ON THE NONNEGATIVITY OF THE COEFFICIENTS OF SOME POLYNOMIALS OCCURRING IN THE THEORY OF CUBICAL SPHERES

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Abstract. As part of his recent research into Cubical Spheres, G. Hetyei gave an inductive proof that a certain sequence of polynomials have nonnegative coefficients. In a poster session at the FPSAC'96 conference he asked for a combinatorial interpretation of these coefficients. In this short note we show that known results in rook theory imply that the coefficients count permutations subject to certain constraints.

Résumé. Dans un de ces derniers articles portant sur les sphères cubiques, G. Hetyei a donné une preuve par récurrence du fait que les coefficients d'une certaine suite de polynômes sont non-négatifs. A l'occasion du colloque SFCA'96 G. Hetyei a proposé de trouver une interprétation combinatoire de ces coefficients. Dans ce résumé on démontre que certains résultats connus sur la théorie des tours impliquent que les dits coefficients comptent bien des permutations sujettes à certaines contraintes.

At the recent FPSAC '96 conference held at the University of Minnesota, G. Hetyei presented a poster session entitled "Invariants of Cubical Spheres" [Het1], a full length version of which appeared in this journal [Het2]. The following lemma was part of his session.

Lemma *Let $\Phi_{a,b,c}(x)$ denote the formal power series $\sum_{k\geq 0} k^a(k+1)^b(k+2)^c x^k$, where a, b and c are natural numbers. Then the following hold.*

(i) We have

$$\Phi_{a,b,c}(x) = \frac{A_{a,b,c}(x)}{(1-x)^{a+b+c+1}}, \tag{1}$$

where $A_{a,b,c}(x)$ is a polynomial with integer coefficients, of degree at most $a+b+c$.

(ii) If $b>0$ then the degree of $A_{a,b,c}(x)$ is at most $a+b+c-1$.

(iii) If $b>0$ or $c=0$ then $A_{a,b,c}(x)$ has only nonnegative coefficients.

Hetyei's proof used induction and differential equations satisfied by $\Phi_{a,b,c}(x)$. As a challenge he asked for a combinatorial interpretation of part *(iii)* of his lemma. In this note we use rook theory to show that if $b>0$ or $c=0$, the coefficients of $A_{a,b,c}(t)$ count permutations subject to certain constraints.

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If B is an n-column Ferrers board with column heights $h_1 \le h_2 \le \cdots \le h_n$, let $r_k(B)$ be the number of ways of placing k non-attacking rooks on B, and if $h_n \le n$, let $t_k(B)$ be the number of ways of placing n non-attacking rooks on the $n \times n$ square chessboard containing B, with exactly k rooks on B. For more detailed descriptions of these objects see [Sta] and [Rio]. Our proof depends on the following theorem.

$$\sum_{k=0}^{\infty} x^k \prod_{i=1}^{n} (k + h_i - i + 1) = \frac{1}{(1-x)^{n+1}} \sum_{k=0}^{n} x^k t_{n-k}(B). \tag{2}$$

Eq. (2) is the $q = 1$ case of a result of Garsia and Remmel [GaRe,p.259].

Let B be the board which starts off with c columns with heights $2, 3, \ldots, c+1$, followed by b columns with heights $c+1, c+2, \ldots, c+b$, followed by a columns with heights $c+b, c+b+1, \ldots, c+b+a-1$. Then $h_{a+b+c} = a+b+c-1$, the left-hand side of (2) equals the left-hand side of (1), and part (iii) of the lemma follows. The condition $b > 0$ or $c = 0$ is necessary to ensure B has non-decreasing column heights.

To deduce part (ii) of the lemma, note that if $c > 0$, the first column of B has height 2, which implies $t_0(B) = t_1(B) = 0$, since any placement of $a+b+c$ rooks must have a rook in row 1 and a rook in row 2. Thus the degree of A is at most $a+b+c-2$. If $c = 0$ and $b > 0$, the first column of B has height 1 so we get $t_0(B) = 0$, with the degree of A being at most $a+b+c-1$.

Remarks:

If $b = 0$, let B be the board which starts with a columns with heights $0, 1, \ldots, a-1$, followed by c columns with heights $a+2, a+3, \ldots, a+c+1$. In this case $h_{a+b+c} = a+b+c+1$, so the combinatorial definition of the t_k doesn't apply, but (2) still holds if we define the t_k via Riordan and Kaplansky's inclusion-exclusion identity [KaRi]

$$\sum_{k=0}^{n} k! r_{n-k}(B)(x-1)^{n-k} = \sum_{k=0}^{n} x^k t_k(B).$$

In [HOW] it is shown that a result of F. Brenti [Bre, Thm. 4.4.1, p.43] implies that for any Ferrers board satisfying $h_n \le n$, the hit polynomial $\sum_{k=0}^{n} t_k(B) x^k$ has only real zeros.

EXTENDED FRENCH ABSTRACT (TRANSLATION BELOW): At the recent FPSAC '96 conference held at the University of Minnesota, G. Hetyei presented a poster session entitled "Invariants of Cubical Spheres" [Het1], a full length version of which appeared in this journal [Het2]. Part of his session involved the polynomials $A_{a,b,c}(x)$ defined by

$$A_{a,b,c}(x) := (1-x)^{a+b+c+1} \sum_{k \ge 0} k^a (k+1)^b (k+2)^c x^k,$$

where a, b and c are natural numbers. For $b > 0$ or $c = 0$ he required that the coefficients of this polynomial be nonnegative, which he was able to prove by an inductive argument. As a combinatorial challenge, he asked for a combinatorial interpretation of these coefficients. Here we use known results in rook theory to show these coefficients count permutations subject to certain constraints.

If B is an n-column Ferrers board with column heights $h_1 \leq h_2 \leq \cdots \leq h_n$, where $h_n \leq n$, let $t_k(B)$ be the number of ways of placing n non-attacking rooks on the $n \times n$ square chessboard containing B, with exactly k rooks on B. Our proof depends on the following theorem, which is the $q = 1$ case of a result of Garsia and Remmel.

$$(1-x)^{n+1} \sum_{k=0}^{\infty} x^k \prod_{i=1}^{n} (k + h_i - i + 1) = \sum_{k=0}^{n} x^k t_{n-k}(B).$$

By an appropriate choice of the h_i, the left-hand side above reduces to $A_{a,b,c}(x)$, and the result follows immediately. END OF EXTENDED ABSTRACT

EXTENDED ABSTRACT IN FRENCH

Résumé détaillé.

Au colloque SFCA'96, qui a eu lieu à l'université du Minnesota, G. Hetyei a présenté un article intitulé "Invariants des sphères cubiques" [Het1], dont une version complète apparaît dans ce journal. L'auteur y définit les polynômes $A_{a,b,c}(x)$ comme suit:

$$A_{a,b,c}(x) := (1-x)^{a+b+c+1} \sum_{k \geq 0} k^a (k+1)^b (k+2)^c x^k,$$

où a, b et c sont des nombres naturels. De plus, dans les cas $b > 0$ ou $c = 0$ G. Hetyei démontre par récurrence que les coefficients sont non-négatifs. Par la suite il propose de trouver une interprétation combinatoire à ces coefficients. Dans cet article, on utilise des résultats connus sur la théorie des tours pour montrer que les dits coefficients comptent bien des permutations sujettes à certaines contraintes.

Si B est un tableau de Ferrers ayant n colonnes d'hauteurs $h_1 \leq h_2 \leq \cdots \leq h_n$, où $h_n \leq n$, soit $t_k(B)$ le nombre de façons de placer n tours non-attaquantes sur un échiquier de taille $n \times n$ contenant B, avec exactement k tours sur B. Notre preuve dépend du théorème suivant, qui est le cas $q = 1$ d'un résultat de Garsia et Remmel.

$$(1-x)^{n+1} \sum_{k=0}^{\infty} x^k \prod_{i=1}^{n} (k + h_i - i + 1) = \sum_{k=0}^{n} x^k t_{n-k}(B).$$

Pour un choix approprié des valeurs h_i, le côté gauche de l'équation ci-dessus se réduit à $A_{a,b,c}(x)$, et le résultat en découle aisément. END OF EXTENDED ABSTRACT IN FRENCH

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