1} \right\rangle \in \mathbb{N}[q,t], \tag{173}$$

the result now follows. □

Both Corollary 3 and Corollary 8 prove special cases of a recent conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov known as the *shuffle conjecture*, which is described in detail in [HHL+]. We will provide a brief description of this here. It builds on a conjecture of Haglund and Loehr [HL02] which gives a combinatorial formula for the Hilbert Series $\mathcal{H}_n(q,t)$ in terms of statistics on parking functions. We will view a parking function $P = P(B,C,n)$ as a triple (B,C,n), where B is a Catalan path from $(0,0)$ to (n,n) and C is a placement of the integers 1 through n (or "cars" 1 through n, respectively) in the squares immediately to the right of the N steps of B. We denote the ith row (from the top) of B "$\text{row}_i = \text{row}_i(P)$", and we call the number of lower triangles in row i the length of row_i, denoted by $\text{area}_i = \text{area}_i(P)$, and set $\text{area}(P) = \sum_i \text{area}_i(P)$. If car_i is in row_j, we say $\text{occupant}_j = i$. We require that cars are decreasing down columns, so if $\text{area}_i = \text{area}_{i+1} + 1$, then we must have $\text{occupant}_i > \text{occupant}_{i+1}$.

We define the statistic $\mathrm{dinv}(P)$, or the number of "inversions" of P, to be the number of pairs (i,j), $1 \le i < j \le n$ such that

$$\mathrm{area}_i = \mathrm{area}_j \quad \text{and} \quad \mathrm{occupant}_i > \mathrm{occupant}_j \tag{174}$$

or

$$\mathrm{area}_i = \mathrm{area}_j - 1 \quad \text{and} \quad \mathrm{occupant}_i < \mathrm{occupant}_j \tag{175}$$

An example is provided in Figure 4.

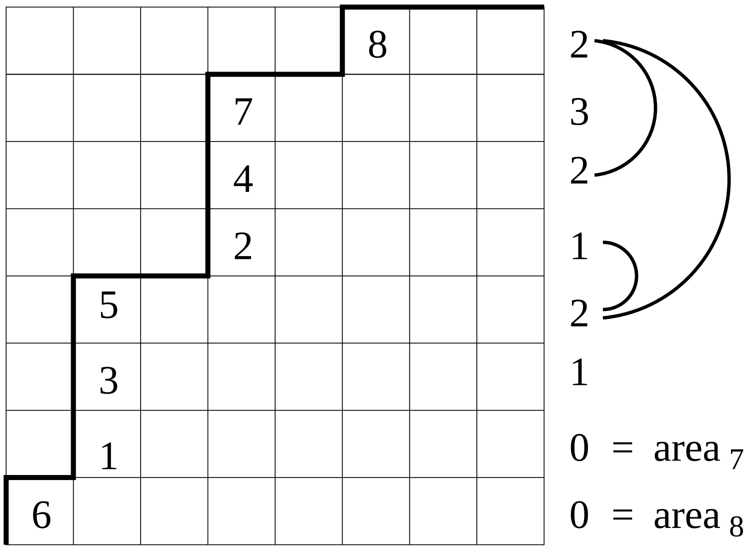

Figure 4: A parking function with dinv = 3 inversions, area = 11, and reading word 78452316

Define the *reading word* of a parking function P to be the permutation on $\{1,\ldots,n\}$ obtained by listing the cars along diagonals, starting with the diagonal farthest from the main diagonal and moving inward. Within a given diagonal we move from top to bottom. For example, the parking function in Figure 4 has reading word 78452316.

Let $\tau = \tau_1\tau_2\cdots\tau_n$ be a permutation. Given sets $A_1, A_2, \ldots A_k$ of pairwise distinct integers whose union is $\{1,2,\ldots,n\}$, we say τ is a shuffle of $A_1, A_2, \ldots, A_k$ if for each triple (i,j,l), with $i,j \in A_l$ and $i < j$, then i occurs before j in τ. For example, 3124 is a shuffle of $\{1,2\},\{3,4\}$. In its simplest form, the shuffle conjecture says that given a partition $\lambda \vdash n$ with parts $\lambda_1,\ldots,\lambda_l$,

$$\langle \nabla e_n, h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}\rangle = \sum_P q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}, \tag{176}$$

where the sum is over all parking functions P whose reading word is a shuffle of $A_1, A_2, \ldots, A_l$, with

$$A_{l-i+1} = \{1 + \sum_{j=1}^{i-1}\lambda_j, 2 + \sum_{j=1}^{i-1}\lambda_j, \ldots, \sum_{j=1}^{i}\lambda_j\} \tag{177}$$

For example, the conjecture predicts that $\langle \nabla e_4, h_3h_2\rangle$ equals the sum of $q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}$ over all parking functions whose reading word is a shuffle of $A_1 = \{4, 5\}$ and $A_2 = \{1, 2, 3\}$. If $\lambda = 1^n$, then the sum is over all parking functions on n cars, and (176) reduces to the conjectured formula for $\mathcal{H}_n(q, t)$ in [HL02].

We now show how to derive the case $l = 2$ of the shuffle conjecture from Theorem 10. Given integers b_1, B_1, B_2 with $1 \le b_1 \le B_1$, $0 < B_2$, let $A_2 = \{1, 2, \ldots, B_2\}$ and $A_1 = \{B_2 + 1, B_2 + 2, \ldots, B_2 + B_1\}$. Furthermore set $n = B_1 + B_2$ and define

$$M(B_1, B_2, b_1) = \sum_P q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}, \tag{178}$$

where the sum is over all parking functions whose reading words are shuffles of A_1, A_2, which have b_1 of the cars from A_1 in rows of length 0, and in addition have car_n in row_n (i.e. n is in the bottom row). The following proposition is implicit in [HHL+, Eq. 62]. We give a self-contained proof, using ideas from the proof of [HL02, Thm. 1], since we will soon use similar reasoning to derive other results.

Proposition 2

$$M(B_1, 0, b_1) = \delta_{B_1,b_1}, \tag{179}$$

and if $B_2 > 0$,

$$M(B_1, B_2, b_1) = \sum_{j\ge 2} \sum_{\substack{b_2,b_3,\ldots,b_j>0\\ b_2+b_4+\ldots=B_2\\ b_3+b_5+\ldots=B_1-b_1}} \begin{bmatrix} b_1+b_2-1\\ b_2\end{bmatrix}_q \begin{bmatrix} b_2+b_3-1\\ b_3\end{bmatrix}_q \cdots \begin{bmatrix} b_{j-1}+b_j-1\\ b_j\end{bmatrix}_q t^F, \tag{180}$$

where $F = b_3 + b_4 + 2(b_5 + b_6) + 3(b_7 + b_8) + \ldots$.

Proof. It is clear combinatorially that $M(B_1, 0, b_1) = \delta_{B_1,b_1}$, since if $B_2 = 0$, then our reading word must be $1, 2, \ldots, n$, which forces all rows to have zero length, and all our cars must be on the main diagonal. If $B_2 > 0$, the summand in (180) corresponds to those parking functions whose reading word is a shuffle of A_1, A_2, and which have b_1 cars from A_1 in rows of length 0, b_3 cars from A_1 in rows of length 1, b_5 in rows of length 2, etc., and have $b_2, b_4, b_6, \ldots$ cars from A_2 in rows of length $0, 1, 2, \ldots$, respectively. F, the power of t, is the area for these parking functions. To see how q^{dinv} generates the q-binomial coefficients, begin by placing the b_1 cars in rows of length 0. Note that the shuffle condition forces these to be the cars $n - b_1 + 1, \ldots, n$. Then insert the b_2 cars, also in rows of length 0, in any possible way, but recalling that car_n must occupy the bottom row. From the definition of dinv, this generates the $\left[\begin{smallmatrix} b_1+b_2-1\\ b_2\end{smallmatrix}\right]_q$ term. Next we insert the b_3 cars in rows of length 1. Each car in this set is smaller than any of the cars in the b_1 set, so cannot be placed above these cars without violating the parking function condition. They are all larger, however, than the cars from the b_2 set, so they can be placed above any of these cars. After inserting these cars, all of the b_3 cars must be in rows above the largest car in the b_2 set; otherwise the b_2 and b_3 cars can be interleaved in all possible ways, thus generating the $\begin{bmatrix} b_2+b_b-1 \\ b_3 \end{bmatrix}_q$ term. We continue in this way until some $b_k = 0$, which forces b_{k+1} to equal 0 (else $\begin{bmatrix} b_k+b_{k+1}-1 \\ b_{k+1} \end{bmatrix}_q = 0$), and hence $b_i = 0$ for $i > k$, and obtain (180). □

Now if $B_2 > 0$ and $b_1 < B_1$, by (180)

$$M(B_1, B_2, b_1) = \sum_{j\ge 2} \sum_{\substack{b_2,b_3,\ldots,b_j>0 \\ b_2+b_4+\ldots=B_2 \\ b_3+b_5+\ldots=B_1-b_1}} \begin{bmatrix} b_1+b_2-1 \\ b_2 \end{bmatrix}_q \begin{bmatrix} b_2+b_3-1 \\ b_3 \end{bmatrix}_q \cdots \begin{bmatrix} b_{j-1}+b_j-1 \\ b_j \end{bmatrix}_q t^F \tag{181}$$

$$= \sum_{b_2=1}^{B_2} \begin{bmatrix} b_1+b_2-1 \\ b_2 \end{bmatrix}_q t^{b_3+b_5+b_7+\ldots} \sum_{j\ge 3} \sum_{\substack{b_3,b_4\ldots,b_j>0 \\ b_3+b_5+\ldots=B_1-b_1 \\ b_4+\ldots=B_2-b_2}} \begin{bmatrix} b_2+b_3-1 \\ b_3 \end{bmatrix}_q \cdots \begin{bmatrix} b_{j-1}+b_j-1 \\ b_j \end{bmatrix}_q t^{F'}, \tag{182}$$

where $F' = b_4 + b_5 + 2(b_6 + b_7) + \ldots$. It follows that if $B_2 > 0$ and $b_1 < B_1$,

$$M(B_1, B_2, b_1) = t^{B_1-b_1} \sum_{b_2}^{B_2} \begin{bmatrix} b_1+b_2-1 \\ b_2 \end{bmatrix}_q M(B_2, B_1-b_1, b_2). \tag{183}$$

By inspection this recurrence also holds with $b_1 = B_1$. Hence by Theorem 10, $M(B_1, B_2, b_1)$ satisfies the same recurrence and initial conditions as H_{B_1,b_1,B_2}, and so we have

Theorem 11

$$M(n, j, k) = \left\langle \Delta_{h_j e_n} E_{n,k}, s_n \right\rangle. \tag{184}$$

Corollary 9 *Eq. (176) is true if $l = 2$.*

Proof. If car$_n$ is in the bottom row, it doesn't contribute any inversions or area. Thus

$$\sum_{b_1=1}^{B_1} M(B_1, B_2, b_1) \tag{185}$$

equals the right-hand side of (176), with $n = B_1 + B_2 - 1$ and $\lambda = B_2, B_1 - 1$. On the other hand, from Theorem 11 and the proof of Corollary 8,

$$\sum_{b_1=1}^{B_1} M(B_1, B_2, b_1) = \sum_{b_1=1}^{B_1} \left\langle \Delta_{h_{B_2} e_{B_1}} E_{B_1,b_1}, s_{B_1} \right\rangle \tag{186}$$

$$\left\langle \Delta_{h_{B_2} e_{B_1}} e_{B_1}, s_{B_1} \right\rangle \tag{187}$$

$$= \left\langle \nabla e_{B_1+B_2-1}, h_{B_2} h_{B_1-1} \right\rangle. \tag{188}$$

□

There is a more general form of the shuffle conjecture, which involves replacing any subset of the h_{λ_i}'s in the the left-hand side of (176) with e_{λ_i}'s (and also involves the "parameter m" extension [GH96], [Hai98] of Diagonal Harmonics). A special case of this more general form says that

$$\langle \nabla e_n, e_{n-d} h_d \rangle = \sum_P q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}, \tag{189}$$

where the sum is over all parking functions whose reading word is a shuffle of $\{1, 2, \ldots, d\}$ and $\{n, n-1, \ldots, n-d+1\}$. For $1 \le k \le n$, let $\mathcal{T}_{n,d,k}$ denote the subset of this sum involving Catalan paths with k rows of length 0. The reader may find it an interesting exercise to show that if we define

$$T_{n,d,k}(q,t) = \sum_{P \in \mathcal{T}_{n,d,k}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}, \tag{190}$$

then $T_{n,d,k}(q,t) = S_{n,d,k}(q,t)$ since it satisfies the same recurrence and initial conditions.

In [EHKK03] it is shown that

$$S_{n,d}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{\mathrm{dinv}(\Pi)} t^{\mathrm{area}(\Pi)}, \tag{191}$$

where $\mathrm{dinv}(\Pi)$ is a certain statistic on Schröder paths which involves inversions. The reader may also want to verify that $\mathrm{dinv}(\Pi) = \mathrm{dinv}(P(\Pi))$, where $P(\Pi)$ is the parking function obtained by replacing the d D steps of Π by NE pairs, to form a Catalan path, then placing the cars $n-d+1$ through n where these D steps used to be, and the cars $n-d$ through 1 in the remaining spaces, in such a way that the reading word is a shuffle of $\{n-d+1, \ldots, n\}$ and $\{n-d, \ldots, 1\}$. Π uniquely defines $P(\Pi)$, and clearly $\mathrm{area}(\Pi) = \mathrm{area}(P(\Pi))$. Thus Corollary 3 is equivalent to (189).

There is a more general form of (183), which relates to the right-hand side of (176) for arbitrary l. Let's first consider the case $l = 3$. Let

$$M(B_1, B_2, B_3, b_1, b_2) = \sum_P q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}, \tag{192}$$

where this time the sum is over all P whose reading word is a shuffle of sets of size B_3, B_2, B_1, and which have b_1 cars from the B_1 block and b_2 cars from the B_2 block in rows of length 0. As before, we require that car_n is in the bottom row. By the argument in the $l = 2$ case, if in addition to the b_1, b_2 cars we want b_3 cars from set B_3 in rows of length 0, we will get a factor of

$$\begin{bmatrix} b_1 + b_2 + b_3 - 1 \\ b_2, b_3 \end{bmatrix}_q \tag{193}$$

where

$$\begin{bmatrix} A+B+C \\ B,C \end{bmatrix}_q = \frac{[A+B+C]!}{[A]![B]![C]!} \tag{194}$$

is the q-trinomial coefficient. If we want b_4, b_5, b_6 cars from sets B_1, B_2, B_3 in the rows of length 1, then we will also generate the factors

$$\begin{bmatrix} b_2+b_3+b_4-1 \\ b_4 \end{bmatrix}_q \begin{bmatrix} b_3+b_4+b_5-1 \\ b_5 \end{bmatrix}_q \begin{bmatrix} b_4+b_5+b_6-1 \\ b_6 \end{bmatrix}_q. \tag{195}$$

The main difference between the $l=2$ and $l=3$ cases is that if $l=3$, we could have say $b_2 = 0$ but $b_3 > 0$, and the term $\left[\begin{smallmatrix} b_1+b_2+b_3-1 \\ b_2,b_3 \end{smallmatrix}\right]_q$ would not be 0. However, if say $b_i = b_{i+1} = 0$, then we must have $b_k = 0$ for $k > i+1$. Thus we have

$$M(B_1,B_2,B_3,b_1,b_2) = \sum_{j\geq 3} \sum_{b_3,b_4,\ldots,b_{j-1}\geq 0, b_j>0} t^F \tag{196}$$

$$\times \begin{bmatrix} b_1+b_2+b_3-1 \\ b_2,b_3 \end{bmatrix}_q \begin{bmatrix} b_2+b_3+b_4-1 \\ b_4 \end{bmatrix}_q \begin{bmatrix} b_3+b_4+b_5-1 \\ b_5 \end{bmatrix}_q \cdots, \tag{197}$$

where $F = b_4+b_5+b_6+2(b_7+b_8+b_9)+\ldots$ and in the sum over $b_3,\ldots,b_j$ we require $b_i \geq 0$, $b_i + b_{i+1} > 0$, $b_1+b_4+\ldots = B_1$, $b_2+b_5+\ldots = B_2$ and $b_3+b_6+\ldots = B_3$.

Now if we try and write $M(B_1,B_2,B_3,b_1,b_2)$ recursively by first summing over b_3 then expressing the remaining sum over $b_4, b_5, \ldots, b_j$ in terms of $M(B_2,B_3,B_1-b_1,b_2,b_3)$ as in the $l=2$ case, we run into the problem that b_2 may equal 0, while in $M(B_1,B_2,B_3,b_1,b_2)$ we require $b_1 > 0$. We can get around this problem by breaking the recurrence into two cases, one where $b_2 = 0$ and one where $b_2 > 0$. In the first case, since $b_2 = 0$ we must have $b_3 > 0$ and we get

$$M(B_1,B_2,B_3,b_1,0) = \sum_{b_3\geq 1, b_4\geq 0} \begin{bmatrix} b_1+b_3-1 \\ b_3 \end{bmatrix}_q \begin{bmatrix} b_3+b_4-1 \\ b_4 \end{bmatrix}_q t^{b_4+b_5+b_7+b_8+\ldots} \tag{198}$$

$$\times \sum_{j\geq 5} \sum_{b_5,b_6,\ldots,b_{j-1}\geq 0, b_j>0} \begin{bmatrix} b_3+b_4+b_5-1 \\ b_5 \end{bmatrix}_q \begin{bmatrix} b_4+b_5+b_6-1 \\ b_6 \end{bmatrix}_q \cdots t^{b_6+b_7+b_8+2(b_9+b_{10}+b_{11})+\ldots} \tag{199}$$

$$= \sum_{b_3\geq 1, b_4\geq 0} \begin{bmatrix} b_1+b_3-1 \\ b_3 \end{bmatrix}_q t^{B_1-b_1+B_2} \sum_{j\geq 5} \tag{200}$$

$$\times \sum_{b_5,b_6,\ldots,b_{j-1}\geq 0, b_j>0} \begin{bmatrix} b_3+b_4+b_5-1 \\ b_4,b_5 \end{bmatrix}_q \begin{bmatrix} b_4+b_5+b_6-1 \\ b_6 \end{bmatrix}_q \cdots t^{b_6+b_7+b_8+2(b_9+b_{10}+b_{11})+\ldots} \tag{201}$$

$$= \sum_{b_3\geq 1, b_4\geq 0} \begin{bmatrix} b_1+b_3-1 \\ b_3 \end{bmatrix}_q t^{B_1-b_1+B_2} M(B_3,B_1-b_1,B_2,b_3,b_4). \tag{202}$$

In the case where $b_2 > 0$, we get

$$M(B_1, B_2, B_3, b_1, b_2) = \sum_{b_3 \geq 0} \begin{bmatrix} b_1 + b_2 + b_3 - 1 \\ b_2, b_3 \end{bmatrix}_q t^{b_4+b_7+\dots} \tag{203}$$

$$\times \sum_{j \geq 4} \sum_{b_4, b_5, \dots, b_{j-1} \geq 0, b_j > 0} \begin{bmatrix} b_2 + b_3 + b_4 - 1 \\ b_4 \end{bmatrix}_q \begin{bmatrix} b_3 + b_4 + b_5 - 1 \\ b_5 \end{bmatrix}_q \cdots t^{b_5+b_6+b_7+2(b_8+b_9+b_{10})+\dots} \tag{204}$$

$$= \sum_{b_3 \geq 0} \begin{bmatrix} b_1 + b_2 + b_3 - 1 \\ b_2, b_3 \end{bmatrix}_q t^{B_1 - b_1} \frac{[b_2 - 1]![b_3]!}{[b_2 + b_3 - 1]!} \tag{205}$$

$$\times \sum_{b_4, b_5, \dots, b_{j-1} \geq 0, b_j > 0} \begin{bmatrix} b_2 + b_3 + b_4 - 1 \\ b_3, b_4 \end{bmatrix}_q \begin{bmatrix} b_3 + b_4 + b_5 - 1 \\ b_5 \end{bmatrix}_q \cdots t^{b_5+b_6+b_7+2(b_8+b_9+b_{10})+\dots} \tag{206}$$

$$= \sum_{b_3 \geq 0} \begin{bmatrix} b_1 + b_2 + b_3 - 1 \\ b_3 \end{bmatrix}_q t^{B_1 - b_1} \frac{[b_2 - 1]![b_3]!}{[b_2 + b_3 - 1]!} M(B_2, B_3, B_1 - b_1, b_2, b_3). \tag{207}$$

For general l, define

$$M(B_1, \dots, B_l, b_1, \dots, b_{l-1}) = \sum_P q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}, \tag{208}$$

where the sum is over all P which are shuffles of blocks of length $B_l, \dots, B_1$, with car $n = B_l + \dots + B_1$ in row n, and with b_i cars from set B_i in rows of length 0, $1 \leq i \leq l - 1$. To write this recursively, assume we also have b_l cars from set B_l in rows of length 0, and let m be the smallest number satisfying $1 < m \leq l$ for which $b_m > 0$. By the argument in the $l = 3$ case, we obtain

$$M(B_1, \dots, B_l, b_1, 0, \dots, 0, b_m, b_{m+1}, \dots, b_{l-1}) \sum_{b_l, b_{l+1} \dots, b_{m+l-2} \geq 0} \begin{bmatrix} b_1 + b_m + \dots + b_l - 1 \\ b_m, \cdots, b_l \end{bmatrix}_q \tag{209}$$

$$\times \begin{bmatrix} b_m + \dots + b_{l+1} - 1 \\ b_{l+1} \end{bmatrix}_q \cdots \begin{bmatrix} b_m + \dots + b_{m+l-2} - 1 \\ b_{m+l-2} \end{bmatrix}_q t^{F'} \frac{[b_m - 1]! \cdots [b_{m+l-2}]!}{[b_m + \dots + b_{m+l-2} - 1]!} \tag{210}$$

$$\times M(B_m, B_{m+1}, \dots, B_l, B_1 - b_1, B_2, \dots, B_{m-1}, b_m, b_{m+1} \dots, b_{m+l-2}), \tag{211}$$

where $F' = B_1 - b_1 + B_2 + \dots + B_{m-1}$.

The author doesn't know if $M(B_1, \dots, B_l, b_1, \dots, b_{l-1})$ can be expressed in terms of the ∇ operator for $l > 2$.

5 A formula for $E_{n,n-1}$

In this section we sketch a proof of the following result, which shows Conjecture 1 is true when $k = n - 1$.

Theorem 12 *For $\lambda \vdash n$,*

$$\langle \nabla E_{n,n-1}, s_\lambda \rangle = t \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \sum_{\substack{T \in SYT(\lambda) \\ n \text{ is below } n-1 \text{ in } T}} q^{maj(T)-n+1}. \tag{212}$$

Proof. Let Y be the linear operator defined on the modified Macdonald basis via

$$Y \tilde{H}_\nu = \Pi_\nu \tilde{H}_\nu. \tag{213}$$

Another way to express (55) is

$$\nabla E_{n,k} = t^{n-k}(1-q^k) Y \left(\sum_{\nu \vdash n-k} \frac{\tilde{H}_\nu T_\nu h_k[\frac{X}{1-q}]}{w_\nu} \right) \tag{214}$$

$$= t^{n-k}(1-q^k) Y \left(h_{n-k}[\frac{X}{M}] h_k[\frac{X}{1-q}] \right) \quad \text{(using (88))}, \tag{215}$$

which holds for $1 \le k \le n$. Using (42) we get

$$\nabla E_{n,n-1} = \frac{t(1-q^{n-1})}{(1-t)(1-q)(q;q)_{n-1}} Y(X \tilde{H}_{n-1}) \tag{216}$$

$$= \frac{t(1-q^{n-1})}{(1-t)(1-q)(q;q)_{n-1}} Y(d^{e_1}_{n,n-1} \tilde{H}_n + d^{e_1}_{(n-1,1),n-1} \tilde{H}_{n-1,1}). \tag{217}$$

The case $\nu = n-1$ of [BGHT99, Eq. (1.39)] implies

$$d^{e_1}_{n,n-1} = \frac{1-t}{q^{n-1}-t} \quad \text{and} \quad d^{e_1}_{(n-1,1),n-1} = \frac{1-q^{n-1}}{t-q^{n-1}}. \tag{218}$$

Plugging these into (217) and using the fact that

$$\Pi_n = (q;q)_{n-1} \quad \text{and} \quad \Pi_{(n-1,1)} = (q;q)_{n-2}(1-t) \tag{219}$$

we get

$$\nabla E_{n,n-1} = \frac{t(1-q^{n-1})}{(1-t)(1-q)(q;q)_{n-1}} \left(\frac{1-t}{q^{n-1}-t} \tilde{H}_n (q;q)_{n-1} + \frac{1-q^{n-1}}{t-q^{n-1}} \tilde{H}_{(n-1,1)} (q;q)_{n-2}(1-t) \right) \tag{220}$$

$$= \frac{t(1-q^{n-1})}{(1-q)} \left(\frac{\tilde{H}_n - \tilde{H}_{(n-1,1)}}{q^{n-1}-t} \right) \tag{221}$$

$$= \frac{t(1-q^{n-1})}{(1-q)} \left(\sum_{\lambda \vdash n} s_\lambda \frac{\tilde{K}_{\lambda,n} - \tilde{K}_{\lambda,(n-1,1)}}{q^{n-1}-t} \right). \tag{222}$$

Eq. (212) can now be derived from known identities for $\tilde{K}_{\lambda,\mu}$ when μ is a hook [Ste94]. □

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