FURTHER INVESTIGATIONS INVOLVING ROOK POLYNOMIALS WITH ONLY REAL ZEROS

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Abstract. We study the zeros of two families of polynomials related to rook theory and matchings in graphs. One of these families is based on the cover polynomial of a digraph introduced by Chung and Graham [ChGr]. Another involves a version of the "hit polynomial" of rook theory, but which applies to weighted matchings in (non-bipartite) graphs. For both of these families we prove a result which is analogous to a theorem of of the author, K. Ono, and D. G. Wagner, namely that for Ferrers boards the hit polynomial has only real zeros. We also show that for each of these families there is a general conjecture involving arrays of numbers satisfying inequalities which contains these theorems as special cases. We provide evidence for the truth of these conjectures by proving other special cases and discussing computational experiments.

1. Introduction

Polynomials with only real zeros arise often in combinatorial theory. The relevance of this property is demonstrated by Newton's Inequality [HLP,p.52] , which says that if $\sum_{k=0}^{n} z^k b_k$ is a polynomial with only real zeros, then

$$\frac{b_k^2}{\binom{n}{k}^2} \geq \frac{b_{k-1}b_{k+1}}{\binom{n}{k-1}\binom{n}{k+1}}.$$

Other inequalities satisfied by the b_k, assuming $b_k \geq 0$ for all k, were discovered by Aissen, Schoenberg, and Whitney [ASW].

If A is an $n \times n$ matrix, the *permanent* of A, denoted $\mathrm{per}(A)$, is defined as

$$\mathrm{per}(A) := \sum_{\sigma \in S_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Let J_n denote the $n \times n$ matrix of all ones. Recently the author, K. Ono, and D. G. Wagner made the following conjecture [HOW]:

Monotone Column Permanent (MCP) Conjecture. *Let A be a real $n \times n$ matrix whose entries are weakly increasing down columns, i.e. $a_{i,j} \leq a_{i+1,j}$. Then all of the zeros of $per(zA+J_n)$ are real.*

The MCP Conjecture has a natural interpretation in terms of *rook theory*. A placement of rooks on the squares of A is nonattacking if no two rooks are in the same column, and no two are in the same row. Define the *weight* of such a placement to be the product of the entries in A which are under the rooks, and define the kth rook number $r_k(A)$ to be the sum of these weights over all nonattacking placements of k rooks on A. When $n = 2$ these rook numbers are

$$r_2(A) = a_{11}a_{22} + a_{12}a_{21}, \quad r_1(A) = a_{11} + a_{12} + a_{21} + a_{22},$$

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and by convention $r_0(A) := 1$ for all A. Note that $r_n(A)$ is per(A), and more generally $r_k(A)$ is the sum of all $k \times k$ permanental minors of A. If each a_{ij} is zero or one, then A is called a *board*, and $r_k(A)$ is the number of placements of k nonattacking rooks on the nonzero entries of A. Nijenhuis [Nij] proved that the rook polynomial $\sum_k z^k r_k(A)$ has only real zeros for any real matrix A with nonnegative entries, and in particular for any board.

The *hit polynomial* $T(z;A)$ of A is defined by

$$T(z;A) := \sum_{k=0}^{n} k! r_{n-k}(A)(z-1)^{n-k}. \tag{1}$$

Denote the coefficient of z^k in (1) by $t_k(A)$. If A is a board, then $t_k(A)$ is called the kth *hit number*, and equals the number of ways of placing n nonattacking rooks on A where exactly k rooks lie on nonzero entries [KaRi]. An old result of Laguerre says that if $\sum_k z^k b_k$ is a polynomial with only real zeros, then the polynomial $\sum_k z^k b_k/k!$ also has only real zeros (in general the polynomial $\sum_k z^k b_k k!$ will not have only real zeros). Hence if $T(z;A)$ has only real zeros, so does the rook polynomial, but not necessarily vice-versa.

The question of for which boards does $T(z;A)$ have only real zeros was investigated in [HOW], with the following result.

Definition. *A Ferrers board is a board with the property that $a_{ij} = 1 \implies a_{kp} = 1$ for $1 \le k \le i$ and $j \le p \le n$, i.e. all the entries of A to the right and above square (i,j) also equal 1. We let $c_i(A)$ denote the height of (the number of ones in) the ith column of A. If $c_n \le n$, we say A is admissible (later we will be working with $N \times n$ matrices with $N \ge n$). We can represent a Ferrers board pictorially by outlining the region of the matrix equal to 1, as in Fig. 1.*

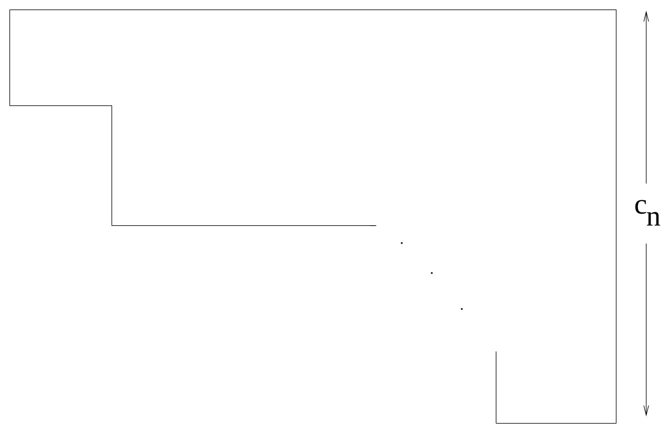

FIGURE 1. A Ferrers board with n columns.

Theorem 1.1. *(Haglund-Ono-Wagner) If A is an admissible Ferrers board, then $T(z;A)$ has only real zeros.*

Note that by expanding out per$((z-1)A + J_n)$ in powers of $z-1$ we get the identity

$$\text{per}((z-1)A + J_n) = T(z;A).$$

By starting with a Ferrers board, flipping the matrix upside down and permuting the columns accordingly (which doesn't change the permanent) we see that Theorem 1.1 implies that the MCP Conjecture is true if $a_{ij} \in \{0,1\}$. This, together with Nijenhuis' result, led to the MCP Conjecture.

Let W be a strictly upper-triangular, $n \times n$ matrix. We will often call W a weighted graph, since W can be identified with the graph on n vertices with weight w_{ij} on the edge between vertices i and j. If $w_{ij} \in \{0, 1\}$, a weighted graph becomes a graph. A *matching* in a weighted graph is a selection of edges no two of which share a common vertex. The weight of a matching is the product of the weights of the edges in the matching, and the kth matching number $m_k(W)$ is the sum of these weights over all matchings with k edges. Although it was originally defined slightly differently, for our purposes we call $\sum_k z^k m_k(W)$ the *matching polynomial*. For example, if W is the weighted version of the complete graph on 4 vertices K_4 of Fig. 2, the matching polynomial of W is

$$1 + z(w_{12} + w_{13} + w_{14} + w_{23} + w_{24} + w_{34}) + z^2(w_{12}w_{34} + w_{13}w_{24} + w_{14}w_{23}).$$

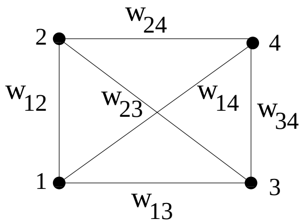

FIGURE 2. A weighted version of K_4.

If n is even, a matching with $n/2$ edges is called a *perfect matching*, and the function $m_{n/2}(W)$ is known as the *Hafnian* of W, denoted $\mathrm{Hf}(W)$. Hafnians occur in the study of plane partitions [Kup].

Prior to the work of Nijenhuis, Heilmann and Lieb [HeLi] proved that the matching polynomial of any simple graph with nonnegative, real edge-weights has only real zeros. It was later realized that the matching polynomial reduces to the rook polynomial when the graph is bipartite, and so Nijenhuis' result also follows from the Heilmann-Lieb Theorem. For more recent work on this theorem see [Wag2].

In this paper we show that analogs of Theorem 1.1 hold for two other polynomials connected to matchings in graphs and digraphs. In Section 2 we state these theorems, and for each one we introduce a corresponding general conjecture which contains it as a special case. One of these is expressible in terms of the Hafnian of an array, and is related to the matching polynomial in the same way that the MCP conjecture is related to the rook polynomial. Another involves generalized rook numbers which count cycles in digraphs, as studied by Chung and Graham, Chow, Dworkin, and Gessel [ChGr], [Cho], [Dwo], [Ges1], [Ges2]. Section 3 contains various technical lemmas which we utilize in Sections 4 and 5 to prove the analogs of Theorem 1.1 as well as other special cases of these conjectures. In particular, we show (Theorem 2.2) that for threshold graphs a stronger result than the Heilmann-Lieb Theorem holds. Computational evidence for the conjectures is discussed in Section 6.

2. Analogs of Theorem 1.1 and the MCP Conjecture

We begin by fixing some notation; $(x)_k$ will denote the rising factorial $x(x+1)\cdots(x+k-1)$, while LHS and RHS are abbreviations for "left hand side" and "right hand side", respectively. $\mathbb{P}$ denotes the positive integers, and $\mathbb{N}$ the nonnegative integers.

Definition. *A shifted Ferrers board G is a strictly upper-triangular $n \times n$ matrix of zeros and ones which is weakly decreasing across rows and down columns, i.e.*

$$g_{ij} \geq g_{i,j+1} \quad 1 \leq i < j \leq n-1$$

and

$$g_{ij} \geq g_{i+1,j} \quad 2 \leq i+1 < j \leq n.$$

Let $b_i(G), 1 \leq i \leq n-1$ denote the length of (the number of ones in) the ith row from the bottom of G (so that the top row of G has length $b_{n-1}(G)$). Note that $b_{i-1}(G) < b_i(G)$ for $2 \leq i \leq n-1$ and that $b_1(G) = 1$ if and only if G is the complete graph on n vertices K_n (otherwise $b_1(G) = 0$).

Shifted Ferrers boards are known in graph theory as *threshold graphs*, and have been the subject of substantial investigation over the past twenty years. Our interest in these objects comes from the following result of Reiner and White [ReWh].

Theorem 2.1. *(Reiner-White). Let G be a threshold graph on n vertices. Then*

$$\sum_{k=0}^{n-1} x(x-2)\cdots(x-2k+2)m_{n-1-k}(G) = \prod_{i=1}^{n-1}(x+b_i(G)-2i+2). \tag{2}$$

Proof: Reiner and White's proof uses induction. Here we include a combinatorial proof, along the lines of Goldman, Joichi, and White's original proof of their well-known result [GJW]:

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(A) = \prod_{i=1}^{n}(x+c_i-i+1), \tag{3}$$

where A is a Ferrers board whose ith column has height c_i.

Replacing x by $2x$ and dividing both sides of (2) by 2^n we get

$$\sum_{k=0}^{n-1} x(x-1)\cdots(x-k+1)\frac{m_{n-1-k}(G)}{2^{n-1-k}} = \prod_{i=1}^{n-1}(x+b_i(G)/2-i+1). \tag{4}$$

Form an extended board $G(x)$ by adjoining an $x \times n-1$ rectangle of squares to the right of G, as in Fig. 3. Consider the number of ways of placing $n-1$ nonattacking rooks on $G(x)$, subject to the following conditions:

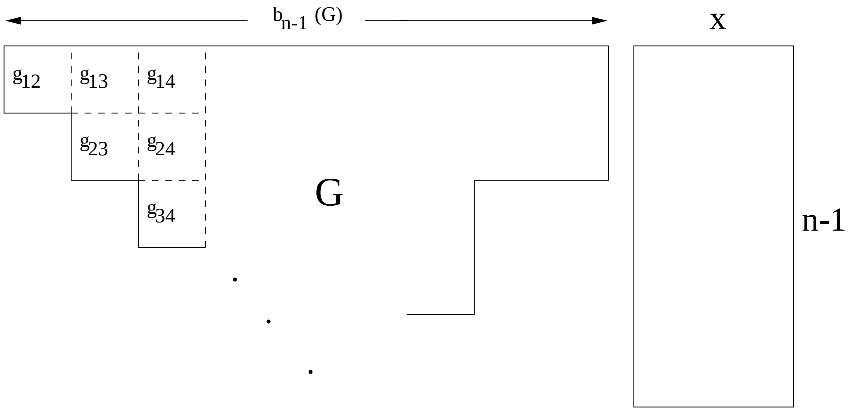

Figure 3. The extended board $G(x)$. A rook placed on square $(2,3)$ of G would attack everything in the second row from the top of G and the second row of the $x \times n-1$ rectangle, as well as squares $(1,2),(1,3),(3,4),(3,5),\ldots,(3,n)$ of G.

(a) a rook placed on square (i,j) of G has weight $g_{ij}/2$, and attacks any square of G with any coordinate equal to i or j, as well as the squares of the $x \times n-1$ rectangle in its row.
(b) rooks placed on the $x \times n-1$ rectangle attack everything in $G(x)$ in their row and column, and have weight one, as in a traditional rook placement.

We count the number of ways of placing $n-1$ nonattacking rooks on $G(x)$ in two different ways, with the weight of each rook placement equal to the product of the weights of the individual rooks. Begin by placing a rook in the bottom row of $G(x)$ in $x + b_1(G)/2$ ways. If this rook is placed on the $x \times n-1$ rectangle, it attacks one square in the next row up, of weight one. If it is placed on G, it attacks two squares in the next row up, both of weight $1/2$. In either case, the number of ways of placing a rook in the next row up is now $x + b_2(G)/2 - 1$. Continuing in this way we generate the RHS of (4). On the other hand, we could start by placing $n-1-k$ rooks on G in $m_{n-1-k}(G)/2^{n-1-k}$ ways. For any such placement there are $x(x-1)\cdots(x-k+1)$ ways to place the remaining k rooks on the $x \times n-1$ rectangle, and we thus generate the LHS of (4). Since both sides of (4) are polynomials in x which agree at infinitely many values of x, they must be equal for all x. $\square$

Definition. *Let W be a strictly upper-triangular $n \times n$ matrix (i.e. a weighted graph on n vertices). For n a nonnegative integer, let $n!!$ denote the product of all the positive odd integers which are less than or equal to n. (If n is even, $n!! = \frac{n!}{2^{n/2}(n/2)!}$.) Define $Q(z;W)$ by*

$$Q(z;W) := \sum_{k \geq 0} m_k(W)(z-1)^k(n-2k)!!.$$

The polynomial $Q(z;W)$ plays a role in recent joint work of the author and J. Remmel [HaRe]. It can be thought of as a K_n-version of the hit polynomial. Note that if n is odd, we can add an empty vertex to W without changing $Q(z;W)$. Assuming n is even, $Q(z;W)$ can be expressed as $Hf((z-1)W + J_n)$, which follows from the fact that the number of ways to extend a k-edge matching of K_n to a perfect matching of K_n is $(n-2k)!!$.

In Section 4 we prove the following analog of Theorem 1.1.

Theorem 2.2. *Let G be a threshold graph. Then $Q(z;G)$ has only real zeros.*

In view of the MCP Conjecture, one could expect Theorem 2.2 to be a special case of a more general phenomenon, which we now describe.

Definition. *The ith hook of K_n is the set of all squares (u,v) of K_n where either $u = i$ or $v = i$, i.e. the set of all squares $(i,v), i < v \leq n$ or $(u,i), 1 \leq u < i$. Note that each square is in two hooks. Given a graph G on n vertices, the ith hook of G is the set of all squares which are in both the ith hook of K_n and are also in G. If we travel along a hook of G we say we are moving in the positive direction if we are moving either upwards or to the left. Given an upper-triangular array W of real numbers, we say W is monotone, with respect to a hook of G, if, as we traverse the hook in a positive direction, the values of the corresponding squares of W are weakly increasing. We say W is G-monotone if for each square β of G, W is monotone with respect to one of the two hooks of G containing β.*

Conjecture 2.3. *Let W be a K_n-monotone array, with n even. Then*

$$Hf(zW + J_n)$$

has only real zeros.

We now show that the special case of Conjecture 2.3 where $w_{ij} \in \{0,1\}$ follows from Theorem 2.2. We claim that if $w_{ij} \in \{0,1\}$ and W is K_n-monotone, then W is isomorphic to a threshold graph. If W is monotone with respect to all of the hooks of K_n, the claim is obvious. Assume, for example, that W is not monotone with respect to the nth hook of K_n. Then for each square in the nth column, W must be monotone with respect to the other hook it is in. This forces W to be monotone with respect to all of the hooks except for the nth. Hence W must be an array obtained by starting with a threshold graph, then changing some subset of the squares in the nth column from 1's to 0's. The resulting graph is isomorphic to a threshold graph. A similar argument works if W is not monotone with respect to some hook other than the nth.

The following conjecture combines both the MCP Conjecture and Conjecture 2.3.

Conjecture 2.4. *Let n be an even integer. Let G be either K_n or the complete bipartite graph $K_{n/2,n/2}$, with W a G-monotone array. Then the polynomial*

$$\sum_{C} \prod_{(i,j)\in C} (zw_{ij} + 1)$$

has only real zeros, where the sum is over all perfect matchings C of G.

If G is $K_{n/2,n/2}$, we can identify the relevant squares of W with an $n/2 \times n/2$ square grid, i.e. a matrix. Perfect matchings of G are in bijection with terms in $\mathrm{per}(zW + J_{n/2})$. The condition of W being G-monotone translates into each square of this matrix being in a weakly increasing column or in a weakly increasing row. It is easy to show that this forces all of the columns (or all of the rows) to be weakly increasing, and we end up with the MCP Conjecture.

Perhaps other infinite classes of graphs can plausibly be substituted in for G in Conjecture 2.4 (it is false with G replaced by the complete tripartite graph $K_{2,2,2}$).

We now turn to digraph polynomials. For the remainder of this article A will denote an $N \times n$ matrix, with $N \geq n$. A placement C of nonattacking rooks on the squares of A can be identified with a weighted digraph $D(C)$ with N vertices as follows; if square (i, j) contains a rook, include a directed edge from vertex i to vertex j with weight a_{ij}. Such a digraph decomposes into a disjoint union of directed paths and cycles. See Fig. 4.

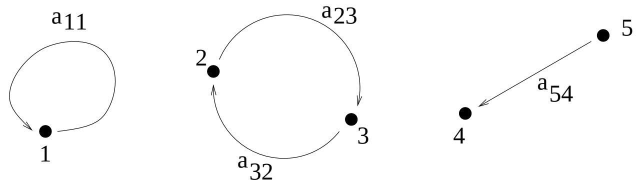

FIGURE 4. The digraph corresponding to the placement of rooks on squares $(1, 1)$, $(3, 2)$, $(2, 3)$, and $(5, 4)$ of the 5×4 matrix A.

Definition. *For a given placement C of nonattacking rooks on the squares of A, let $cyc(C)$ denote the number of cycles of $D(C)$, and $wt(C)$ the weight of C. Set*

$$r_k(y; A) := \sum_{\substack{C \\ k \text{ rooks}}} y^{\mathrm{cyc}(C)} wt(C).$$

For example, if A is a 3×2 matrix, then

$$r_2(y; A) = y^2 a_{11}a_{22} + y(a_{11}a_{32} + a_{21}a_{12} + a_{31}a_{22}) + a_{21}a_{32} + a_{31}a_{12},$$

$$r_1(y; A) = y(a_{11} + a_{22}) + a_{12} + a_{21} + a_{31} + a_{32},$$

and by convention $r_0(y; A) := 1$ for all A.

Chung and Graham introduced what they call the *cover polynomial* of a digraph [ChGr]. They defined it via a deletion-contraction construction, and then showed that it could be expressed as

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(y;A),$$

where A is the board corresponding to the digraph A (and $N=n$). Chow introduced a symmetric function which contains the cover polynomial as a special case [Cho]. In [Hag2] the author studied a different generalization, defined as follows.

Definition. *For any $N \times n$ matrix A, let*

$$T(x,y,z;A) := \sum_{k=0}^{n} (x)_k r_{n-k}(y;A)(z-1)^{n-k}.$$

Note that if $N=n$, $T(1,1,z;A)=T(z;A)$ and $(-1)^n T(-x,y,0;A)$ is the cover polynomial.

Definition. *Let A be the Ferrers board of Fig. 1, where $c_n(A) \le N$. Set*

$$s_i(A) = \begin{cases} y & \text{if} \quad c_i(A) \ge i, \\ 1 & \text{else}. \end{cases}$$

In Section 5 we prove the following result. For $x=y=1$, it reduces to Theorem 1.1. Below it we state the general conjecture which contains it as a special case.

Theorem 2.5. *Let A be the Ferrers board of Fig. 1. Assume $y>0$ and either*

(a) $\quad x > c_i(A) - i + s_i(A) \qquad$ *for $1 \le i \le n$*

or

(b) $\quad x < 1-n$

or

(c) $\quad x = y$ *and A is admissible.*

Then as a polynomial in z, $T(x,y,z;A)$ has only real zeros.

Conjecture 2.6. *Let A be a $N \times n$ matrix, $N \ge n$, of nonnegative real numbers which are weakly increasing across rows and weakly decreasing down columns, i.e. $a_{i,j} \le a_{i,j+1}$ and $a_{ij} \ge a_{i+1,j}$. Assume $y>0$ and either*

(a) $\quad x > N-1+y$

or

(b) $\quad x < 1-n$

or

(c) $\quad x = y$ *and $N=n$.*

Then as a polynomial in z, $T(x,y,z;A)$ has only real zeros.

Note that Theorem 2.5 implies that Conjecture 2.6 is true if $a_{ij} \in \{0,1\}$.

3. The Methods

In this section we collect various results concerning polynomials with only real zeros which we use in later sections to prove special cases of our conjectures.

Theorem 3.1: **Transformations which preserve the property of having only real zeros.** *Assume* $f(z) := \sum_k z^k b_k$ *and* $g(z) := \sum_k z^k d_k$ *are two polynomials of degree* n *having only real zeros, and furthermore that all the zeros of* g *are of the same sign. Then*

(a) $\sum_{k=0}^{n} z^k b_{n-k}$ *has only real zeros.* (*reversal*)

(b) $\sum_{k=0}^{n} z^k b_k/(y)_k$ *has only real zeros if* $y > 0$ (*Laguerre [Lag,p.201]*).

(c) $\sum_{k=0}^{n} z^k b_k d_k k!$ *has only real zeros* (*Schur [Sch]*).

(d) $\sum_{k=0}^{n} z^k b_k d_k/\binom{n}{k}$ *has only real zeros* (*Szegö [Sze], [PoSz, p.61, Prob. 154]*).

Note that transformation (c) follows from (a), (b), and (d). The special case of the MCP Conjecture where $a_{i,j} := p_i q_j$, with p_i, q_j real numbers and q_j nonnegative, reduces to a statement equivalent to transformation (d).

Theorem 3.1 (b) gives an example of a *factor sequence*, i.e. a sequence γ_k, $k \geq 0$ of real numbers with the property that for any polynomial $\sum_{k=0}^{n} b_k z^k$ with only real zeros, $\sum_{k=0}^{n} b_k \gamma_k z^k$ also has only real zeros. Laguerre derived this from the following more general result [Lag].

Theorem 3.2. *(Laguerre) Let* $E(z)$ *be an entire function whose Weierstrass factorization is of the form*

$$E(z) = exp(-\gamma z^2 + \beta z) \prod_{i=1}^{\infty} (1 + \frac{z}{\beta_k}) exp(-z/\beta_k),$$

where $\gamma \geq 0$, $\beta_k > 0$, $\beta \geq 0$, *and* $\sum_k 1/\beta_k^2 < \infty$. *Then* $\gamma_k = E(k)$, $k \geq 0$ *is a factor sequence.*

Corollary 3.3. *(See also [PoSz,p.63]). Let* $m \in \mathbb{P}$. *Then*

$$\gamma_k = \frac{k!}{(mk)!}, \qquad k \geq 0$$

is a factor sequence.

Proof: Let $E(z) = \Gamma(z+1)/\Gamma(mz+1)$ in Theorem 3.2. □

We call a finite sequence γ_k, $0 \leq k \leq n$ of real numbers a *degree-n factor sequence* if for any polynomial $\sum_{k=0}^{n} b_k z^k$ of degree n with only real zeros, $\sum_{k=0}^{n} b_k \gamma_k z^k$ also has only real zeros. Theorem 3.1 (d) implies that γ_k is a degree-n factor sequence if and only if $\sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ has only real zeros, all of the same sign.

As a further example of the use of these transformations, we include the following observation, which follows directly from Theorem 3.1 (d), but which the author has not seen before in the literature. Let $n \in \mathbb{N}$ and assume

$${}_{m+1}F_m \left[\begin{matrix} -n, & a_1, & \ldots, & a_m \\ & b_1, & \ldots, & b_m \end{matrix} ; z \right]$$

and

$${}_{k+1}F_k \left[\begin{matrix} -n, & c_1, & \ldots, & c_k \\ & d_1, & \ldots, & d_k \end{matrix} ; z \right]$$

are two terminating hypergeometric series with only real zeros, one of whose zeros are all of the same sign. Then

$${}_{m+k+1}F_{m+k} \left[\begin{matrix} -n, & a_1, & \ldots, & a_m, & c_1, & \ldots, & c_k \\ & b_1, & \ldots, & b_m, & d_1, & \ldots, & d_k \end{matrix} ; z \right]$$

has only real zeros.

Lemma 3.4. *Let $t(z) := \sum_{k=0}^{n} b_k z^k$ be a polynomial with only real zeros, all of the same sign, and let $[n/2]$ be the greatest integer less than or equal to $n/2$. Assume that*

$$h(z) := \sum_{k=0}^{[n/2]} \binom{n}{2k} \beta_k z^k$$

has only real, nonpositive zeros. Then

$$\sum_{k=0}^{[n/2]} z^k b_{2k} \beta_k$$

has only real zeros.

Proof: Apply Theorem 3.1 (d) with $f(z) := h(-z^2)$ and $g(z) := t(z)$. □

If f is a polynomial of degree n with real zeros $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and g is a polynomial of degree $n-1$ with real zeros $\beta_1 \leq \beta_2 \ldots \leq \beta_{n-1}$, g is said to *interlace* f if $\alpha_i \leq \beta_i \leq \alpha_{i+1}$ for $1 \leq i \leq n-1$. The next lemma can be proven using the techniques from Section 3 of [Wag1].

Lemma 3.5. *Assume that f and g are two polynomials with only real zeros and that g interlaces f. Then g interlaces $f+g$.*

The following lemma is similar to a result of Brenti [Bre, Thm. 2.4.4]. We denote the degree of a polynomial f by $\deg(f)$.

Lemma 3.6. *Let h and g be two polynomials with only real zeros, and with positive leading coefficients a and b, respectively. Assume $deg(h) = deg(g) + 1$ and furthermore that g interlaces h. Let x, ζ, β be real numbers with $\beta \geq 0$, $\beta \geq \zeta/x$, and either $x > 0$ or $x < -b/a$. Assume that all of the zeros of h are nonpositive if $x > 0$ and nonnegative if $x < -b/a$. Then*

$$f := (xz + \zeta)h + z(z + \beta)g$$

has only real zeros. In addition, if all the zeros γ of h satisfy $-\beta \leq \gamma \leq 0$, then h interlaces f.

Proof: If γ is a zero of h of multiplicity m, then the interlacing hypothesis on g forces γ to be a zero of g of multiplicity at least $m-1$. It follows that we can assume without loss of generality that all the zeros of h are simple. We will prove the lemma under the assumptions that β is not equal to any zero of h and $\beta > \zeta/x$, since the general result then follows by continuity.

There are several cases to consider, depending on whether or not $h(0)$ is zero, x is positive or negative, and whether $-\beta$ is less than all the zeros of h, between two consecutive, negative zeros of h, or larger than all the negative zeros of h. Since the same argument, with minor alterations, works in all the cases, we will prove only four of the cases in detail.

For any nonzero real number w, let

$$\operatorname{sign}(w) := \begin{cases} 1 & \text{if } w > 0, \\ -1 & \text{if } w < 0 . \end{cases}$$

Case 1) $h(0) > 0$, $x > 0$, and $-\beta$ is between two consecutive, negative zeros of h: If γ is a negative zero of h, then $\operatorname{sign}(f(\gamma)) = \operatorname{sign}(g(\gamma))$ if $\gamma < -\beta$, and $\operatorname{sign}(f(\gamma)) = -\operatorname{sign}(g(\gamma))$ if $\gamma > -\beta$. Thus we have a zero of f between any two consecutive, negative zeros γ_1, γ_2 of h, unless $-\beta$ is between γ_1 and γ_2. Under this assumption, since $x > 0$ and $-x\beta + \zeta < 0$, the interlacing hypothesis on g forces $\operatorname{sign}(h(-\beta)) = -\operatorname{sign}(g(\gamma_2))$ and $\operatorname{sign}(h(-\beta)) = \operatorname{sign}(g(\gamma_1))$. This implies $\operatorname{sign}(f(-\beta)) = -\operatorname{sign}(f(\gamma_i)), i = 1, 2$. Hence f has (at least) two zeros between γ_1 and γ_2 and we have now obtained $\deg(h) + 1$ real zeros for f.

Case 2) $h(0) > 0$, $x > 0$, and $-\beta$ is larger than all the zeros of h: At all the zeros of h, $\operatorname{sign}(f) = \operatorname{sign}(g)$. Thus between any two consecutive zeros of h, f has at least one zero. Since $h(-\beta) > 0$,

we have $f(-\beta) < 0$, and thus f has at least one zero larger than the largest zero of h. This already gives us $\deg(h)$ real zeros for f.
Case 3) $h(0) = 0$, $x < -b/a$: At a positive zero of h, $\operatorname{sign}(f) = \operatorname{sign}(g)$. The assumption on x implies the leading coefficient of f is negative, and we get the required number of real zeros for f by interlacing, together with degree considerations.
Case 4) $x > 0$, all the zeros γ of h satisfy $-\beta \leq \gamma \leq 0$, and $h(0) = 0$: We have $f(0) = 0$ and $g(0) \geq 0$. Let γ denote the largest negative zero of h. The interlacing hypothesis implies $g(\gamma) < 0$, and hence $f(\gamma) > 0$. This forces f to have two zeros larger than γ, at least one of which is 0. Extending this argument we get a zero of f between any two consecutive zeros of h. Since the degree of f is one more than the degree of h and their leading coefficients are both positive, we also get a zero of f which is less than the smallest zero of h, which completes the proof. □.

We will also make use of another method of Brenti for proving polynomials have only real zeros [Bre,p.43].

Theorem 3.7. *(Brenti) Let $f(x) = \sum_{k=0}^{n} \binom{x+k}{n} b_k$ be a polynomial with only real zeros, with smallest zero $\lambda(f)$ and largest zero $\Lambda(f)$. If all the integers in the intervals $[\lambda, -1]$ and $[0, \Lambda]$ are also zeros of f, then all the zeros of $\sum_{k=0}^{n} b_k x^k$ are real.*

In [HOW] Theorem 1.1 was proven as follows; use (1), the definition of the hit numbers $t_j(B)$, and the binomial theorem to rewrite the LHS of (3) as

$$\sum_{j=0}^{n} \binom{x+j}{n} t_j(A).$$

It is not hard to show that the RHS of (3) satisfies the conditions on $f(x)$ in the statement of Theorem 3.7, and Theorem 1.1 follows.

4. Results on Conjecture 2.3

Definition. *Let G be a threshold graph. Viewing G as a shifted Ferrers board, let $\alpha = \alpha(G)$ denote the rightmost "corner square" of G, that is the lowest square in the rightmost nonzero column of G. Furthermore let G_α denote the threshold graph obtained by removing square α from G, and let $G/\overline{\alpha}$ denote the threshold graph obtained by deleting both of the hooks that contain α from G, and collapsing the remaining graph. If G is the empty graph with n vertices, let $G/\overline{\alpha}$ denote the empty graph on $n-2$ vertices.*

Theorem 4.1. *Let n be an even integer. Then for any threshold graph G, $Hf(G + zJ_n)$ has only real zeros, and is interlaced by $Hf(G/\overline{\alpha} + zJ_{n-2})$.*

Proof: Note that

$$Hf(G + zJ_n) = \sum_{k=0}^{n/2} z^{n/2-k} m_k(G)(n-2k)!!.$$

Abbreviate $Hf(G + zJ_n)$ by $H(z; G)$.

The proof is by induction on ordered pairs (n, # of squares of G). If G is the empty board on n vertices, the theorem follows trivially. So assume G is nonempty.
Case 1) α is not in the top row.
By grouping matchings according to whether edge α is used or not we see that

$$m_k(G) = m_k(G_\alpha) + m_{k-1}(G/\overline{\alpha}).$$

Multiplying by $z^{n/2-k}(n-2k)!!$ and summing over k we get

$$H(z;G) = H(z;G_\alpha) + \sum_{k\geq 1} z^{(n-2)/2-(k-1)} m_{k-1}(G/\overline{\alpha})(n-2-2(k-1))!!$$

or

$$H(z;G) = H(z;G_\alpha) + H(z;G/\overline{\alpha}).$$

If β is the square just above α, i.e. $\beta = \alpha(G_\alpha)$, then $G_\alpha/\overline{\beta} = G/\overline{\alpha}$. Thus by the induction hypothesis, $H(z;G/\overline{\alpha})$ interlaces $H(z;G_\alpha)$. Lemma 3.5 now implies $H(z;G/\overline{\alpha})$ interlaces $H(z;G)$.
Case 2) α is in the top row.
To create a k-edge matching of G, we can either choose no edges in the top row in $m_k(G/\overline{\alpha})$ ways or choose $k-1$ edges below the top row in $m_{k-1}(G/\overline{\alpha})$ ways, each of which eliminates $2(k-1)$ edges in the top row. Thus

$$m_k(G) = m_k(G/\overline{\alpha}) + (b_{n-1}(G) - 2(k-1))m_{k-1}(G/\overline{\alpha}).$$

Multiplying by $z^{n/2-k}(n-2k)!!$ and summing over k we get

$$H(z;G) = \sum_{k\geq 0} z^{n/2-k} m_k(G/\overline{\alpha})(n-2k)!! + \sum_{k\geq 1} z^{(n-2)/2-(k-1)} m_{k-1}(G/\overline{\alpha})(b_{n-1}(G) - 2(k-1))(n-2-2(k-1))!!$$

$$= z\sum_{k\geq 0} z^{(n-2)/2-k} m_k(G/\overline{\alpha})(n-2-2k)!!(n-2k-1) + \sum_{j\geq 0} z^{(n-2)/2-j} m_j(G/\overline{\alpha})(b_{n-1}(G) - n + 2 + n - 2 - 2j))(n-2-2j)!!$$

$$= z\sum_{k\geq 0} z^{(n-2)/2-k} m_k(G/\overline{\alpha})(n-2-2k)!!(n-2k) - zH(z;G/\overline{\alpha}) + (b_{n-1}(G) - n + 2)H(z;G/\overline{\alpha}) + \sum_{j\geq 0} z^{(n-2)/2-j} m_j(G/\overline{\alpha})(n-2-2j)(n-2-2j)!!$$

$$= 2z\frac{d}{dz}zH(z;G/\overline{\alpha}) + (-z + b_{n-1}(G) - n + 2)H(z;G/\overline{\alpha}) + 2z\frac{d}{dz}H(z;G/\overline{\alpha})$$

$$= H(z;G/\overline{\alpha})(z + b_{n-1}(G) - n + 2) + 2z(z+1)\frac{d}{dz}H(z;G/\overline{\alpha}).$$

By induction we can assume that $H(z;G/\overline{\alpha})$ has only real zeros, which by Rolle's Theorem is interlaced by $H'(z;G/\overline{\alpha})$. Any real zero of $H(z;G/\overline{\alpha})$ must be between -1 and 0 (otherwise all the terms in the Hafnian have the same sign). This and the fact that $b_{n-1}(G) \leq n-1$ allow us to apply Lemma 3.6 with $h = H(z;G/\overline{\alpha})$ and $\beta = 1$ to conclude that $H(z;G/\overline{\alpha})$ interlaces $H(z;G)$. □

Corollary 4.2. *Theorem* 2.2 *is true.*

Remark: Theorem 3.7 can be applied to the RHS of (4) with the result that for any threshold graph G on n vertices

$$\sum_{k\geq 0} m_k(G)(n-1-k)!z^k \tag{5}$$

has only real zeros. The following proposition shows that Theorem 2.2 is a stronger statement.

Proposition 4.3. *For $m \in \mathbb{P}$,*

$$\gamma_k = \frac{(2m-1-k)!}{(2m-2k)!!}, \qquad k \geq 0$$

is a degree-m factor sequence.

Proof: Let n be an even integer and set

$$F_n(z) := \sum_{k=0}^{n/2} m_k(K_n)(n-1-k)!z^k.$$

Now

$$F_n(z) = \sum_{k=0}^{n/2} \binom{n}{2k} \frac{(2k)!}{2^k k!}(n-1-k)!z^k$$

(since there are $(2k)!!$ k-edge matchings involving a given subset of $2k$ vertices)

$$= \frac{2^{-n/2}}{(n/2)!} n! \sum_{k=0}^{n/2} \binom{n/2}{k} \frac{2^{n/2-k}(n/2-k)!(n-1-k)!}{(n-2k)!} z^k.$$

By (5) $F_n(z)$ has only real zeros, which can also be shown by expressing $F_n(z)$ as a terminating ${}_2F_1$ hypergeometric series and using known facts about ultraspherical polynomials [Rai]. Thus

$$\sum_{k=0}^{n/2} \binom{n/2}{k} \frac{(n-1-k)!}{(n-2k)!!} z^k$$

has only real zeros. Proposition 4.3 now follows from Theorem 3.1 (d). $\square$

We now prove another special case of Conjecture 2.3. If $w_{ij} = x_i x_j$ for some set of nonnegative reals $x_1, x_2, \ldots, x_n$, after a short calculation $Hf(zW + J_n)$ becomes

$$\sum_{k=0}^{[n/2]} e_{2k}(X)(2k)!!(n-2k)!!z^k, \tag{6}$$

where $e_k(X)$ is the kth elementary symmetric function of the x_i. By Lemma 3.4, (6) will have only real zeros provided

$$\sum_{k=0}^{n/2} \binom{n}{2k} (2k)!!(n-2k)!!z^k$$

has only real zeros, which is the $G = K_n$ case of Theorem 2.2.

Remark: By starting with (6), applying the $m = 2$ case of Corollary 3.3, Theorem 3.1 (a), and Corollary 3.3 again, we get the known result [Bre, p.10], that under the assumptions of Lemma 3.4

$$\sum_k z^k b_{2k}$$

has only real zeros.

Theorem 4.4. *Conjecture* 2.3 *is true if* $n \leq 4$.

Proof: From the definition of $Hf(W + zJ_n)$ we can assume $w_{ij} \geq 0$, since we can replace z by $z + c$ for some $c > 0$ if necessary. If $n = 2$, the conjecture is trivial. If $n = 4$, we consider the various possible linear orderings on the w_{ij} for K_4-monotone W. One such ordering is

$$0 \leq w_{14} \leq w_{24} \leq w_{34} \leq w_{23} \leq w_{13} \leq w_{12}. \tag{7}$$

Set $w_{14} = t_1$, $w_{24} = t_1 + t_2$, $w_{34} = t_1 + t_2 + t_3$, $\ldots$, and $w_{12} = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$. Then (7) is equivalent to $t_i \geq 0$ for $1 \leq i \leq 6$. A short calculation shows that the discriminant of $Hf(zW + J_4)$ can be written in the form

$$(t_2 + 2t_3 - 2t_5 - t_6)^2 + 12t_3t_5,$$

which is clearly nonnegative for $t_i \geq 0$. There are five other linear orderings which produce different Hafnians, and for each of these there is a simple expression for the discriminant which is clearly nonnegative. $\square$

Definition. *For* $p \in \mathbb{P}$, *we call an array* B *of the following shape*

$$\begin{array}{ccccccccccc}
b_{1,2}(1) & b_{1,2}(2) & \ldots & b_{1,2}(p) & b_{1,3}(1) & \ldots & b_{1,3}(p) & \ldots & b_{1,n}(1) & \ldots & b_{1,n}(p) \\
 & & & & b_{2,3}(1) & \ldots & b_{2,3}(p) & \ldots & b_{2,n}(1) & \ldots & b_{2,n}(p) \\
 & & & & & & & \vdots & \vdots & & \vdots \\
 & & & & & & & & b_{n-1,n}(1) & \ldots & b_{n-1,n}(p)
\end{array}$$

a "p-shifted Ferrers board" if $b_{ij} \in \{0, 1\}$ *and* B *is weakly decreasing across rows and down columns. Note that a* 1*-shifted Ferrers board is just a shifted Ferrers board. Furthermore let* $r_k(B)$ *denote the number of ways of placing* k *nonattacking rooks on* B, *where a rook on the square with weight* $b_{ij}(s)$ *attacks everything in its row and column, and the squares with weights* $b_{k,i}(m)$, $1 \leq k < i$, $1 \leq m \leq p$.

The proof of Theorem 2.1 can be generalized in a straightforward way to obtain Theorem 4.5 below.

Theorem 4.5. *Let* B *be a p-shifted Ferrers board. Then*

$$\sum_{k=0}^{n-1} x(x-1)\cdots(x-k+1)\frac{r_{n-1-k}(B)}{(p+1)^{n-1-k}} = \prod_{i=1}^{n-1}(x + b_i(B)/(p+1) - i + 1),$$

where $b_i(B)$ *is the length of the ith row (starting from the bottom and counting up) of* B.

Corollary 4.6. *Let* B *be a p-shifted Ferrers board. Then*

$$\sum_{k \geq 0} k! z^k r_{n-1-k}(B)$$

has only real zeros.

Proof: Apply Theorem 3.7 to Theorem 4.5. $\square$

Unfortunately, the author doesn't know of any natural combinatorial objects counted by these rook placements if $p > 1$.

5. Results on Conjecture 2.6

In this section we prove Theorem 2.5, as well as other special cases of Conjecture 2.6. We begin with several definitions and lemmas, which are used in proving Theorem 5.4, which contains part (c) of Theorem 2.5. We then use a different method to prove Theorem 5.6, which implies the other parts of Theorem 2.5.

Definition. *Let $A(h_1, d_1; \ldots ; h_m, d_m)$ denote the Ferrers board of Fig. 5, with $c_i(A)$ the height of the ith column. If $h_p > 0$ for some p in the range $1 \leq p \leq m$, let A_p stand for the Ferrers board obtained from A by decreasing d_p and h_p by one each. Furthermore let*

$$PR(w, y; A) := \prod_{i=1}^{n} (w + c_i(A) - i + s_i(A)).$$

(PR stands for product). Abbreviate $h_1 + h_2 + \ldots + h_i$ and $d_1 + d_2 + \ldots + d_i$ by H_i and D_i, respectively. Set

$$\tau_p(A) := H_p - D_{p-1} + s_u(A),$$

where $u := D_{p-1} + 1$.

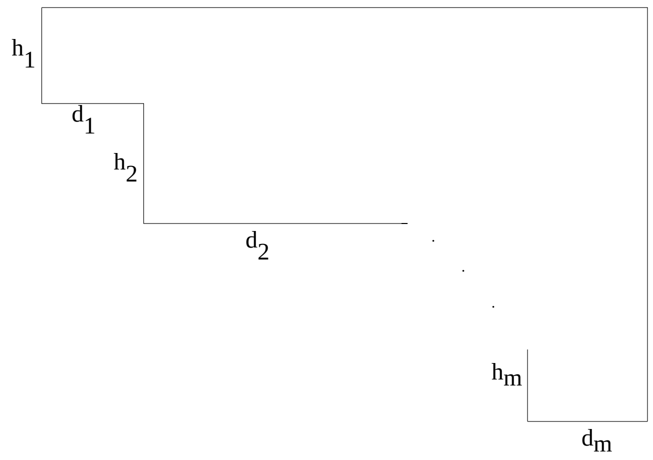

Figure 5. The Ferrers board $A(h_1, d_1; \ldots ; h_m, d_m)$, where $d_i \in \mathbb{P}, h_i \in \mathbb{N}$ for $1 \leq i \leq m$. The first d_1 columns have height h_1, the next d_2 have height h_2, etc..

The following observation, which occurs in [Dwo] and [EHR], will prove useful in the sequel.

Observation 5.1. *Let B be a Ferrers board whose nth column has height c_n. Let $\hat{B}$ be the Ferrers board obtained by deleting the nth column of B. Let $\hat{C}$ be any placement of nonattacking rooks on the squares of $\hat{B}$. Then if $c_n \geq n$, there is exactly one square β in the nth column of B which satisfies*

a) β is not attacked by any of the rooks in $\hat{C}$

and

b) the placement C consisting of $\hat{C}$ union a rook on β satisfies $cyc(C) = cyc(\hat{C}) + 1$.

If $c_n < n$, there is no such square.

The following lemma is motivated by a result of Simion [Sim], who used a similar technique to prove that polynomials whose coefficients count compositions of multisets by number of parts have only real zeros. (These polynomials are special cases of hit polynomials, a fact which led to the discovery of Theorem 1.1).

Lemma 5.2. *Let $x, y \in \mathbb{R}$ and let A be the Ferrers board of Fig. 5. Assume $h_p > 0$ for some p in the range $1 \leq p \leq m$. Then*

$$T(x,y,z;A) = z^{\tau_p(A)-x}(1-z)^{n+x}\frac{d}{dz}\left[z^{1+x-\tau_p(A)}(1-z)^{1-n-x}T(x,y,z;A_p)\right]. \tag{8}$$

Proof: The following result can be easily derived by combining Observation 5.1 with the same combinatorial technique used in [GJW] to prove (3):

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(y;A) = PR(x,y;A). \tag{9}$$

See [Dwo] for conditions on when the LHS of (9) factors for boards obtained by permuting the columns of a Ferrers board. Using (9), the $j=n$ case of [Hag2,Thm. 2.2] reduces to

$$T(x,y,z;A) = z^{-x}(z-1)^{n+x}\sum_{k=0}^{\infty}\binom{x+k-1}{k}PR(k,y;A)z^{-k},$$

and

$$T(x,y,z;A_p) = z^{-x}(z-1)^{n-1+x}\sum_{k=0}^{\infty}\binom{x+k-1}{k}PR(k,y;A_p)z^{-k}.$$

Since

$$PR(k,y;A) = PR(k,y;A_p)(\tau_p(A)-1+k),$$

we get

$$T(x,y,z;A) = z^{-x}(z-1)^{n-1+x}(z-1)\sum_{k=0}^{\infty}\binom{x+k-1}{k}z^{-k}PR(k,y;A_p)(\tau_p(A)-1+k)$$

$$\begin{aligned}
&= z^{-x}(z-1)^{n-1+x}(z-1)\sum_{k=0}^{\infty}\binom{x+k-1}{k}z^{-k}PR(k,y;A_p)(\tau_p(A)-1)\\
&\qquad - z^{-x}(z-1)^{n-1+x}z(z-1)\frac{d}{dz}\sum_{k=0}^{\infty}\binom{x+k-1}{k}z^{-k}PR(k,y;A_p)
\end{aligned}$$

$$= (\tau_p(A)-1)(z-1)T(x,y,z;A_p) - z(z-1)z^{-x}(z-1)^{n+x-1}\frac{d}{dz}\left[T(x,y,z;A_p)z^{x}(z-1)^{1-x-n}\right]$$

$$\begin{aligned}
&= (\tau_p(A)-1)(z-1)T(x,y,z;A_p) - z^{1-x}(z-1)^{n+x}[T'(x,y,z;A_p)z^{x}(z-1)^{1-x-n}\\
&\qquad + T(x,y,z;A_p)xz^{x-1}(z-1)^{1-x-n} + T(x,y,z;A_p)z^{x}(1-x-n)(z-1)^{-x-n}]
\end{aligned}$$

$$= T'(x,y,z;A_p)z(1-z) + T(x,y,z;A_p)\left[(z-1)(\tau_p(A)-1) - x(z-1) - z(1-x-n)\right]$$

$$= T'(x,y,z;A_p)z(1-z) + T(x,y,z;A_p)\left[z(\tau_p(A)-1-x-1+x+n) - \tau_p(A)+1+x\right] \tag{10}$$

which is exactly what we get by performing the differentiation on the RHS of (8). $\square$

Lemma 5.3. *Let A be the Ferrers board of Fig. 5. Assume $y > 0$ and $(x)_n \neq 0$. Then as a polynomial in z,*

$$deg(T(x,y,z;A)) = deg(T(x,y,z;A_p)) + 1.$$

Proof : Let $\mathrm{Max}(A) = \max\{k : r_k(y;A) \neq 0\}$, which equals $\max\{k : r_k(1;A) \neq 0\}$ since $y > 0$. Now $(x)_n \neq 0$ implies $\deg(T(x,y,z;A)) = \mathrm{Max}(A)$ and $\deg(T(x,y,z;A_p)) = \mathrm{Max}(A_p)$. The $q = 1$ case of [Hag1,Theorem 4.38] says

$$r_k(1;A) = (n + H_p - D_{p-1} - k)r_{k-1}(1;A_p) + r_k(1;A_p)$$

which implies

$$r_{\mathrm{Max}(A_p)+2}(1;A) = 0.$$

On the other hand, after placing a rook on the corner of the rectangular region of A where the border segments of lengths h_p and d_p meet, the unattacked portion of A remaining has shape A_p, which shows

$$r_{\mathrm{Max}(A_p)+1}(1;A) \neq 0. \quad \square$$

In order to prove Theorem 5.4, we also require the following result of Gessel [Ges2] (see also [Hag2, eq. (21)]), which holds for any admissible board A;

$$T(y,y,z;A) = \sum_{\substack{C \\ n \text{ rooks on } A}} z^{\#\text{ of rooks on nonzero squares of } A}\, y^{\mathrm{cyc}(C)}. \tag{11}$$

Theorem 5.4. *Let A be the Ferrers board of Fig. 5; assume $y > 0$ and A is admissible. Then as a polynomial in z, $T(y,y,z;A)$ has only real, nonpositive zeros. All the zeros are simple, except possibly for a multiple zero at 0. Furthermore, $T(y,y,z;A)$ is interlaced by $T(y,y,z;A_p)$ for those p for which $h_p > 0$.*

Proof: We proceed by induction on the number of zeros of $T(y,y,z;A)$. If $T(y,y,z;A)$ is constant, then by Lemma 5.3, A_p is undefined for all p and Theorem 5.4 is vacuously true.

If $\deg(T(y,y,z;A)) = 1$ then $T(y,y,z;A_p)$ is constant. Either $m = 1, p = 1, h_1 = 1$, and $d_1 = n$, or $m = 2$, p=2, and for some $1 \leq j < n$, $h_1 = 0, d_1 = n-j, h_2 = 1$, and $d_2 = j$. In the case $m = 1$

$$T(y,y,z;A) = (z-1)(y)_{n-1}(n-1+y) + (y)_n = (y)_n z.$$

In the case $m = 2$,

$$T(y,y,z;A) = (z-1)(y)_{n-1}j + (y)_n.$$

This has a nonpositive zero if and only if $y \geq j + 1 - n$, which follows since j is at most $n-1$ and $y > 0$.

We now assume by induction that $T(y,y,z;A_p)$ has l real nonpositive zeros, with $l > 0$, and that these zeros are distinct except possibly for a zero, of multiplicity say β, at zero. Writing $T(y,y,z;A_p)$ as $z^\beta F(z)$ and performing the differentiation on the RHS of (8) we get

$$T(y,y,z;A) = z^\beta(1 + x - \tau_p + \beta)F(z) + z^{\beta+1}G(z)$$

for some polynomial G. Thus the order of the zero of $T(y,y,z;A)$ at 0 is at least β.

Applying Rolle's Theorem to the RHS of (8) we get a zero of $T(y,y,z;A)$ between any two negative zeros of $T(y,y,z;A_p)$. Let α be the smallest of the zeros of $T(y,y,z;A_p)$. By (11), no zero of $T(y,y,z;A_p)$ can be positive if $y > 0$. If $\alpha = 0$, by the previous paragraph at most one of the zeros of $T(y,y,z;A)$ is nonzero, thus all its zeros must be real. Hence we can assume $\alpha < 0$. From the $x = y$ case of (10), $T(y,y,\alpha;A) = \alpha(1-\alpha)T'(y,y,\alpha;A_p)$ which has the opposite sign of $T'(y,y,\alpha;A_p)$ (recall $T'(y,y,\alpha;A_p) \neq 0$ since the negative zeros are simple by the induction hypothesis). Note that the coefficient of the highest power of z in $T(y, y, z; A)$ is $(y)_{n-\mathrm{Max}(A)} r_{\mathrm{Max}(A)}(y; A)$, which is positive since $y > 0$, as is the coefficient of the highest power of z in $T(y, y, z; A_p)$. Combining these observations with Lemma 5.3 easily leads to the conclusion that $T(y, y, z; A)$ has a real zero less than α.

Let ζ be the largest of the nonzero zeros of $T(y, y, z; A_p)$. For any negative zero σ of $T(y, y, z; A_p)$, (10) implies $T(y, y, \sigma; A)$ and $T'(y, y, \sigma; A_p)$ have opposite signs, which shows that between any two consecutive, negative zeros of $T(y, y, z; A_p)$, $T(y, y, z; A)$ has an odd number of zeros (counting multiplicity). Similarly there are an odd number of zeros less than α. We have already accounted for all the zeros but one, which must therefore be real. Since we cannot have any positive zeros, the last zero γ must satisfy $\zeta < \gamma \leq 0$. $\square$

By making minor modifications to the proof of Theorem 5.4, one can also derive the following result. We omit the details, since it is a special case of Theorem 5.6 below (except for the interlacing condition and other minor hypotheses) which we will prove by a simpler method. (The author does not know how to prove Theorem 5.4 by this simpler method, hence Lemma 5.2 appears to be needed to prove part (c) of Theorem 2.5).

Theorem 5.5. *Let A be the Ferrers board of Fig. 5. Assume $y > 0$ and either*
(a) $\quad x > c_i(A) - i + s_i(A) \quad$ *for* $1 \leq i \leq n$
or
(b) $\quad x < 1 - n$.
Then as a polynomial in z, all the zeros of $T(x, y, z; A)$ are real and simple. Furthermore $T(x, y, z; A_p)$ interlaces $T(x, y, z; A)$ for those p for which $h_p > 0$. In case (a) all the zeros are negative; in case (b) they are all greater than 1.

We now introduce a general type of matrix for which we can obtain recurrences of a convenient type for the polynomial $T(x, y, z; A)$. Although they don't have a particularly elegant description, we state Theorem 5.6 in terms of these general matrices since this may provide some insight into why Conjecture 2.6 and the MCP Conjecture seem to hold, as the entries of these matrices satisfy inequalities on the columns.

Definition. *Let B be a Ferrers board with n columns. Let $V := v_{ij}$ be a $c_n(B) \times n$ matrix of nonnegative real numbers, weakly decreasing down columns. Assume there exist real numbers x_j such that for $2 \leq j \leq n$ and $1 \leq i \leq c_{j-1}(B)$, $v_{ij} = x_j$. Let $B(V)$ be the $c_n(B) \times n$ matrix $(B(V))_{ij} := b_{ij} v_{ij}$. An example of such a matrix is*

$$\begin{bmatrix} v_{11} & x_2 & x_3 & x_4 \\ v_{21} & x_2 & x_3 & x_4 \\ & v_{32} & x_3 & x_4 \\ & & & v_{44} \\ & & & v_{54} \end{bmatrix},$$

with $v_{11} \geq v_{21} \geq 0$, $x_2 \geq v_{32} \geq 0$, $x_3 \geq 0$, and $x_4 \geq v_{4,4} \geq v_{54} \geq 0$.

Theorem 5.6. *Let $B(V)$ be as above. Assume $y > 0$ and either*
(a) $x > c_i(B) - i + s_i(B)$ for $1 \leq i \leq n$
or
(b) $x < 1 - n$.
Then $T(x, y, z; B(V))$ has only real zeros.

Proof: Abbreviate $c_i(B)$ by c_i, and define α to be the sum

$$\alpha := \begin{cases} v_{c_{n-1}+1,n} + \ldots + v_{c_n,n} + x_n(c_{n-1} + 1) & \text{if } c_n < n, \\ v_{c_{n-1}+1,n} + \ldots + v_{c_n,n} + x_n(c_{n-1} + y) & \text{if } c_{n-1} \geq n, \\ v_{c_{n-1}+1,n} + \ldots + v_{c_n,n} + (y-1)v_{n,n} + x_n(c_{n-1} + 1) & \text{if } c_n \geq n \text{ and } c_{n-1} < n. \end{cases}$$

Let $\hat{B}$ be the matrix, with $n-1$ columns, obtained by deleting the nth column of $B(V)$. Let

$$H(z) := \sum_{k=0}^{n-1} (x)_{n-1-k} z^{n-1-k} r_k(y; \hat{B});$$

note that $H(z)$ has only real zeros if and only if $T(x, y, z; \hat{B})$ does.

Using Observation 5.1, the following recurrence is easily derived:

$$r_k(y; B(V)) = r_k(y; \hat{B}) + r_{k-1}(y; \hat{B})(\alpha - kx_n).$$

Thus

$$\begin{aligned} \sum_{k=0}^{n} (x)_{n-k} z^{n-k} r_k(y; B(V)) &= \sum_{k=0}^{n} (x)_{n-k} z^{n-k} (r_k(y; \hat{B}) + r_{k-1}(y; \hat{B})(\alpha - kx_n)) \\ &= z^{2-x} \frac{d}{dx}(z^x H(z)) + (\alpha - nx_n)H(z) + x_n z \frac{d}{dz} H(z) \\ &= (xz + \alpha - nx_n)H(z) + z(z + x_n)\frac{d}{dz}H(z). \end{aligned}$$

Theorem 5.6 will now follow by induction on n if we can show the hypotheses of Lemma 3.6 are satisfied with $\beta = x_n$, $\zeta = \alpha - nx_n$, $h = (-1)^{n-1}H(z)$, $g = (-1)^{n-1}H'(z)$, $a = (-1)^{n-1}(x)_{n-1}$, and $b = (-1)^{n-1}(x)_{n-1}(n-1)$ (to start the induction we need only note that if $n = 1$, $T(x, y, z; B(V))$ is a linear polynomial).

If $x < 1 - n (= -b/a)$, write $x = 1 - n - d$, $d > 0$. Clearly $(x)_k(-1)^k > 0$ for $0 \le k \le n$, and thus all the nonzero zeros of $H(z)$ have the opposite sign as x. We also need to show that $x_n + (\alpha - nx_n)/(n-1+d) \ge 0$. Since $0 \le v_{m,n} \le x_n$ for all m,

$$x_n + (\alpha - nx_n)/(n-1+d) \ge x_n + \frac{x_n}{n-1+d} \times \begin{cases} c_{n-1} + 1 - n & \text{if } c_n < n, \\ c_{n-1} + y - n & \text{if } c_{n-1} \ge n, \\ c_n - n & \text{if } c_n \ge n \text{ and } c_{n-1} < n. \end{cases}$$

In all these cases, the RHS is easily seen to be nonnegative.

If $x > c_1 - 1 + s_1$ then in particular $x > 0$. Now

$$(nx_n - \alpha)/x + x_n = \begin{cases} \frac{(n-c_{n-1}-1)x_n - (v_{c_{n-1}+1,n} + \dots + v_{c_n,n})}{x} + x_n & \text{if } c_n < n, \\ \frac{x_n(n-c_{n-1}-y) - (v_{c_{n-1}+1,n} + \dots + v_{c_n,n})}{x} + x_n & \text{if } c_{n-1} \ge n, \\ \frac{x_n(n-c_{n-1}-1) - (v_{c_{n-1}+1,n} + \dots + v_{c_n,n}) - yv_{n,n} + v_{n,n}}{x} + x_n & \text{if } c_n \ge n \text{ and } c_{n-1} < n. \end{cases}$$

$$\ge \begin{cases} x_n \frac{n-c_{n-1}-1-(c_n-c_{n-1})}{x} + x_n & \text{if } c_n < n, \\ x_n \frac{n-c_{n-1}-y-(c_n-c_{n-1})}{x} + x_n & \text{if } c_{n-1} \ge n, \\ x_n \frac{(n-c_{n-1}-1)-(c_n-c_{n-1}-1)-y}{x} + x_n & \text{if } c_n \ge n, \text{ and } c_{n-1} < n. \end{cases}$$

Using the hypotheses on x, the RHS above is easily seen to be positive in all three cases. $\square$

Corollary 5.7. *Theorem 2.5 is true.*

Proof: Let all the $v_{ij} = 1$ in Theorem 5.6 to prove parts (a) and (b). Part (c) follows from Theorem 5.4. $\square$.

Corollary 5.8. *For any Ferrers board B, the polynomial*

$$\sum_k r_k(y;B)z^k$$

has only real zeros if $y > 0$.

Proof: Set $y = x$ in Theorem 3.1 (b) and apply this to part (a) of Theorem 2.5 (with x sufficiently large). □

Example 5.9: Let B be the n-column Ferrers board with $c_i(B) = i$, $1 \leq i \leq n$. There is a well-known bijection between placements of $n-k$ nonattacking rooks on B and set partitions of $\{1,2,\ldots,n+1\}$ into $k+1$ blocks [Sta, p.75]. Under this bijection, a rook occupies square (i,i) if and only if i and $i+1$ are in the same block of the corresponding set partition. The number of such rooks is the number of cycles, so by the $V = J_n$ case of Theorem 5.6, if $y \geq 0$ and either $x \geq y$ or $x \leq 1-n$,

$$\sum_{k=0}^{n}(x)_k z^k \sum_{\substack{\pi \\ k+1 \text{ blocks}}} y^{\# \text{ of } i \text{ such that } i \text{ and } i+1 \text{ are in the same block}}$$

has only real zeros. Here the inner sum is over all set partitions π of $\{1,2,\ldots,n+1\}$ into $k+1$ blocks.

Another special case of Conjecture 2.6 we can prove, which doesn't follow immediately from Theorems 5.4, or 5.6, is when $A = J_n(X)$, where $(J_n(X))_{ij} := x_j, 1 \leq i,j \leq n$, for some sequence of (possibly negative) real numbers x_i. The proof starts by using Observation 5.1 and induction to obtain

$$r_k(y;J_n(X)) = (y+n-k)_k e_k(X).$$

Hence

$$T(x,y,z;J_n(X)) = \sum_{k=0}^{n}(x)_k e_{n-k}(X)(y+n-k)_{n-k}z^k = (y)_n \sum_{k=0}^{n} \frac{(x)_k}{(y)_k} e_{n-k}(X)z^k. \tag{12}$$

If $x_i = 1$ for all i, (12) has only real zeros for x and y satisfying the conditions of Conjecture 2.6 by the $B(V) = J_n$ case of Theorem 5.6 (this also follows from known properties of Jacobi polynomials [Rai, p.254-255]). Now letting $b_k = e_{n-k}(X)$ and $d_k = \binom{n}{k}(x)_k/(y)_k$ in Theorem 3.1 (d) completes the proof.

6. Computational Evidence

Conjectures 2.3 and 2.6 have been verified for over 50,000 matrices each, of various sizes ranging from 3×3 to 10×10, using Maple. In addition Conjecture 2.6 has been proven for $n \leq 2$ by the same method used to prove Theorem 4.4. Conjecture 2.3 is open for $n \geq 6$, while Conjecture 2.6 is open for $n \geq 3$. The $n = 3$ case of the MCP Conjecture has recently been proven by R. Mayer [May], but remains open for $n \geq 4$.

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