CYCLES AND PERFECT MATCHINGS

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Abstract. Fan Chung and Ron Graham (*J. Combin. Theory Ser. B* **65** (1995), 273-290) introduced the cover polynomial for a directed graph and showed that it was connected with classical rook theory. M. Dworkin (*J. Combin. Theory Ser. B* **71** (1997), 17-53) showed that the cover polynomial naturally factors for directed graphs associated with Ferrers boards. The authors (*Adv. Appl. Math.* **27** (2001), 438-481) developed a rook theory for shifted Ferrers boards where the analogue of a rook placement is replaced by a partial perfect matching of K_{2n}, the complete graph on $2n$ vertices. In this paper, we show that an analogue of Dworkin's result holds for shifted Ferrers boards in this setting. We also show how cycle-counting matching numbers are connected to cycle-counting "hit numbers" (which involve perfect matchings of K_{2n}).

Introduction

Let B_{2n} be the board pictured in Fig. 1.

Let (i,j) denote the square in the i-th row and j-th column of B_{2n}, so $B_{2n} = \{(i,j) : 1 \leq i < j \leq 2n\}$. Let K_{2n} denote the complete graph on vertices $\{1,2,\dots,2n\}$. A perfect matching of K_{2n} is a set of n edges of K_{2n} where no two edges have a vertex in common. Given a perfect matching m of K_{2n}, we let $p_m = \{(i,j) : i < j \text{ and } \{i,j\} \in m\}$. For example, if $m = \{\{1,4\},\{2,7\},\{3,5\},\{6,8\}\}$ is a perfect matching of K_8, then p_m is pictured in Fig. 2.

For a given board $B \subseteq B_{2n}$, we say that a subset $p \subseteq B$ is a *rook placement of B* if there is a perfect matching m of K_{2n} such that $p \subseteq p_m$. We let $M_k(B)$ denote the set of all k element perfect matchings of B and we call $m_k(B) = |M_k(B)|$ the *k-th rook number of B*. We let $F_{k,2n}(B) = \{p_m : |p_m \cap B| = k \text{ and } m \text{ is a perfect matching of } K_{2n}\}$. We call $f_{k,2n}(B) = |F_{k,2n}(B)|$ the *k-th hit number of B*. Haglund and Remmel [HR] proved the following relationship between the hit numbers and the rook numbers of a board $B \subseteq B_{2n}$.

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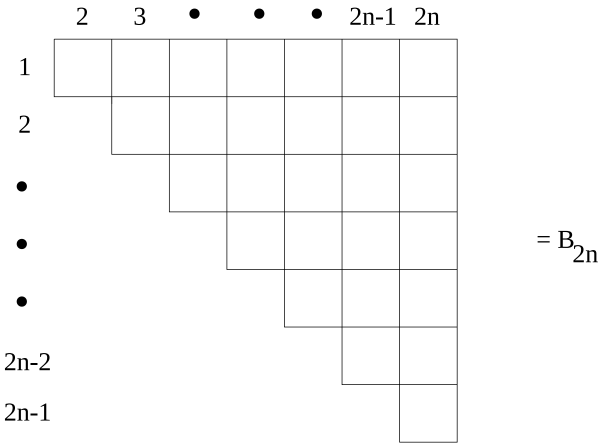

FIGURE 1

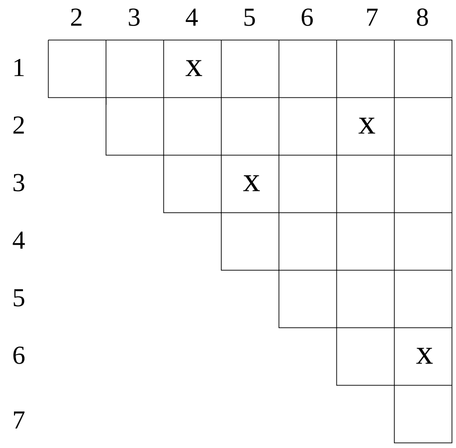

FIGURE 2

Theorem 1.

$$\sum_{k=0}^{n} m_k(B)(n-k)!!(z-1)^k = \sum_{k=0}^{n} f_{k,2n}(B)z^k \tag{1}$$

where $k!! = \prod_{i=1}^{k}(2i-1)$.

For any sequence $a_1, \ldots, a_{2n-1}$ such that $a_i \leq 2n-i$ for all i, we let $B(a_1, \ldots, a_{2n-1}) = \{(i, i+j) : 1 \leq j \leq a_i\}$. We say that $B(a_1, \ldots, a_{2n-1})$ is a Ferrers board if $2n-1 \geq a_1 \geq a_2 \geq \cdots \geq a_{2n-1} \geq 0$ and the nonzero entries of $a_1, \ldots a_{2n-1}$ are strictly decreasing. For example, $B(5,3,2,1,0,0,0) \subseteq B_8$ is pictured in Fig. 3.

Now Reiner and White [RW] proved that if $F = B(a_1, \ldots, a_{2n-1})$ is a Ferrers

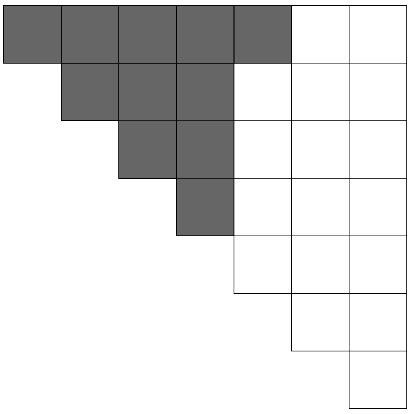

FIGURE 3

board contained in B_{2n}, then

$$\sum_{k=0}^{n} m_k(F)(x) \downdownarrows_{2n-1-k} = \prod_{i=1}^{2n-1} (x + a_{2n-i} - 2i + 2) \tag{2}$$

where $(x) \downdownarrows_k = x(x-2)(x-4)\cdots(x-2k+2)$. Haglund and Remmel [HR] defined a q-analogue of the rook numbers $m_k(B,q)$ for any board $B \subseteq B_{2n}$ and showed that if $F = B(a_1, \ldots, a_{2n-1}) \subseteq B_{2n}$ is a Ferrers board, then

$$\sum_{k=0}^{n} m_k(F,q)[x] \downdownarrows_{2n-1-k} = \prod_{i=1}^{2n-1} [x + a_{2n-i} - 2i + 2] \tag{3}$$

where $[n] = 1+q+\ldots+q^{n-1} = (1-q^n)/(1-q)$ and $[x] \downdownarrows_k = [x][x-2]\cdots[x-2k+2]$. In fact, Haglund and Remmel proved that (2) holds if F is a nearly Ferrers board with a_i squares in row i. Here a board N is nearly Ferrers if whenever $(i,j) \in N$, then $\{(s,i),(s,j) : s < i\}$ are also contained in N.

In this paper, we prove another extension of Reiner and White's formula. Given a board $B \subseteq B_{2n}$ and a placement $p \in M_k(B)$, we form a graph $G_{2n}(p) = (V_{2n}(p), E_{2n}(p))$ where the vertex set $V_{2n}(p) = \{1, \ldots, 2n\}$ and the edge set $E_{2n}(p) = \{\{2i-1, 2i\} : i = 1, \ldots, n\} \cup \{\{i,j\} : (i,j) \in p\}$. We note that $G_{2n}(p)$ may have multiple edges. That is, if for some i, $(2i-1, 2i) \in p$, then we shall think of $G_{2n}(p)$ as having two edges from $2i-1$ and $2i$ and we shall think of these edges as forming a cycle. For example if p is the placement pictured in Fig. 4, $G_{2n}(p)$ has two cycles, namely $(3,5,6,7,8,4)$ and $(1,2)$.

Note, however, that since no two edges of $\{\{i,j\} : (i,j) \in p\}$ share a common vertex, it follows that each vertex i of $G_{2n}(p)$ is contained in at most two edges and hence i can be a vertex of at most one closed path (cycle) of $G_{2n}(p)$. We let

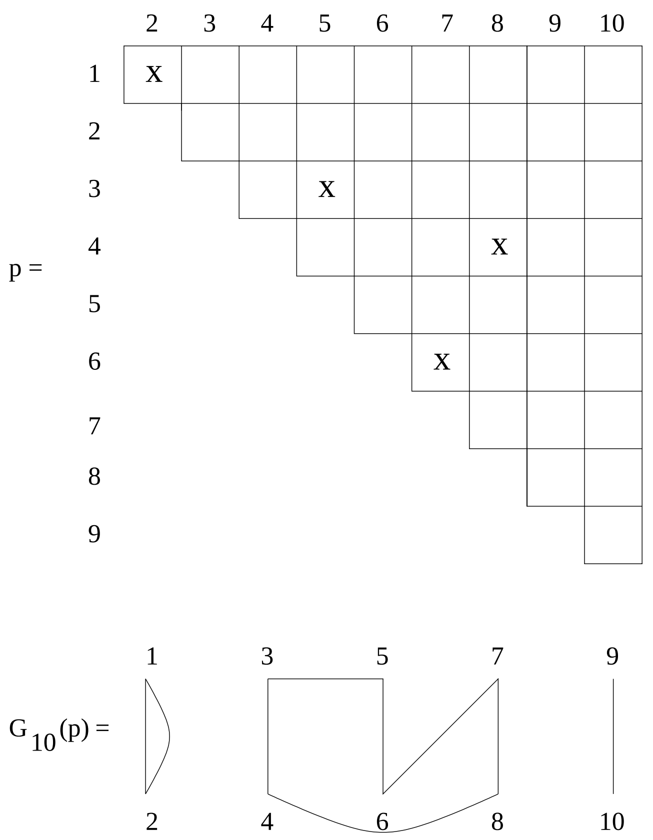

FIGURE 4

cy(p) denote the number of cycles of $G_{2n}(p)$. Note that if $B \subseteq B_{2n}$, then B is also contained in B_{2n+2}. However the only difference between $G_{2n}(p)$ and $G_{2n+2}(p)$ is that $G_{2n+2}(p)$ has an extra edge $\{2n+1, 2n+2\}$ which is disjoint from $G_{2n}(p)$. Thus the number of cycles of $G_{2n}(p)$ equals the number of cycles of $G_{2n+2}(p)$. Thus cy(p) depends only on p and not on n. We then let

$$cm_k(B, \alpha) = \sum_{p \in M_k(B)} \alpha^{\mathrm{cy}(p)} \tag{4}$$

and we call $cm_k(B, \alpha)$ the k-th cycle-rook number of B. For example, if $F = B(4, 2, 0, 0, 0)$, then $cm_2(F, \alpha) = 2 + 2\alpha$ as can be seen from Fig. 5 where we have pictured p and $G_6(p)$ for the four elements of $M_2(F)$.

Let $\sigma \in S_n$ be a permutation and let $m(\sigma)$ denote the perfect matching of K_{2n} consisting of edges $\{2i-1, 2\sigma_i\}$, $1 \leq i \leq n$. One easily verifies that $\mathrm{cy}(m(\sigma))$ equals the number of cycles of σ, so the function cy can be viewed as a generalization of the number of cycles of a permutation.

The major result of this paper is to prove the following factorization theorem.

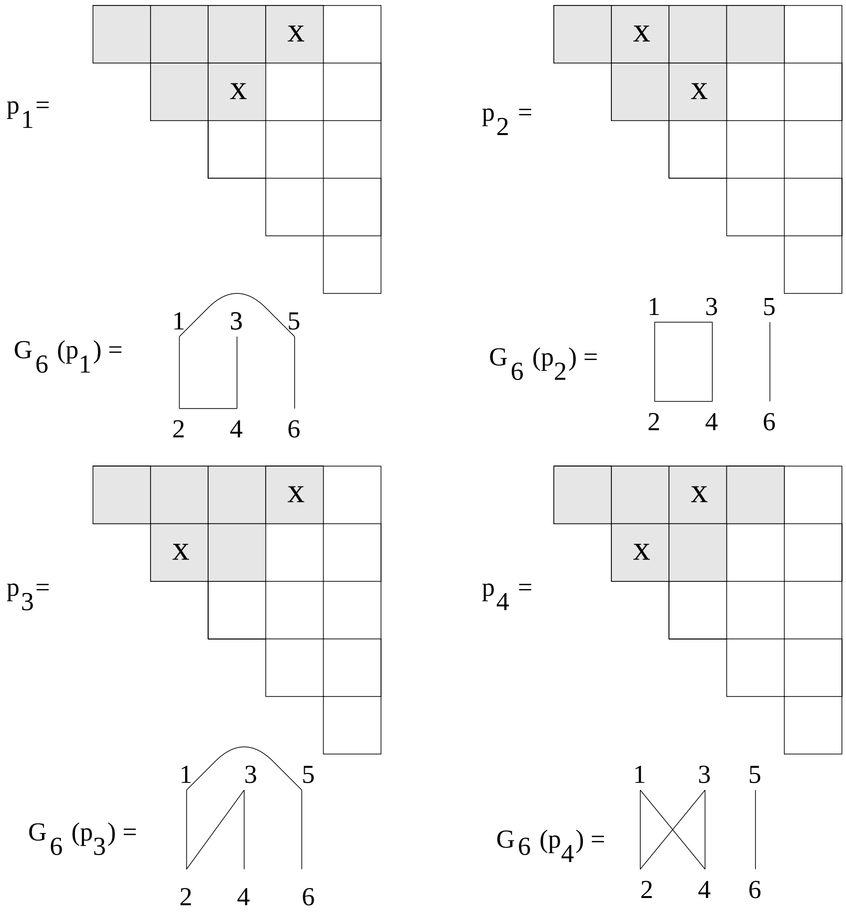

FIGURE 5

Theorem 2. *Let $B = B(a_1, \dots, a_{2n-1}) \subseteq B_{2n}$ be a Ferrers board. Then*

$$\sum_{k=0}^{n} cm_k(B,\alpha)(x) \downdownarrows_{2n-1-k} = \prod_{i=1}^{2n-1} (x + d_{2n-i}(B,\alpha) - 2i + 2) \tag{5}$$

where

$$d_{2n-1}(B,\alpha) = \begin{cases} 0 & \text{if } a_{2n-1} = 0 \\ \alpha & \text{if } a_{2n-1} = 1 \end{cases}$$

and for $j = 1, \dots, n-1$,

(i) $$d_{2j-1}(B,\alpha) = d_{2j}(B,\alpha) = 0 \text{ if } a_{2j-1} = a_{2j} = 0,$$

(ii) $d_{2j-1}(B,\alpha) = a_{2j-1} + \alpha - 1$ *and* $d_{2j}(B,\alpha) = a_{2j}$ *if* $a_{2j-1} > 0$
and it is not the case that both a_{2j-1} is even and $a_{2j-1} = a_{2j} + 1$,

and

(iii) $d_{2j-1}(B,\alpha) = a_{2j-1} + \alpha$ *and*

$$d_{2j}(B,\alpha) = a_{2j} - 1 \text{ if } a_{2j-1} > 0, a_{2j-1} \text{ is even, and } a_{2j-1} = a_{2j} + 1.$$

We note that when we set $\alpha = 1$ in (5), $cm_k(B,1) = m_k(B)$ and $d_k(B,1) = a_k$ unless $k \in \{2j-1, 2j\}$ for some j where $a_{2j-1} > 0$, a_{2j-1} is even, and $a_{2j-1} = a_{2j}+1$. However, in the latter case,

$$\begin{aligned}
(x + a_{2j} - 2(2(n-j)) + 2)(x + a_{2j-1} - 2(2(n-j)+1) + 2) &= \\
(x + a_{2j} - 4(n-j) + 2)(x + a_{2j-1} - 4(n-j)) &= \\
(x + (a_{2j-1}+1) - 4(n-j))(x + (a_{2j}-1) - 4(n-j) + 2) &= \\
(x + d_{2j-1}(B,1) - 4(n-j))(x + d_{2j}(B,1) - 4(n-j) + 2)&.
\end{aligned}$$

Thus (5) reduces to (2) when we set $\alpha = 1$.

In Section 2 we show (Theorem 3) that for certain special boards $cm_k(B,\alpha)$ has a compact expression as a product of linear factors in α. In Section 3 (Theorem 4) we derive a version of Theorem 1 involving the cm_k and cycle-counting versions of the f_k.

In [CG], Chung and Graham introduced the *cover polynomial* of a directed graph, which has interesting connections to rook theory. Let G be a bipartite graph on the sets of vertices $\{1, 2, \ldots, n\}$ and $\{1', 2', \ldots n'\}$. We can associate a directed graph $D(G)$ on n vertices to G by including an edge from u to v in $D(G)$ if and only if there is an edge between u and v' in G. To each k-edge matching p in G we associate the corresponding set $e(p)$ of k directed edges in $D(G)$, which will consist of a disjoint union of cycles and paths. With this in mind, the cover polynomial of $D(G)$ can be expressed as

$$\sum_{k=0}^{n} x(x-1)(x-2)\cdots(x-k+1) r_{n-k}(G,y),$$

where $r_j(G,y)$ is the sum, over all j-edge matchings p of G, of $y^{\mathrm{cy}(e(p))}$, where $\mathrm{cy}(e(p))$ is the number of cycles of $e(p)$.

Theorem 2 can be thought of as a "type B_n" analogue of a result of Dworkin [D]. He showed that the cover polynomial factors as a product of linear factors when the directed graph corresponds to a Ferrers board of classical shape. (He also showed that the cover polynomial sometimes factors when you permute the columns of a Ferrers board, an issue we will not address in our setting).

1. Proof of Theorem 2

We proceed with the proof of Theorem 2. Let $B_{2n,x}$ denote the board B_{2n} with x columns of height $2n-1$ added to the right of B_{2n}, as in Fig. 6.

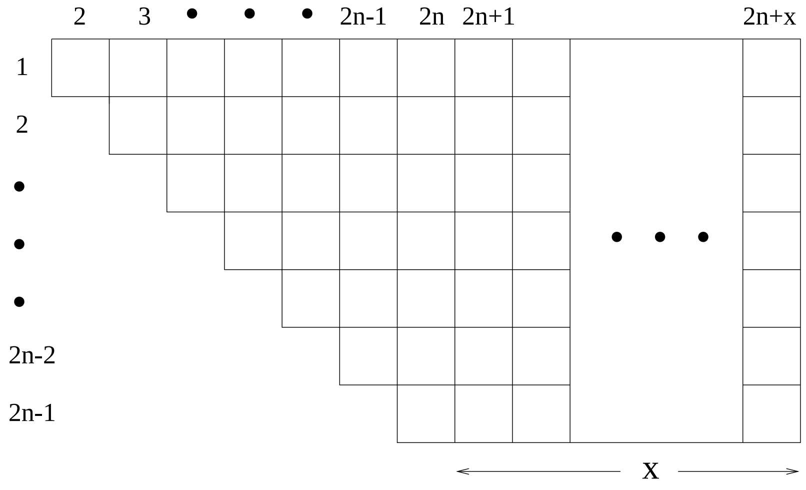

FIGURE 6: THE BOARD $B_{2n,x}$

We shall follow [HR] and consider the set of all placements of $2n-1$ nonattacking rooks in $B_{2n,x}$. That is, if a rook r is on square $(i,j) \in B_{2n}$, then r attacks all cells in row i and column j other than (i,j) plus all cells in $a^{2n}_{(i,j)} = \{(s,t) \in B_{2n} : |\{s,t\} \cap \{i,j\}| = 1\}$. However, if r is on cell $(i,j) \in B_{2n,x} - B_{2n}$, then the cells that r attacks relative to a rook placement p depends on the other rooks in $p \cap (B_{2n,x} - B_{2n})$. That is, if (i,j) is the position of the lowest rook r_1 in $p \cap (B_{2n,x} - B_{2n})$, then r_1 attacks all cells in row i and column j other than (i,j) plus all cells in column $j-1$ if $2n+1 < j$. If $j = 2n+1$, then r_1 attacks all cells in row i and column j plus all cells in column $2n+x$. In general, if (i,j) is the position of the k-th lowest rook r_k in $p \cap (B_{2n,x} - B_{2n})$, then r_k attacks all cells in row i and column j other than (i,j) plus all cells in the first column occurring in the following list of columns $j-1, j-2, \ldots, 2n+1, 2n+x, 2n+x-1, \ldots, j+1$ that contains a square which is not attacked by any of the $k-1$ lower rooks in $B_{2n,x} - B_{2n}$. Note that this means that each rook r in $p \cap (B_{2n,x} - B_{2n})$ will attack all cells in two columns of $B_{2n,x} - B_{2n}$. That is, if r is in cell (i,j), r attacks all cells in column j other than (i,j). It then looks for the first column $s > 2n$ to the left of column j which has a cell that is not attacked by a lower rook in $p \cap (B_{2n,x} - B_{2n})$. If there is no such column, then r starts at column $2n+x$ and looks for the rightmost column s which has a square which is not attacked by any lower rook in $p \cap (B_{2n,x} - B_{2n})$. Note we are guaranteed that such a column exists if $x > 4n-2$. Then r attacks all cells in column s as well. Our definition of a Ferrers board also ensures that each rook $r \in p$ that lies in B also attacks the squares in two columns of B which lie above r, namely, the squares in column i and column j. For example, consider the placement p pictured in Fig. 7 consisting of 3 rooks, $r_1 \in (7,10)$, $r_2 \in (5,11)$, and $r_3 \in (3,7)$. We have indicated all cells attacked by r_i by placing an i in such cells.

Now let B be a board contained in B_{2n} and assume that $x \geq 4n-2$. We let

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1		3				3		1	1	2						2
2		3				3		1	1	2						2
3			3	3	3	r 3	3	1,3	1,3	2,3	3	3	3	3	3	2,3
4						3		1	1	2						2
5					2	2,3	2	1,2	1,2	r 2	2	2	2	2	2	2
6						3		1	1	2						2
7							1,3	1	r 1	1,2	1	1	1	1	1	1,2

FIGURE 7

$\mathcal{N}_{2n,x}(B)$ denote the set of all placements p of $2n-1$ rooks in $B_{2n,x}$ such that no cell which contains a rook in p is attacked by another rook in p and any rook r in $B_{2n} \cap p$ is an element of B. We claim that (5) arises from two different ways of counting

$$\sum_{p \in \mathcal{N}_{2n,x}(B)} \alpha^{\mathrm{cy}(p \cap B)}. \tag{6}$$

Note that our definition ensures that if $p \in \mathcal{N}_{2n,x}(B)$, then $p \cap B \in M_k(B)$ where $k = |p \cap B|$ so that $\mathrm{cy}(p \cap B)$ is defined.

First suppose that we fix a rook placement $\tilde{p} \in M_k(B)$. We claim that the number of ways to extend $\tilde{p}$ to a rook placement $p \in \mathcal{N}_{2n,x}(B)$ such that $p \cap B = \tilde{p}$ is $(x)\downarrow\downarrow_{2n-1-k}$. That is, there are $2n-1-k$ rows in $B_{2n,x} - B_{2n}$ that have no squares that are cancelled by a rook in $\tilde{p}$. Say the rows are $1 \leq R_1 < \cdots < R_{2n-1-k} \leq 2n-1$. We then have x choices of where to put a rook r_{2n-1-k} in row $R_{2n-1-k} \cap (B_{2n,x} - B_{2n})$. Then r_{2n-1-k} will attack two squares in row $R_{2n-k} \cap (B_{2n,x} - B_{2n})$ so that once we have placed r_{2n-1-k}, we will have $x-2$ choices of where to place a rook r_{2n-k} in row $R_{2n-k} \cap (B_{2n,x} - B_{2n})$. Then r_{2n-1-k} and r_{2n-k} will attack a total of 4 squares in $R_{2n-k+1} \cap (B_{2n,x} - B_{2n})$ so that once we have placed r_{2n-1-k} and r_{2n-k}, we will have $(x-4)$ choices of where to place a rook r_{2n-k+1} in $R_{2n-k+1} \cap (B_{2n,x} - B_{2n})$. Continuing on in this way, it is easy to see that the number of such p is $(x)\downarrow\downarrow_{2n-1-k}$. Thus

$$\begin{aligned}\sum_{p \in \mathcal{N}_{2n,x}(B)} \alpha^{\mathrm{cy}(p \cap B)} &= \sum_{k=0}^{n} \sum_{\tilde{p} \in M_k(B)} \alpha^{\mathrm{cy}(\tilde{p})} (x)\downarrow\downarrow_{2n-1-k} \\ &= \sum_{k=0}^{n} cm_k(B)(x)\downarrow\downarrow_{2n-1-k}\,. \end{aligned} \tag{7}$$

Next consider the number of ways to place a rook r_{2n-1} in row $2n-1$. Clearly there are x choices to place a rook in $B_{2n,x} - B_{2n}$ that lie in row $2n-1$. If

$a_{2n-1} = 1$, then there is one additional choice namely placing the rook r_{2n-1} in square $(2n-1, 2n)$, then the edge $\{2n-1, 2n\}$ will complete a cycle in $G_{2n}(p \cap B)$ for any placement $p \in \mathcal{N}_{2n,x}(B)$ that has r_{2n-1} on cell $(2n-1, 2n)$. Thus the row $2n-1$ contributes a factor of x to (6) if $a_{2n-1} = 0$ and a factor of $(\alpha + x)$ to (6) if $a_{2n-1} = 1$.

Next for any $j \in \{1, \dots, n-1\}$, we want to consider the contribution of possible placements of the rooks in rows $2j-1$ and $2j$ to (6). That is, suppose that we fix a placement p' of nonattacking rooks $r_{2j+1}, r_{2j+2}, \dots, r_{2n-1}$ in rows $2j+1, 2j+2, \dots, 2n-1$ respectively. Note that

$$\prod_{i=1}^{2n-1} (x + d_{2n-i}(B,\alpha) - 2i + 2) = \prod_{j=1}^{2n-1} (x + d_j(B,\alpha) - 2(2n-1-j))$$

so that we must show that the contribution to (6) from the possible placements of the rooks in rows $2j-1$ and $2j$ is

$$(x + d_{2j}(B,\alpha) - (2n-1-2j))(x + d_{2j-1}(B,\alpha) - (2n-2j)).$$

Note that each of these rooks will attack two cells in $B \cup (B_{2n,x} - B_{2n})$ that lie in row $2j$ and two cells in $B \cup (B_{2n,x} - B_{2n})$ that lie in row $2j-1$. There are three cases.

<u>Case 1</u> $a_{2j-1} = a_{2j} = 0$

Note that if $a_{2j-1} = 0$, then $a_i = 0$ for all $2j-1 \le i \le 2n-1$. This means that all the rooks $r_{2j+1}, \dots, r_{2n-1}$ must lie in $B_{2n,x} - B_{2n}$. Thus for $i \in \{2j+1, \dots, 2n-1\}$, r_i attacks two cells in $B_{2n,x} - B_{2n}$ in row $2j$ and two cells in $B_{2n,x} - B_{2n}$ in row $2j-1$. Thus there are a total of $x - 2(2n-1-2j)$ cells in row $2j$ which are not attacked by a rook in $\hat{p}$ so that we have $x - 2(2n-1-2j) = (x + d_{2j}(B,\alpha) - 2(2n-1-2j))$ places to put rook r_{2j}. Once we have placed rook r_{2j}, it will attack two additional cells in $B_{2n,x} - B_{2n}$ which lie in row $2j-1$ so that we will have $(x - 2(2n-1-2j) - 2) = (x + d_{2j-1}(B,\alpha) - 2(2n-2j))$ ways to place a rook in $B_{2n,x} - B_{2n}$ which lies in row $2j-1$. Thus the contribution to (6) from the placements of rooks r_{2j-1} and r_{2j} in rows $2j-1$ and $2j$ is $(x + d_{2j-1}(B,\alpha) - 2(2n-2j))(x + d_{2j}(B,\alpha) - 2(2n-1-2j))$ in this case.

<u>Case 2</u> $a_{2j-1} > 0$ and it is not the case that both a_{2j-1} is even and $a_{2j-1} = a_{2j} + 1$.

In this case, there are a total of $x + a_{2j} - 2(2n-1-2j)$ cells of $B \cup (B_{2n,x} - B_{2n})$ which lie in row $2j$ and are not attacked by any rook in p'. Thus there are $(x + a_{2j} - 2(2n-1-2j)) = (x + d_{2j}(B,\alpha) - 2(2n-1-2j))$ ways to place the rook r_{2j}. Note that if r_{2j} is placed in B, say on cell $(2j, s)$, then $\mathrm{cy}(p' \cap B) = \mathrm{cy}((p' \cap B) \cup \{(2j, s)\})$. That is, the only difference between the graphs $G_{2n}(p' \cap B)$ and $G_{2n}((p' \cap B) \cup \{(2j, s)\})$ is that $G_{2n}((p' \cap B) \cup \{(2j, s)\})$ has an extra edge from $2j$ to s. However, by construction, there is no edge e in $G_{2n}((p' \cap B) \cup \{(2j, s)\})$ which involves vertex $2j-1$ other that the edge $\{2j-1, 2j\}$. That is, the only edges in $G_{2n}(p' \cap B)$ that are not of the form $\{2i-1, 2i\}$ must connect vertices from $\{2j+1, \dots, 2n\}$.

Thus adding the edge $\{2j, s\}$ to $G_{2n}(p' \cap B)$ cannot complete a cycle. Once we have placed r_{2j}, it will cancel 2 additional cells of $B \cup (B_{2n,x} - B_{2n})$ that lie in row $2j-1$. Thus there will be a total of $(x + a_{2j-1} - 2(2n-2j))$ cells of $B \cup (B_{2n,x} - B_{2n})$ which lie in row $2j-1$ which are not attacked by any of the rooks $r_{2j}, r_{2j+1}, \ldots, r_{2n-1}$. We claim there is exactly one way to place the rook r_{2j-1} to result in a placement p'' of the rooks $r_{2j-1}, r_{2j}, r_{2j+1}, \ldots, r_{2n-1}$ such that $\mathrm{cy}(p'' \cap B) = 1 + \mathrm{cy}(p' \cap B)$. That is, let p^* be the placement consisting of our rooks $r_{2j}, r_{2j+1}, \ldots, r_{2n-1}$ and consider the edges of $G_{2n}(p^* \cap B)$ that involve vertex $2j$. There is of course the edge $\{2j-1, 2j\}$. If there is another such edge, it must be of the form $\{2j, s_1\}$ with $s_1 > 2j$. Then s_1 is connected to s_2 by an edge in $G_{2n}(p^* \cap B)$ where $s_2 = s_1 - 1$ if s_1 is even or $s_2 = s_1 + 1$ if s_1 is odd. If there is another edge out of s_2, it must be of the form $\{s_2, s_3\}$ where $s_3 > 2j$ and s_3 will be connected to s_4 where $s_4 = s_3 - 1$ if s_3 is even and $s_4 = s_3 + 1$ if s_3 is odd. We can continue on in this way producing a sequence of edges $\{2j, s_1\}, \{s_1, s_2\}, \ldots, \{s_{2t-1}, s_{2t}\}$ in $G_{2n}(p^* \cap B)$ such that for all $1 \leq i \leq t$, $\{s_{2i-1}, s_{2i}\}$ is an edge of the form $\{2l-1, 2l\}$ and there is no edge other than $\{s_{2t-1}, s_{2t}\}$ which has s_{2t} as a vertex. Now let q be the maximum element of $s_1, \ldots, s_{2t}$. Clearly q must be even since whenever $2i-1 \in \{s_1, \ldots, s_{2t}\}$, $2i \in \{s_1, \ldots, s_{2t}\}$. Thus either there is an edge $\{i, q\}$ or $\{i, q-1\}$ in $G_{2n}(p^* \cap B)$ where $2j \leq i \leq q-2$. Since B is a Ferrers board, this means that $(2j, q)$ or $(2j, q-1)$ is in B. We claim that $(2j-1, s_{2t}) \in B$. That is, if $q > s_{2t}$, then $(2j-1, q-1) \in B$ since B is a Ferrers board and hence $(2j-1, s_{2t}) \in B$. If $q = s_{2t}$, then we know that $(2j-1, q-1) \in B$ since $(2j, q-1) \in B$. Now if $(2j-1, q) \notin B$, then it must be that B ends at column $q-1$ in rows $2j-1$ and $2j$. But then $a_{2j} = q-1-2j$ and $a_{2j-1} = q-1-(2j-1) = q-2j$. Thus if $(2j-1, q) \notin B$, then a_{2j-1} is even since q is even and $a_{2j-1} = a_{2j} + 1$ which we have explicitly ruled out. Hence in either case, we can conclude that $(2j-1, s_{2t}) \in B$. Note $(2j-1, s_{2t})$ is not attacked by any of the rooks $r_{2j}, r_{2j+1}, \ldots, r_{2n-1}$ since there is only one edge with vertex s_{2t} in $G_{2n}(p^* \cap B)$. Thus if we place the rook r_{2j-1} in $(2j-1, s_{2t})$, then we will complete a cycle $(2j, s_1, \ldots, s_{2t}, 2j-1)$ so that $\mathrm{cy}(p'' \cap B) = 1 + \mathrm{cy}(p^* \cap B) = 1 + \mathrm{cy}(p' \cap B)$. If we place r_{2j-1} in any other nonattacked square, we won't create a new cycle so that $\mathrm{cy}(p'' \cap B) = \mathrm{cy}(p^* \cap B) = \mathrm{cy}(p' \cap B)$. Thus in this case, the placement of the rook r_{2j-1} contributes a factor of $(x + a_{2j-1} - 1 + \alpha - 2(2n-1-2j) - 2) = (x + d_{2j-1}(B, \alpha) - 2(2n-2j))$ to (6). Of course, there may be no other edge in $G_{2n}(p^* \cap B)$ with vertex $2j$ other than $\{2j-1, 2j\}$. In this case, the only way to create a cycle is to place the rook r_{2j-1} in $(2j-1, 2j)$. Note that $(2j-1, 2j) \in B$ since $a_{2j-1} \geq 1$. Thus once again, the placement of the rook r_{2j-1} contributes a factor of $(x + a_{2j-1} - 1 + \alpha - 2(2n-2j))$ to (6).

It follows that in case 2, the possible placements of the rooks r_{2j} and r_{2j-1} contribute a factor of $(x + d_{2j}(B, \alpha) - 2(2n-1-2j))(x + d_{2j-1}(B, \alpha) - 2(2n-2j))$ to (6) as desired.

<u>Case 3</u> $a_{2j-1} > 0$, a_{2j-1} is even, and $a_{2j-1} = a_{2j} + 1$.

Note that in this case both rows $2j-1$ and $2j$ must end at column $2j-1+a_{2j-1}$ which is odd since a_{2j-1} is even. Thus let $2j-1+a_{2j-1} = 2r-1$.

The difference between case 2 and case 3 is that, in case 2, no matter how we placed the rook r_{2j} in row $2j$, there was one and only one way to place the rook r_{2j-1} in row $2j-1$ to complete a cycle. In case 3, there is one exception to this fact. That is and fix a placement $\bar{p}$ of nonattacking rooks $r_{2j+1}, \ldots, r_{2n-1}$ in rows $2j+1, \ldots, 2n-1$ respectively. Then consider the graph $G_{2n}(\bar{p} \cap B)$, and the vertex $2r-1$. There is of course one edge which has $2r$ as a vertex, namely $\{2r-1, 2r\}$. If there is another edge which has $2r-1$ as a vertex, then it must be of the form $(2r-1, t_1)$ where $t_1 \in \{2j+1, \ldots, 2r-2\}$. That is, since $(2j-1, 2r) \notin B$ and B is a Ferrers board, $(i, 2r) \notin B$ for any $i > 2j-1$ and hence $(i, s) \notin B$ for any $i \geq 2j-1$ and $s \geq 2r$. Thus in $G_{2n}(\bar{p} \cap B)$, the only edges involving the vertices $2r, \ldots, 2n$ are $\{2u-1, 2u\}$ for $u = r, \ldots, n$. Then t_1 is connected to t_2 where $t_2 = t_1 - 1$ if t_1 is even and $t_2 = t_1 + 1$ if $t_1$is odd. Now if there is another edge out of t_2 other than $\{t_1, t_2\}$, it must be of the form $\{t_2, t_3\}$ where $t_3 \in \{2j+1, \ldots, 2r-2\}$. Then there will be an edge out of t_3, namely $\{t_3, t_4\}$ where $t_4 = t_3 - 1$ if t_3 is even and $t_4 = t_3 + 1$ if t_3 is odd. We can continue on in this way to construct a sequence of edges $\{2r, 2r-1\}$, $\{2r-1, t_1\}$, $\{t_1, t_2\}, \ldots, \{t_{2q-1}, t_{2q}\}$ of $G_{2n}(\bar{p} \cap B)$ where for $i = 1, \ldots, q$, $\{t_{2i-1}, t_{2i}\}$ is an edge of the form $\{2u-1, 2u\}$. Note that $2r, 2r-1, t_1, \ldots, t_{2q}$ is not a cycle since the only edge involving $2r$ in $G_{2n}(\bar{p} \cap B)$ is $\{2r-1, 2r\}$. Moreover it must be the case that $t_1, \ldots, t_{2q} \subseteq \{2j+1, \ldots, 2r-2\}$ and that there is no edge out of t_{2q} other than $\{t_{2q-1}, t_{2q}\}$. It follows that $(2j, t_{2q})$ is not attacked by any rook in $\bar{p}$ and $(2j, t_{2q}) \in B$ since $t_{2q} \leq 2r-2$. Now if we place r_{2j} in cell $(2j, t_{2q})$ and construct the sequence of edges $\{2j, s_1\}, \{s_1, s_2\}, \ldots, \{s_{2t-1}, s_{2t}\}$ as described in case 2, then it is easy to see that $s_{2t-1} = 2r-1$ and $s_{2t} = 2r$. In this case, the only way to complete a cycle by the placement of r_{2j-1} in row $2j-1$ is to place r_{2j-1} in $(2j-1, 2r)$. But $(2j-1, 2r) \notin B$! Thus there is no way to complete a cycle by the placement of r_{2j-1} in $B \cup (B_{2n,x} - B_{2n})$. Similarly if there is no edge out of $2r-1$ other than $\{2r-1, 2r\}$ in $G_{2n}(\bar{p} \cap B)$, then by placing r_{2j} in $(2j, 2r-1)$, the sequence of edges $\{2j, s_1\}, \ldots, \{s_{2t-1}, s_{2t}\}$ constructed as in case 2 will simply be $\{2j, 2r-1\}, \{2r-1, 2r\}$ and once again there will be no way to place the rook r_{2j-1} in $B \cup (B_{2n,x} - B_{2n})$ to complete a cycle. If we do not place r_{2j} on cell $(2j, t_{2q})$, we can use the same argument that we used in case 2 to see that there is one and only one way to place the rook r_{2j-1} in $B \cup (B_{2n,x} - B_{2n})$ to complete a cycle. Hence there are $(x + a_{2j} - 2(2n-1-2j))$ ways to place rook r_{2j} in row $2j$. For all but one of them the factor contributed to (6) by the placement of the rook r_{2j-1} in row $2j-1$ is $(x + a_{2j-1} - 2(2n-2j) + \alpha - 1)$. For the other placement of r_{2j} in row $2j$, there is no way to place r_{2j-1} to complete a cycle so the placement of r_{2j-1} contributes a factor of $(x + a_{2j-1} - 2(2n-2j))$ to (6). Thus the total contribution to (6) caused by the placements of r_{2j} and r_{2j-1} in case 3 is

$$\begin{aligned}
(x + a_{2j} - 1 - 2(2n-1-2j))&(x + a_{2j-1} + \alpha - 1 - 2(2n-2j)) \\
&+ (x + a_{2j-1} - 2(2n-2j)) = \\
(x + a_{2j} - 1 - 2(2n-1-2j))&(x + a_{2j-1} + \alpha - 2(2n-2j)) \\
- (x + a_{2j} - 1 - 2(2n-1-2j)) &+ (x + a_{2j-1} - 2(2n-2j)) = \\
(x + a_{2j} - 1 - 2(2n-1-2j))&(x + a_{2j-1} + \alpha - 2(2n-2j)) \\
- (x + a_{2j} - 1 - 2(2n-1-2j)) &+ (x + a_{2j} + 1 - 2(2n-2j)) = \\
(x + a_{2j} - 1 - 2(2n-1-2j))&(x + a_{2j-1} + \alpha - 2(2n-2j)) = \\
(x + d_{2j}(B,\alpha) - 2(2n-1-2j))&(x + d_{2j-1}(B,\alpha) - 2(2n-2j)).
\end{aligned}$$

It follows that

$$\sum_{p \in \mathcal{N}_{2n,x}(B)} \alpha^{\mathrm{cy}(p \cap B)} = \prod_{i=1}^{2n-1} (x + d_{2n-i}(B,\alpha) - 2i + 2) \tag{8}$$

which combined with (7) proves Theorem 2. □

2. Special values of the cycle matching numbers

Let $D_k = \{(i,j) \in B_{2n} : j \leq k\}$. Thus D_k consists of the first k columns of B_{2n}. We can use Theorem 2 to prove the following.

Theorem 3. *For any* $2 \leq r \leq n$,

$$(i) \qquad cm_k(D_{2r},\alpha) = \begin{cases} \binom{r}{k}(\alpha + 2r - 2) \downarrow\downarrow_k & \text{for } 0 \leq k \leq r \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$(ii) \qquad cm_k(D_{2r-1},\alpha) = \begin{cases} \binom{r-1}{k}(\alpha + 2r - 2) \downarrow\downarrow_k & \text{for } 0 \leq k \leq r-1 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Proof: By our previous remarks preceding the definition of the k-th cycle-rook number of a board B, it is enough to compute $cm_k(B,\alpha)$ relative to the smallest n such that $B \subseteq B_{2n}$. Thus for fixed n, we need only prove our formulas for $D_{2n} = B_{2n}$ and D_{2n-1} which is the board that results from B_{2n} by removing the last column.

First we consider the case of B_{2n}. It is easy to see that $cm_k(B_{2n},\alpha)$ is a polynomial in α of degree k. That is, if $p \in M_k(B_{2n})$, then $G_{2n}(p)$ has k edges in addition to the edges $\{\{2i-1,2i\} : i = 1,\ldots,n\}$ that are in the graph of any placement. Thus we can form a maximum of k cycles with these extra k edges. Indeed, the only way to have k cycles in such a $G_{2n}(p)$ is to add a subset of k edges from $\{\{2i-1,2i\} : i = 1,\ldots,n\}$. That is, p must be of the form

$\{(2i_1-1,2i_1),\ldots,(2i_k-1,2i_k)\}$ where $1 \le i_1 < i_2 < \cdots < i_k \le n$. Since there are $\binom{n}{k}$ placements of this form, it follows that

$$cm_k(B_{2n},\alpha) = \binom{n}{k}\alpha^k + \sum_{j=0}^{k-1} a_{j,k}\alpha^j$$

for some nonnegative integers $a_{0,k},\ldots,a_{k-1,k}$. Thus to prove that $cm_k(B_{2n},\alpha) = \binom{n}{k}(\alpha+2n-2)\downarrow\downarrow_k$, we need only show that $(\alpha+2n-2)\downarrow\downarrow_k$ divides $cm_k(B_{2n},\alpha)$.

First observe that if $B = B_{2n}$ in Theorem 2, then $d_{2n-1}(B_{2n},\alpha) = \alpha$ and for $j = 1,\ldots,n-1$, $d_{2n-2j}(B_{2n},\alpha) = 2j$ and $d_{2n-(2j+1)}(B_{2n},\alpha) = 2j+\alpha$. Thus for $j = 0,\ldots,n-1$,

$$(x + d_{2n-(2j-1)}(B_{2n},\alpha) - 2(2j-1) + 2) = (x + \alpha - 2j)$$

and for $j = 1,\ldots,n-1$,

$$(x + d_{2n-2j}(B_{2n},\alpha) - 2(2j) + 2) = (x - 2j + 2).$$

Thus Theorem 2 gives that

$$\sum_{k=0}^{n} cm_k(B_{2n},\alpha)(x)\downarrow\downarrow_{2n-1-k} = (x)\downarrow\downarrow_{n-1}(x+\alpha)\downarrow\downarrow_n . \tag{11}$$

Dividing both sides of (11) by $(x)\downarrow\downarrow_{n-1}$ and then replacing x by $x+2n-2$ we get

$$\sum_{k=0}^{n} cm_k(B_{2n},\alpha)(x)\downarrow\downarrow_{n-k} = (x+\alpha+2n-2)\downarrow\downarrow_n . \tag{12}$$

Here $(x)\downarrow\downarrow_0 = 1$ by definition. We shall prove that $(\alpha+2n-2)\downarrow\downarrow_{n-k}$ divides $cm_{n-k}(B_{2n},\alpha)$ by induction on k. Setting $x = 0$ in (12) yields that

$$cm_n(B_{2n},\alpha) = (\alpha+2n-2)\downarrow\downarrow_n \tag{13}$$

which is the base step of our induction. Next assume that $(\alpha+2n-2)\downarrow\downarrow_{n-j}$ divides $cm_{n-j}(B_{2n},\alpha)$ for $j = 0,\ldots,k-1$. Then we know that $cm_{n-j}(B_{2n},\alpha) = \binom{n}{n-j}(\alpha+2n-2)\downarrow\downarrow_{n-j}$ for $j = 0,\ldots,k-1$. If we set $x = 2k$ in (12), we get

$$\sum_{i=0}^{k} cm_{n-i}(B_{2n},\alpha)(2k)\downarrow\downarrow_i = (2k+\alpha+2n-2)\downarrow\downarrow_n .$$

Solving for $cm_{n-k}(B_{2n},\alpha)$ yields

$$\begin{aligned} cm_{n-k}(B_{2n},\alpha) &= \frac{1}{(2k)\downarrow\downarrow_k} \\ &\times \left[(2k+\alpha+2n-2)\downarrow\downarrow_n - \sum_{i=0}^{k-1}\binom{n}{i}(\alpha+2n-2)\downarrow\downarrow_{n-i}(2k)\downarrow\downarrow_i\right]. \end{aligned} \tag{14}$$

Clearly $(2n-2+\alpha)\downarrow\downarrow_{n-k}$ divides the right-hand side of (14) and hence we can conclude that $cm_{n-k}(B_{2n},\alpha) = \binom{n}{n-k}(\alpha+2n-2)\downarrow\downarrow_{n-k}$. Thus by induction, $cm_j(B_{2n},\alpha) = \binom{n}{j}(\alpha+2n-2)\downarrow\downarrow_j$ for all j.

The proof of Theorem 3 for D_{2n-1} is almost the same. That is, $cm_n(D_{n-1},\alpha)=0$ since any placement $p \in M_n(B_{2n})$ must have a rook in the last column of B_{2n}. Next we can argue as before that for $0 \le k \le n-1$,

$$cm_k(D_{2n-1},\alpha) = \binom{n-1}{k}\alpha^k + \sum_{j=0}^{k} b_{j,k}\alpha^j$$

for some nonnegative integers $b_{0,k},\ldots,b_{k-1,k}$. That is, if $p \in M_k(D_{2n-1})$, the maximum number of cycles that can occur in $G_{2n}(p)$ is k and the only way that we can get k cycles in such a $G_{2n}(p)$ is if $p = \{(2i_1-1,2i_1),\ldots,(2i_k-1,2i_k)\}$ for some $1 \le i_1 < \cdots < i_k \le n-1$. Thus to prove that $cm_k(D_{2n-1},\alpha) = \binom{n-1}{k}(\alpha+2n-2)\downarrow\downarrow_k$, we need only show that $(\alpha+2n-2)\downarrow\downarrow_k$ divides $cm_k(D_{2n-1},\alpha)$.

It is easy to check that $d_{2n-1}(D_{2n-1},\alpha)=0$ and for $j=1,\ldots,n-1$,

$$d_{2n-(2j+1)}(D_{2n-1},\alpha) = \alpha+2j-2 \text{ and } d_{2n-2j}(D_{2n-1},\alpha) = 2j-2.$$

Hence $(x-d_{2n-1}(D_{2n-1},\alpha)-2+2) = x$ and for $j=1,\ldots,n-1$

$$(x+d_{2n-(2j+1)}(D_{2n-1},\alpha)-2(2j+1)+2) = (x+\alpha-2j-2)$$

and

$$(x+d_{2n-2j}(D_{2n-1},\alpha)-2(2j)+2) = (x-2j).$$

Thus for D_{2n-1}, Theorem 2 becomes

$$\sum_{k=0}^{n-1} cm_k(D_{2n-1},\alpha)(x)\downarrow\downarrow_{2n-1-k} = (x)\downarrow\downarrow_n (x+\alpha-2)\downarrow\downarrow_{n-1}. \tag{15}$$

If we divide both sides of (15) by $(x)\downarrow\downarrow_n$ and replace x by $x+2n$, we get

$$\sum_{k=0}^{n-1} cm_{n-1-k}(D_{2n-1},\alpha)(x)\downarrow\downarrow_k = (\alpha+2n-2)\downarrow\downarrow_{n-1}. \tag{16}$$

We can then use (16) to prove that $(\alpha+2n-2)\downarrow\downarrow_{n-1-k}$ divides $cm_{n-1-k}(D_{2n-1},\alpha)$ by induction on k exactly as before. $\Box$.

3. A cycle version of Theorem 1

For a board $B \subseteq B_{2n}$, set

$$cf_{k,2n}(B,\alpha) = \sum_{p_m \in F_{k,2n}(B,\alpha)} \alpha^{\mathrm{cy}(p_m)}.$$

Note that if $\alpha=1$, the $cf_{k,2n}(B,\alpha)$ reduce to the $f_{k,2n}(B)$ from (1). We will prove the following.

Theorem 4. *Let B be a board contained in B_{2n}. Then*

$$\sum_{k=0}^{n} cm_k(B,\alpha)\alpha(\alpha+2)\cdots(\alpha+2(n-k)-2)(z-1)^k = \sum_{k=0}^{n} cf_{k,2n}(B,\alpha)z^k. \quad (17)$$

Proof: Note that by replacing z by $z+1$ in (17) and taking the coefficient of z^k on both sides, Theorem 4 is equivalent to the fact that for any $k \in \{0,\ldots,n\}$,

$$cm_k(B,\alpha)\alpha(\alpha+2)\cdots(\alpha+2(n-k)-2) = \sum_{i=k}^{n} cf_{i,2n}(B,\alpha)\binom{i}{k}. \quad (18)$$

First we shall prove by induction on k that if $p \in M_{n-k}(B_{2n})$, then

$$\sum_{\substack{q\in M_n(B_{2n})\\ p\subseteq q}} \alpha^{\mathrm{cy}(q)} = \alpha^{\mathrm{cy}(p)}\alpha(\alpha+2)\cdots(\alpha+2k-2). \quad (19)$$

Now if $k=0$, (19) is immediate. Thus assume (19) is true for $j=0,\ldots,k-1$. Then fix $p \in M_{n-k}(B_{2n})$. There are $2k$ elements of $\{1,\ldots,2n\}$, $1 \le i_1 < \cdots < i_{2k} \le 2n$, which are not coordinates of any square in p. There are $2k-1$ ways to extend p to a rook placement by adding a square with i_1 as a coordinate, namely, $q_j = p\cup\{(i_1,i_j)\}$ for $j=2,\ldots,2k$. We claim that there is a $t \in \{2,\ldots,2k\}$ such that

$$\mathrm{cy}(q_j) = \begin{cases} \mathrm{cy}(p) & \text{if } j \in \{2,\ldots,2k\} - \{t\} \\ \mathrm{cy}(p)+1 & \text{if } j = t. \end{cases} \quad (20)$$

We use an argument similar, but not identical to, that of case 2 of Theorem 2 to construct a sequence of distinct vertices $s_1,\ldots,s_{2r+1}$. That is, consider the vertex i_1 in the graph $G_{2n}(p)$. Then let s_1 equal i_1-1 if i_1 is even and equal i_1+1 if i_1 is odd. Hence $\{i_1,s_1\}$ will be an edge of the form $\{2a-1,2a\}$ in $G_{2n}(p)$. If there is another edge in $G_{2n}(p)$ out of s_1, then it will be of the form $\{s_1,s_2\}$ where either (s_1,s_2) or (s_2,s_1) is in p. Then let s_3 be s_2-1 if s_2 is even and be s_2+1 if s_2 is odd. Thus $\{s_2,s_3\}$ will be another edge of the form $\{2b-1,2b\}$ in $G_{2n}(p)$ which is distinct from $\{i_1,s_1\}$ and $\{s_1,s_2\}$. If there is another edge out of s_3, then it will be of the form $\{s_3,s_4\}$ where either (s_3,s_4) or (s_4,s_3) is in p. We then let s_5 be s_4-1 if s_4 is even and s_4+1 if s_4 is odd. Again $\{s_4,s_5\}$ will be an edge of the form $\{2c-1,2c\}$ in $G_{2n}(p)$. Continuing in this way we get a sequence of distinct vertices, $s_1,\ldots,s_{2r+1}$ such that for all $i \le r$, $\{s_{2i},s_{2i+1}\}$ is an edge of the form $\{2d-1,2d\}$ and either (s_{2j-1},s_{2j}) or (s_{2j},s_{2j-1}) is in p and there is only one edge of $G_{2n}(p)$ that contains s_{2r+1}. Thus s_{2r+1} is not a coordinate of any square in p and hence $s_{2r+1}=i_t$ for some $2 \le t \le 2k$.

It is then easy to see that the edge $\{i_1,i_t\}$ will create a new cycle in $G_{2n}(q_t)$ and that an edge $\{i_1,i_j\}$ will not create a new cycle in $G_{2n}(q_j)$ for $j=\{2,\ldots,2k\}-\{t\}$. This establishes (20). But then by induction,

$$\sum_{\substack{q\in M_n(B_{2n})\\ q_i\subseteq q}} \alpha^{\mathrm{cy}(q)} = \alpha^{\mathrm{cy}(q_i)}\alpha(\alpha+2)\cdots(\alpha+2k-4) \quad (21)$$

for $i = 2, \dots, 2k$. Thus

$$\begin{aligned}
\sum_{\substack{q \in M_n(B_{2n}) \\ p \subseteq q}} \alpha^{\mathrm{cy}(q)} &= \sum_{i=2}^{2k} \sum_{\substack{q \in M_n(B_{2n}) \\ q_i \subseteq q}} \alpha^{\mathrm{cy}(q)} \\
&= \sum_{i=2}^{2k} \alpha^{\mathrm{cy}(q_i)} \alpha(\alpha+2)\cdots(\alpha+2k-4) \\
&= \alpha(\alpha+2)\cdots(\alpha+2k-4) \sum_{i=2}^{2k} \alpha^{\mathrm{cy}(q_i)} \\
&= \alpha(\alpha+2)\cdots(\alpha+2k-4)(\alpha+2k-2)\alpha^{\mathrm{cy}(p)}.
\end{aligned}$$

Thus by induction, (19) holds.

It follows that for $B \subseteq B_{2n}$,

$$\begin{aligned}
cm_k(B,\alpha)\alpha(\alpha+2)\cdots(\alpha+2(n-k)+2) &= \sum_{p \in M_k(B)} \sum_{\substack{q \in M_n(B_{2n}) \\ p \subseteq q}} \alpha^{\mathrm{cy}(q)} \\
&= \sum_{i=k}^{n} \sum_{q \in F_{i,2n}(B)} \alpha^{\mathrm{cy}(q)} \sum_{\substack{p \in M_k(B) \\ p \subseteq q}} 1 \\
&= \sum_{i=k}^{n} \binom{i}{k} f_{i,2n}(B,\alpha).
\end{aligned}$$

Thus Theorem 4, in the form of (18), holds. $\square$

References

[CG] F. R. K. Chung and R. L. Graham, *On the cover polynomial of a digraph*, J. Combin. Theory Ser. B **65** (1995), 273–290.

[D] M. Dworkin, *Factorization of the cover polynomial*, J. Combin. Theory Ser. B **71** (1997), 17–53.

[HR] J. Haglund and J. B. Remmel, *Rook theory for perfect matchings*, Adv. Appl. Math. **27** (2001), 438–481.

[RW] V. Reiner and D. White, private communication.

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