Some New Symmetric Function Tools and their Applications

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Abstract

Symmetric Function Theory is a powerful computational device with applications to several branches of mathematics. The Frobenius map and its extensions provides a bridge translating Representation Theory problems to Symmetric Function problems and ultimately to combinatorial problems. Our main contributions here are certain new symmetric function tools for proving a variety of identities, some of which have already had significant applications. One of the areas which has been nearly untouched by present research is the construction of bigraded S_n modules whose Frobenius characteristics has been conjectured from both sides. The flagrant example being the Delta Conjecture whose symmetric function side was conjectured to be Schur positive since the early 90's and there are various unproved recent ways to construct the combinatorial side. One of the most surprising applications of our tools reveals that the only conjectured bigraded S_n modules are remarkably nested by the restriction $\downarrow^{S_n}_{S_{n-1}}$.

1 Introduction

Frobenius constructed a map $\mathcal{F}_n$ from class functions of S_n to degree n homogeneous symmetric polynomials and proved the identity $\mathcal{F}_n\chi^\lambda = s_\lambda[X]$, for all $\lambda \vdash n$. This not only identifies the irreducible

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characters of S_n but permits the translation of problems of S_n representation theory to symmetric function problems and ultimately to purely combinatorial problems. This map has since been extended to other important groups. If $\mathbf{M}$ is a bi-graded S_n module with character $\chi^{\mathbf{M}}(q,t) \in \mathbb{Z}[q,t]$ the symmetric function $\mathcal{F}_{\mathbf{M}}(X;q,t) = \mathcal{F}_n \chi^{\mathbf{M}}(q,t)$, usually referred to as the "*Frobenius characteristic*" of $\mathbf{M}$ is easily shown to have Schur function expansion with coefficients in $\mathbb{N}[q,t]$. We usually refer to this latter property as "*Schur positivity*".

The first named author and Haiman [13] introduced the modified Macdonald basis $\{\tilde{H}_\mu[X;q;t]\}_\mu$ to solve the Macdonald q,t-Kostka conjectures [27] by representation theoretical means. They conjectured that $\tilde{H}_\mu[X;q;t]$ is the Frobenius characteristic of a bi-graded S_n module $\mathbf{M}_\mu$. In fact, $\mathbf{M}_\mu$ was constructed as the linear span of derivatives of an explicit bi-homogeneous determinant $\Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n)$ of bidegree $n(\mu), n(\mu')$. This effort led to the so called "*$n!$-conjecture*". This conjecture was eventually proved in [17] using algebraic geometry. Later, the second named author [20] conjectured a purely combinatorial formula for $\tilde{H}_\mu[X;q;t]$ which was proved in [22]. Neither of these results solves the original q,t-Kostka problem which demanded a combinatorial formula for the coefficients of the Schur expansion of $\tilde{H}_\mu[X;q,t]$. Representation theory only proved these coefficients to be in $\mathbb{N}[q,t]$ and the combinatorial formula only yielded the coefficients of the monomial basis expansion.

Research efforts towards proving the $n!$-conjecture led to the discovery of the bi-graded S_n-module of Diagonal Harmonics $\mathbf{DH}_n$ since it contains all the Modules $\mathbf{M}_\mu$. In fact, $\mathbf{DH_n}$ is the finite dimensional space of polynomials in $(x_1,\ldots,x_n;y_1,\ldots,y_n)$ which are killed by all differential operators $\sum_{i=1}^n \delta_{x_i}^r \delta_{y_i}^s$ with $r+s \geq 1$. A permutation $\in S_n$ acts "*diagonally*" on these polynomials by sending x_i, y_i to $x_{\sigma_i}, y_{\sigma_i}$.

The discovery in [12] of the Frobenius characteristic of $\mathbf{DH_n}$ as the symmetric polynomial on the left

$$a) \quad \mathcal{FDH}_n = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu[X;q,t] M B_\mu(q,t) \Pi_\mu(q,t)}{w_\mu(q,t)} \qquad b) \quad e_n = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X;q,t] M B_\mu(q,t) \Pi_\mu(q,t)}{w_\mu(q,t)}, \tag{1}$$

and its comparison with the expansion on the right of the elementary symmetric function e_n in terms of the $\tilde{H}_\mu$, led to the introduction in [4] of the Nabla operator, which is simply defined by setting

$$\nabla \tilde{H}_\mu[X;q,t] = T_\mu \tilde{H}_\mu[X;q,t] \quad \text{with } T_\mu = t^{n(\mu)} q^{n(\mu')}. \tag{2}$$

The various factors occurring in (1) will be defined in the next section. At this point it suffices to introduce the family of Delta operators which, like Nabla, act diagonally on the modified Macdonald basis. More precisely for each Symmetric polynomial $F[X]$ we set

$$\Delta_F \tilde{H}_\mu[X;q,t] = F[B_\mu(q,t)] \tilde{H}_\mu[X;q,t] \tag{3}$$

In this paper all our symmetric function tools are none other than specializations of a single tool we are about to define. To begin let us set

$$\alpha_n = \frac{(-1)^{n-1}}{[n]_q [n]_t} = \frac{(-1)^{n-1} M}{(1-q^n)(1-t^n)}, \qquad \text{with } M = (1-t)(1-q). \tag{4}$$

This given, our basic new tool may be stated as follows:

Theorem 1. *Let P be a homogeneous symmetric function of degree k, and let Q be a homogeneous symmetric function of degree n. Then*

$$\langle \Delta_{\omega(P)} \alpha_n p_n, Q \rangle = \langle \Delta_{\omega(Q)} \alpha_k p_k, P \rangle, \tag{5}$$

where p_k, p_n denote the power-sum symmetric function.

All the identities proved in this paper are applications of special cases of this single equality, which in turn is an immediate consequence of the Macdonald-Koorwinder reciprocity formula. Thus, in contrast to past formulations, we obtain here a unified derivation of a variety of scalar product identities, some of which have been effective tools for producing important symmetric function identities (for example, [8], [9], [11], [19],[29],[16], [32], [24]).

Part of the motivation for studying the Δ operator is work of Haiman [18], who proved that $\Delta_{s_\lambda e_n} e_n$ is Schur positive for any λ, and furthermore conjectured that $\Delta_{s_\lambda} e_n$ is Schur positive for any λ. There has been a lot of research over the past fifteen years devoted to trying to understand the combinatorial structure of the coefficients when $\Delta_{s_\lambda} e_n$ is expanded in terms of the monomial symmetric functions, for various choices of λ. (Ideally, we would like to understand the Schur coefficients, but so far this has not proved tractable). The original Shuffle Conjecture of the second named author, Haiman, Loehr, Remmel, and Ulyanov [23] gives a combinatorial interpretation for the expansion of $\Delta_{e_n} e_n$ in terms of monomials. The second named author, Remmel, and Wilson [24] recently introduced a generalization of this they called the Delta Conjecture, which gives a combinatorial interpretation for the expansion of $\Delta_{e_k} e_n$ into monomials for any $k, n \in \mathbb{N}$. The Shuffle Conjecture was proved by Carlsson and Mellit in 2015 [6] but the Delta Conjecture is still open.

Using our new tool, we will tie together a mixture of topics conected to the Shuffle Conjecture, the Delta Conjecture, q, t-Narayana numbers and Kreweras numbers. Furthermore, some of our results have implications to a subject that remains largely undeveloped to this date and deserves special mention. We have seen recently the amazing ubiquity of Diagonal Harmonics with its connections to torus knots [25], [28], [15]. Many years ago, the following remarkable identity was discovered

$$(-1)^{n-1} e_1^\perp \nabla p_n = [n]_t [n]_q \nabla e_{n-1}. \tag{6}$$

At about the same time, it was conjectured that the symmetric function $(-1)^{n-1}\nabla p_n$ is a Schur positive symmetric polynomial, for all $n \geq 1$. Now if we have a bigraded S_n module $\mathbf{P_n}$ with Frobenius Characteristic $(-1)^{n-1}\nabla p_n$, the identity in (6) asserts that when we restrict $\mathbf{P_n}$ from S_n to S_{n-1} we obtain n^2 copies of the Diagonal Harmonics Module in $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$. That is, we may write (6) in the form

$$\mathbf{P_n} \downarrow_{S_{n-1}}^{S_n} = [n]_t [n]_q \; \mathbf{DH}_{n-1}. \tag{7}$$

Since $\mathbf{DH}_{n-1}$ has been shown to have dimension n^{n-2}, it follows that $\mathbf{P_n}$ must have dimension n^n. Combinatorially, the objects acted upon by S_n should be words of length n in the alphabet $\mathcal{A}_n = \{1, 2, \ldots, n\}$. However, here we need to decide whether this action should be on the letters or on their positions. Computing the polynomial $(-1)^{n-1}\nabla p_n\big|_{t=q=1}$ in a few special cases immediately rules out the first alternative. The resulting conjecture is that, combinatorially, $\mathbf{P_n}$ should give a bi-graded sign twisted version of the action on words of length n in the alphabet $\mathcal{A}_n$ by positions. Using what we know about the action on Parking Functions with $n-1$ cars, it is a routine exercise to prove, combinatorially, the case $t = q = 1$ of (6). This given, it should remain as a priority of the first magnitude to construct **the** natural bi-graded module $\mathbf{P_n}$ and prove that its Frobenius characteristic is related to the Frobenius Characteristic of $\mathbf{DH}_{n-1}$ according to (6). We feel this mysterious module should be as ubiquitous as $\mathbf{DH}_n$ itself.

But there is more in store here. In fact, one of the applications of our basic tool is the identity

$$(-1)^{n-1} e_1^\perp \Delta_{e_k} p_n = [n]_t [n]_q \, \Delta_{e_{k-1}} e_{n-1}. \tag{8}$$

Here the previous problem gets magnified. Not only do we need to construct **the** bi-graded modules with Frobenius characteristic $(-1)^{n-1}\Delta_{e_k} p_n$ but **those** with characteristic $\Delta_{e_k} e_n$ are still unknown. As a starter we could mimic what we did with $\mathbf{P_n}$ and based on what we combinatorially conjecture for $\Delta_{e_{k-1}} e_{n-1}$ derive at least the particular case $t = q = 1$ of (8). We hope to return to these questions in a later writing.

2 Preliminaries

Given a partition μ drawn as a French diagram, and a cell $c \in \mu$, we let $l(c), a(c), l'(c)$, and $a'(c)$ be the number of cells in μ strictly North, East, South, and West of c, respectively. These give the leg, arm, coleg, and coarm of the cell c. We will use the notation

$$\Pi_\mu(q,t) = \prod_{c\in\mu/(0,0)} (1-q^{a'(c)}t^{l'(c)}), \qquad w_\mu(q,t) = \prod_{c\in\mu} (q^{a(c)}-t^{l(c)+1})(t^{l(c)}-q^{a(c)+1}),$$

$$B_\mu(q,t) = \sum_{c\in\mu} q^{a'(c)}t^{l'(c)}, \qquad \text{and} \qquad M=(1-q)(1-t).$$

We will also need $n(\mu)=\sum_{i=1}^{\ell(\mu)}(i-1)\mu_i$ and $T_\mu(q,t)=q^{n(\mu')}t^{n(\mu)}$.

We assume the reader is familiar with the standard notation involving bases of the ring of symmetric functions, as used in the classic texts [27, Chapter 1] and [31, Chapter 7]; the Schur functions s_λ, the Hall scalar product $\langle , \rangle$ (with respect to which the s_λ are orthonormal), the monomial symmetric functions m_λ, the complete homogeneous symmetric functions h_n, elementary symmetric functions e_n, and power-sums $p_n=\sum_i x_i^n$. Note that $h_n=s_n$ and $e_n=s_{1^n}=m_{1^n}$.

Throughout this article, square brackets denote plethystic substitution, as in $F[E]$, which denotes the plethystic substitution of the expression E into F. If E is a positive alphabet, then $F[E]=F(E)$, the usual symmetric function F evaluated at the alphabet E. For background and further examples of plethystic notation see [21, Chapter 1]. For any symmetric function $F[X]$, let Δ_F be the linear operator defined on the modified Macdonald basis $\widetilde{\mathrm{H}}_\mu$ via

$$\Delta_F \widetilde{\mathrm{H}}_\mu[X;q,t] = F[B_\mu]\widetilde{\mathrm{H}}_\mu[X;q,t]. \tag{9}$$

For example, $B_{(2,1)}(q,t)=1+q+t$, and $\Delta_{h_2}\tilde{H}_{(2,1)} = h_2(1,q,t)\tilde{H}_{(2,1)} = (q^2+t^2+1+qt+q+t)\tilde{H}_{(2,1)}$.

One of the defining properties of modified Macdonald polynomials are the orthogonality relations under the $*$-scalar product. The $*$-scalar product is defined by setting

$$\langle p_\lambda, p_\mu\rangle_* = (-1)^{|\mu|-\ell(\mu)} z_\mu p_\mu[M]\chi(\lambda=\mu),$$

where $\chi(A)$ is 1 if A is true and 0 otherwise. We can rewrite the relation as

$$\langle p_\lambda, p_\mu\rangle_* = \langle p_\lambda, \omega(p_\mu)[MX]\rangle.$$

This gives the relationship between the $*$-scalar product and the Hall scalar product. For any two symmetric functions F and G, we have

$$\langle F[X], G[X]\rangle_* = \langle F[X], \omega(G)[MX]\rangle.$$

The orthogonality relations for modified Macdonald polynomials are given by

$$\langle \widetilde{\mathrm{H}}_\lambda[X], \widetilde{\mathrm{H}}_\mu[X]\rangle_* = \langle \widetilde{\mathrm{H}}_\lambda[X], \omega(\widetilde{\mathrm{H}}_\mu)[MX]\rangle = w_\mu(q,t)\chi(\lambda=\mu). \tag{10}$$

If we let

$$\alpha_n = \frac{(-1)^{n-1}}{[n]_q[n]_t} = \frac{(-1)^{n-1}M}{(1-q^n)(1-t^n)},$$

then we can write the expansion of p_n in terms of the modified Macdonald basis as

$$\alpha_n p_n = \sum_{\mu\vdash n} \frac{M\Pi_\mu}{w_\mu}\widetilde{\mathrm{H}}_\mu[X]. \tag{11}$$

The summands for $\alpha_n p_n$ and e_n (from (1)) differ by a factor of $B_\mu = e_1[B_\mu]$. Thus

$$\Delta_{e_1}\alpha_n p_n = e_n. \tag{12}$$

The expansion for h_n is given [5] by

$$h_n = (-qt)^{1-n} \sum_{\mu \vdash n} \frac{M\Pi_\mu B_\mu(1/q, 1/t)}{w_\mu} \widetilde{\mathrm{H}}_\mu\,.$$

Using the fact that for $\mu \vdash n$, $B_\mu(1/q,1/t)T_\mu = e_{n-1}[B_\mu]$ and $e_n[B_\mu] = T_\mu$, we can write

$$h_n = (-qt)^{1-n} \Delta_{e_n}^{-1} \Delta_{e_{n-1}} \alpha_n p_n,$$

where for any f, $\Delta_f^{-1} \tilde{H}_\mu = \tilde{H}_\mu / f[B_\mu]$.

For any two parititions α and β, Macdonald-Koorwinder reciprocity [27], [14] gives a fundamental relation between the Macdonald polynomial indexed by α and the one indexed by β. Expressed In terms of the $\tilde{H}_\mu$ it says

$$\frac{\widetilde{\mathrm{H}}_\alpha \left[1 + u\left(MB_\beta - 1\right)\right]}{\prod_{c \in \alpha} 1 - uq^{a'(c)} t^{l'(c)}} = \frac{\widetilde{\mathrm{H}}_\beta \left[1 + u\left(MB_\alpha - 1\right)\right]}{\prod_{c \in \beta} 1 - uq^{a'(c)} t^{l'(c)}}. \tag{13}$$

Multiplying both sides by $(1-u)$ and letting $u \to 1$, we get the form of Macdonald reciprocity we will make the most use of:

$$\frac{\widetilde{\mathrm{H}}_\alpha[MB_\beta]}{\Pi_\alpha} = \frac{\widetilde{\mathrm{H}}_\beta[MB_\alpha]}{\Pi_\beta}. \tag{14}$$

We should note that the expansions of e_n and p_n described above are a consequence of the orthogonality relations (10) and (13).

Suppose that for a given symmetric function F, there exists a symmetric function G so that

$$\langle \widetilde{\mathrm{H}}_\mu, F \rangle = G[B_\mu],$$

for all $\mu \vdash n$. Then since $\langle \widetilde{\mathrm{H}}_\mu, s_n \rangle = 1$, it follows that $\langle \widetilde{\mathrm{H}}_\mu, F \rangle = \langle \Delta_G \widetilde{\mathrm{H}}_\mu, s_n \rangle$. This will be useful in the case that $F = e_k h_{n-k}$ since it is known [27, p. 362] that $\langle \widetilde{\mathrm{H}}_\mu, e_k h_{n-k} \rangle = e_k[B_\mu]$. It follows that for any homogeneous symmetric function P of degree n, we have

$$\langle \Delta_{e_k} P, s_n \rangle = \langle P, e_k h_{n-k} \rangle. \tag{15}$$

3 Scalar product identities

In [19], identity (17) below played a key role, and further applications of (17) were derived in [8], [9] and [29]. We now prove the main result in this article, Theorem 1, a more general form of (17) that can be shown to be equivalent to Macdonald reciprocity. After proving it, we devote the rest of the article to exploring applications.

Proof. (Of Theorem 1) It suffices to check this equality over a basis. We will assume $P = \omega\, \widetilde{\mathrm{H}}_\alpha[MX]$ and $Q = \omega\, \widetilde{\mathrm{H}}_\beta[MX]$ for arbitrary partitions $\alpha \vdash k$ and $\beta \vdash n$. Since

$$\Delta_{\omega(P)} \alpha_n p_n = \sum_{\mu \vdash n} \frac{M\Pi_\mu \omega(P)[B_\mu]}{w_\mu} \widetilde{\mathrm{H}}_\mu[X],$$

we have

$$\begin{aligned}
\langle \Delta_{\omega(P)} \alpha_n p_n, Q \rangle &= \sum_{\mu \vdash n} \frac{M\Pi_\mu \omega(\omega\, \widetilde{\mathrm{H}}_\alpha[MX])[B_\mu]}{w_\mu} \langle \widetilde{\mathrm{H}}_\mu[X], \omega\, \widetilde{\mathrm{H}}_\beta[MX] \rangle \\
&= \sum_{\mu \vdash n} \frac{M\Pi_\mu \widetilde{\mathrm{H}}_\alpha[MB_\mu]}{w_\mu} \langle \widetilde{\mathrm{H}}_\mu, \widetilde{\mathrm{H}}_\beta \rangle_* \\
&= \frac{M\Pi_\beta \widetilde{\mathrm{H}}_\alpha[MB_\beta]}{w_\beta} w_\beta \quad = \quad M\Pi_\beta \widetilde{\mathrm{H}}_\alpha[MB_\beta].
\end{aligned}$$

A similar computation gives

$$\langle \Delta_{\omega(Q)} \alpha_k p_k, P \rangle = M \Pi_\alpha \widetilde{\mathrm{H}}_\beta[MB_\alpha].$$

Therefore, equality holds precisely when

$$\Pi_\alpha \widetilde{\mathrm{H}}_\beta[MB_\alpha] = \Pi_\beta \widetilde{\mathrm{H}}_\alpha[MB_\beta],$$

which is another way of writing Macdonald reciprocity (14). □

Remark 1. *Note that if $n = k$, so both P and Q are of homogeneous degree n, we can cancel the factor α_n on both sides of (5), leaving the more compact form*

$$\langle \Delta_{\omega(P)} p_n, Q \rangle = \langle \Delta_{\omega(Q)} p_n, P \rangle. \tag{16}$$

We now list a few significant special cases of Theorem 1.

Corollary 1. *[19] For $k > 0$ and any homogeneous symmetric function Q of degree n, we have*

$$\langle \Delta_{e_{k-1}} e_n, Q \rangle = \langle \Delta_{\omega(Q)} e_k, s_k \rangle. \tag{17}$$

Proof. We have seen that

$$e_n = \Delta_{e_1} \alpha_n p_n.$$

Therefore, using Theorem 1, with $P = h_{k-1} e_1$, we have

$$\begin{aligned}
\langle \Delta_{e_{k-1}} e_n, Q \rangle &= \langle \Delta_{\omega(h_{k-1} e_1)} \alpha_n p_n, Q \rangle \\
&= \langle \Delta_{\omega(Q)} \alpha_k p_k, h_{k-1} e_1 \rangle \\
\text{(by (15))} \quad &= \langle \Delta_{\omega(Q) e_1} \alpha_k p_k, s_k \rangle \\
&= \langle \Delta_{\omega(Q)} e_k, s_k \rangle.
\end{aligned}$$

□

Corollary 2. *For $k > 0$ and any homogeneous symmetric function Q of degree n, we have*

$$\langle \Delta_{h_{k-1}} e_n, Q \rangle = (-qt)^{k-1} \langle \Delta_{\omega Q} h_k, s_{1^k} \rangle. \tag{18}$$

Proof. We begin from the right-hand side. Recall from the preliminaries that we have

$$h_k = (-qt)^{1-k} \Delta_{e_k}^{-1} \Delta_{e_{k-1}} \alpha_k p_k.$$

Then the right-hand side becomes

$$\begin{aligned}
(-qt)^{k-1} \langle \Delta_{\omega(Q)} h_k, s_{1^k} \rangle &= (-qt)^{k-1} \langle (-qt)^{1-k} \Delta_{\omega(Q)} \Delta_{e_k}^{-1} \Delta_{e_{k-1}} \alpha_k p_k, e_k \rangle \\
\text{(by (15))} \quad &= \langle \Delta_{e_k} \Delta_{e_k}^{-1} \Delta_{\omega(Q) e_{k-1}} \alpha_k p_k, s_k \rangle \\
&= \langle \Delta_{\omega(Q) e_{k-1}} \alpha_k p_k, s_k \rangle \\
\text{(by (15))} \quad &= \langle \Delta_{\omega(Q)} \alpha_k p_k, e_{k-1} h_1 \rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{h_{k-1} e_1} \alpha_n p_n, Q \rangle \\
&= \langle \Delta_{h_{k-1}} e_n, Q \rangle.
\end{aligned}$$

□

3.1 Restrictions

Although our central objects of study have been the symmetric functions $\Delta_{s_\lambda} e_n$ for various λ, there are also many nice results involving $(-1)^{n-1}\nabla p_n$. In particular, Sergel [30] has proved the Square Paths Conjecture of Loehr and Warrington, which gives a combinatorial interpretation for the expansion of $(-1)^{n-1}\nabla p_n$ into monomials. One thing that is missing from this theory is a DH_n-type module giving a representation-theoretic interpretation for $(-1)^{n-1}\nabla p_n$, and more generally for the symmetric functions $(-1)^{n-1}\nabla p_n,\ \Delta_{e_k} e_n,\ (-1)^{n-1}\Delta_{e_k} p_n,\ (-1)^{n-1}\Delta_{h_k} p_n$, which all appear to be Schur positive.

Suppose for the moment we have a module $\mathbf{P_n}$ whose Frobenius characteristic $\mathcal{F}_{\mathbf{P_n}}(X;q,t)$ equals $(-1)^{n-1}\nabla p_n$. We will show that

$$(-1)^{n-1} e_1^\perp \nabla p_n = [n]_q [n]_t \nabla e_{n-1}, \tag{19}$$

where $^\perp$ is the operator which is adjoint to multiplication with respect to the Hall scalar product, i.e. for all f, g, h,

$$\langle g^\perp f, h\rangle = \langle f, gh\rangle. \tag{20}$$

Since $e_1^\perp \mathcal{F}_{\mathbf{P_n}}$ gives the Frobenius image of the restriction from an S_n-action to an S_{n-1}-action, we see that $\mathbf{P_n} \downarrow^{S_n}_{S_{n-1}}$ gives n^2 copies of diagonal harmonics. One may hope that $\mathbf{P_n}$ may be represented by some space of polynomials that are killed by certain partial differential operators. We will see that the same property holds for Δ_{e_k};

$$(-1)^{n-1} e_1^\perp \Delta_{e_k} p_n = [n]_q [n]_t \Delta_{e_{k-1}} e_{n-1}.$$

This together with other identities we will derive suggest a remarkable relation between the yet to be discovered-modules corresponding to $(-1)^{n-1}\Delta_{e_k} p_n$, $\Delta_{e_{k-1}} e_{n-1}$ and even $\Delta_{h_{k-1}} e_{n-1}$.

We now explain how to obtain identity (19).

Corollary 3. *For positive integers k and n we have*

$$(-1)^{n-1} e_1^\perp \Delta_{e_k} p_n = [n]_q [n]_t \Delta_{e_{k-1}} e_{n-1}. \tag{21}$$

Proof. The statement is equivalent to saying $e_1^\perp \Delta_{e_k} \alpha_n p_n = \Delta_{e_{k-1}} e_{n-1}$. Now two symmetric functions F and G are equal if and only if $\langle F, P\rangle = \langle G, P\rangle$ for any choice of P. It is easy to see that by applying Theorem 1 and Corollary 1, we have for any homogeneous symmetric function P of degree n,

$$\begin{aligned} \langle e_1^\perp \Delta_{e_k} \alpha_n p_n, P\rangle &= \langle \Delta_{\omega(h_k)} \alpha_n p_n, e_1 P\rangle \\ \text{(by Theorem 1)} \quad &= \langle \Delta_{\omega(P) e_1} \alpha_k p_k, s_k\rangle \\ &= \langle \Delta_{\omega(P)} e_k, s_k\rangle \\ \text{(by Corollary 1 with } Q \to P) \quad &= \langle \Delta_{e_{k-1}} e_n, P\rangle. \end{aligned}$$

□

Eq. (21) can be viewed as a special case of the following result.

Theorem 2. *For positive integers m, n, k we have*

$$h_m^\perp \Delta_{e_k} \alpha_{n+m} p_{n+m} = \Delta_{e_{k-m} h_m} \alpha_n p_n. \tag{22}$$

In particular, when $m = 1$ and n is replaced by $n-1$, we have

$$(-1)^{n-1} e_1^\perp \Delta_{e_k} p_n = [n]_q [n]_t \Delta_{e_{k-1}} e_{n-1}. \tag{23}$$

If instead, in (22), k is replaced by $k+1$ and m by k we have

$$(-1)^{n+k-1} h_k^\perp \Delta_{e_{k+1}} p_{n+k} = [n+k]_q [n+k]_t \Delta_{h_k} e_n \tag{24}$$

Proof. Let P be a homogeneous symmetric function of degree n. Then

$$\begin{aligned}
\langle h_m^\perp \Delta_{e_k} \alpha_{n+m} p_{n+m}, P\rangle &= \langle \Delta_{e_k} \alpha_{n+m} p_{n+m}, h_m P\rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{\omega(P) e_m} \alpha_k p_k, s_k\rangle \\
\text{(by (15))} \quad &= \langle \Delta_{\omega(P)} \alpha_k p_k, e_m h_{k-m}\rangle \\
&= \langle \Delta_{e_{k-m} h_m} \alpha_n p_n, P\rangle.
\end{aligned}$$

□

In a similar way, we can deal with the symmetric function $h_m^\perp \Delta_{h_k} \alpha_{n+m} p_{n+m}$.

Theorem 3. *For positive integers m, n, k we have*

$$h_m^\perp \Delta_{h_k} \alpha_{n+m} p_{n+m} = h_k^\perp \Delta_{h_m e_{k-m}} \alpha_{n+k} p_{n+k}. \tag{25}$$

In particular, when $m = 1$ and n is replaced by $n-1$, we have

$$(-1)^{n-1} e_1^\perp \Delta_{h_k} p_n = [n]_t [n]_q h_k^\perp \Delta_{e_{k-1}} e_{n+k-1}. \tag{26}$$

When $m = k-1$ and n is replaced by $n+1$, we have

$$(-1)^{n+k-1} h_{k-1}^\perp \Delta_{h_k} p_{n+k} = [n+k]_t [n+k]_q h_k^\perp \Delta_{h_{k-1}} e_{n+k+1}. \tag{27}$$

Proof. Let P be a homogeneous symmteric function of degree n. Then

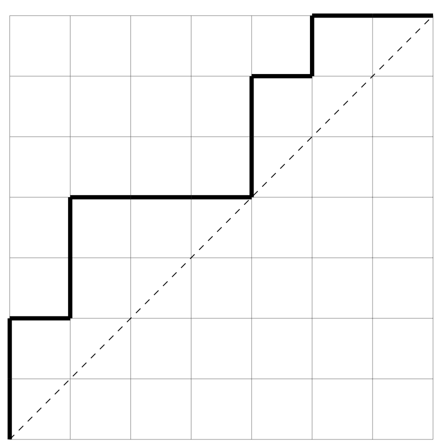
$$\begin{aligned}
\langle h_m^\perp \Delta_{h_k} \alpha_{n+m} p_{n+m}, P\rangle &= \langle \Delta_{h_k} \alpha_{n+m} p_{n+m}, h_m P\rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{\omega(P) e_m} \alpha_k p_k, e_k\rangle \\
\text{(by (15))} \quad &= \langle \Delta_{\omega(P) e_m e_k} \alpha_k p_k, s_k\rangle \\
\text{(by (15))} \quad &= \langle \Delta_{\omega(P) e_k} \alpha_k p_k, e_m h_{k-m}\rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{h_m e_{k-m}} \alpha_{n+k} p_{n+k}, h_k P\rangle \\
&= \langle h_k^\perp \Delta_{h_m e_{k-m}} \alpha_{n+k} p_{n+k}, P\rangle.
\end{aligned}$$

□

4 Combinatorial Applications

4.1 The Delta Conjecture

A Dyck path is a lattice path from $(0,0)$ to (n,n) consisting of unit North and East steps which never goes below the line $y = x$. As a running example, we take the Dyck path D below.



Given a Dyck path π let $a_i = a_i(\pi)$ denote the number of squares in the ith row (from the bottom) which are to the right of π and to the left of the diagonal $y = x$, where $1 \le i \le n$. In the case of D above, we would have

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (0, 1, 1, 2, 0, 1, 1).$$

We let $\operatorname{area}(\pi)$ denote the sum of the a_i. Furthermore let $\operatorname{dinv}(\pi)$ denote the number of pairs (i, j), $1 \le i < j \le n$, with either

$$a_i = a_j \text{ or } a_i = a_j + 1. \tag{28}$$

Next define the *reading order* of the rows of π to be the order in which the rows are listed by decreasing value of a_i, where if two rows have the same a_i-value, the row above is listed first. For the path D above, the reading order is

$$\text{row 4, row 7, row 6, row 3, row 2, row 5, row 1.} \tag{29}$$

Finally let $b_k = b_k(\pi)$ be the number of inversion pairs as in (28) which involve the kth row in the reading order and rows before it in the reading order. For D we have

$$b_1 = 0, b_2 = 1, b_3 = 2, b_4 = 2, b_5 = 3, b_6 = 2, b_7 = 1. \tag{30}$$

Note dinv is the sum of the b_k, and that values of i for which $b_i > b_{i-1}$ correspond to tops of columns in π (where we define $b_0 = -1$ so that $b_1 > b_0$).

One way of defining the q, t-Schröder polynomial from [19] is by setting

$$C_n(q, t, z) = \sum_{i=0}^{n} z^i \langle \nabla e_n, e_{n-i} h_i \rangle = \sum_{\pi} t^{\operatorname{area}(\pi)} q^{\operatorname{dinv}(\pi)} \prod_{\substack{b_i > b_{i-1} \\ i \ge 2}} (1 + z/q^{b_i}). \tag{31}$$

Here the sum is over all Dyck paths π from $(0, 0)$ to (n, n). For example, the weight assigned to D in the right-hand-side of (31) is

$$t^6 q^{11} (1 + z/q)(1 + z/q^2)(1 + z/q^3). \tag{32}$$

In particular, $C_n(q, t, 0)$ is Garsia and Haiman's q, t-Catalan sequence [12].

Let

$$C_n(q, t, w, z) = \sum_{\pi} t^{\operatorname{area}(\pi)} q^{\operatorname{dinv}(\pi)} \prod_{\substack{b_i > b_{i-1} \\ i \ge 2}} (1 + z/q^{b_i}) \prod_{\substack{a_i > a_{i-1} \\ i \ge 2}} (1 + w/t^{a_i}). \tag{33}$$

This four variable Catalan polynomial appears in [24] under the context of the Delta conjecture. Two different conjectured expressions for $C_n(q, t, z, w)$ are given there in terms of the Δ operator. One of these, eq. (34) below, was proved by Zabrocki in [32], and the other, (35) below, was proved by D'Adderio and Wyngaerd [9]. (Without the results in both [32] and [9], it is not known how to show the two Δ-operator expressions for $C_n(q, t, z, w)$ are equal).

Theorem 4 (Zabrocki [32], D'Adderio and Wyngaerd [9])**.** *For integers a, b,*

$$C_n(q, t, z, w)\Big|_{z^a w^b} = \langle \Delta_{h_a e_{n-a}} e_{n-a}, s_{b+1, 1^{n-a-b-1}} \rangle \tag{34}$$

$$(1 + z)(1 + w) C_n(q, t, w, z)|_{z^a w^b} = \langle \Delta_{e_{n-b}} e_n, h_a e_{n-a} \rangle. \tag{35}$$

We also have

$$C_n(q, t, w, z) = C_n(t, q, w, z) = C_n(q, t, z, w). \tag{36}$$

Remark 2. *An equivalent form of (35) is*

$$(1+w)C_n(q,t,w,z)|_{z^a w^b} = \langle \Delta_{e_{n-b}} e_n, s_{a+1,a^{n-b-a-1}} \rangle. \tag{37}$$

There is a simple proof of (36) from (35) using (15).

Proof. Suppose we know that

$$(1+z)(1+w)C_n(q,t,w,z)|_{z^a w^b} = \langle \Delta_{e_{n-b}} e_n, h_a e_{n-a} \rangle.$$

We can rewrite this as

$$\langle \Delta_{e_{n-b}} e_n, h_a e_{n-a} \rangle = \langle \Delta_{e_{n-b} e_{n-a}} e_n, s_n \rangle,$$

which is clearly symmetric in a, b, so $C_n(q,t,z,w) = C_n(q,t,w,z)$. To see the symmetry in q, t, consider the expansion of

$$\Delta_{e_{n-a} e_{n-b}} e_n$$

in terms of the $\tilde{H}_\mu$, using (11) and (12), as a sum over partitions μ. One easily sees that the terms in this sum corresponding to μ and its conjugate are the same, after we interchange q and t, and hence their sum is symmetric in q, t. □

The Delta conjecture of Haglund, Remmel and Wilson [24] gives a combinatorial description of $\Delta_{e_k} e_n$. Let P be a parking function, viewed as a placement of the integers 1 through n just to the right of the North steps of a Dyck path, with strict decrease down columns.

Conjecture 1 (Delta Conjecture [24])**.** *For any integer k, $0 \le k \le n$,*

$$\Delta_{e_{n-k}} e_n = \sum_{\pi} \sum_{P \in PF(\pi)} t^{area(\pi)} q^{dinv(P)} F_{des(read(P)^{-1})} \prod_{\substack{a_i > a_{i-1} \\ i \ge 1}} (1 + w/t^{a_i}) \Big|_{w^k}, \tag{38}$$

where $PF(\pi)$ is the set of parking functions with supporting path π from $(0,0)$ to (n,n); and F is the quasi-symmetric function weight attached to P. (See [21, Chapters 5 and 6] for a definition of $dinv(P)$, and also the quasi-symmetric function weight attached to P.)

The case $k = 0$ of (38) is the Shuffle Theorem of Carlsson and Mellit [6]). The results of D'Adderio and Wyngaerd [9] may be viewed as the Schröder case of the Delta conjecture (it is common usage to refer to the coefficient of a Schur function of hook shape in symmetric functions defined via the Δ operator as the Schröder case, since in [19] it was first shown that the coefficient of a hook Schur function in ∇e_n can be expressed combinatorially in terms of Schröder lattice paths). In related work D'Adderio and Iraci [8] were able to extend the work in [1] and give an interpretation of $\langle \Delta_{e_k} e_n, h_d h_{n-d} \rangle$ in terms of polyominoes, showing it agrees with a predicted combinatorial interpretation from the Delta conjecture.

4.2 Narayana numbers

The Narayana number $N(n,k)$ counts the number of Dyck paths from $(0,0)$ to (n,n) with k columns, i.e. k pairs of consecutive NE steps. It is known that

$$N(n,k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}, \tag{39}$$

and furthermore that $N(n,k) = h_{k-1}$, the kth element of the h-vector of the type A_{n-1} associahedron. The Narayana numbers can be decomposed into Kreweras numbers $\mathrm{Krew}(\lambda)$ which count the number of non-crossing partitions of $\{1,\ldots,n\}$ whose blocks have sizes that rearrange to λ. For a partition λ of length k, we have

$$\mathrm{Krew}(\lambda) = \frac{1}{n+1}\binom{n+1}{m_1(\lambda),\ldots,m_n(\lambda),n-k+1}, \tag{40}$$

where $m_i(\lambda)$ is the number of parts in λ of size i. This also counts (by the Cyclic Lemma) the number of Dyck paths from $(0,0)$ to (n,n) whose columns have lengths that rearrange to λ. This gives

$$N(n,k) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} \mathrm{Krew}(\lambda). \tag{41}$$

Perhaps the most natural q-analog of (41) is given by

$$\begin{aligned} q^{(n+1-k)(n-k)}\frac{1}{[n]_q}\begin{bmatrix} n\\ k-1\end{bmatrix}_q\begin{bmatrix} n\\ k\end{bmatrix}_q \\ = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} q^{(n)(n-\ell(\lambda))-c(\lambda)}\frac{1}{[n+1]_q}\begin{bmatrix} n+1\\ m_1(\lambda),\ldots,m_n(\lambda),n-k+1\end{bmatrix}_q \end{aligned} \tag{42}$$

with $c(\lambda)=\sum \lambda'_j\lambda'_{j+1}$, providing as well a natural q-analog of Kreweras numbers.

There is a q,t version of the Narayana numbers $N(n,k,q,t)$ introduced by Dukes and LeBornge [10], defined there as a weighted sum over parallelogram polyominoes. It was later shown by Aval, D'Adderio, Dukes, Hicks, and Le Borgne [1] that

$$N(n,k,q,t) = \langle \nabla e_{n-1}, h_{k-1}h_{n-k}\rangle. \tag{43}$$

They use the fact that a combinatorial description for the right-hand-side of (43) was proved in [19] (providing a special case of the Shuffle Conjecture).

Now a basic result of Garsia and Haiman is that for any symmetric function F,

$$q^{\binom{n}{2}}\langle \nabla e_n[X], F[X]\rangle\Big|_{t=1/q} = \frac{1}{[n+1]_q}\omega(F)\left[[n+1]_q\right]. \tag{44}$$

It follows that

$$q^{\binom{n-1}{2}}N(n,k,q,1/q) = \frac{1}{[n]_q}e_{k-1}\left[[n]_q\right]e_{n-k}\left[[n]_q\right] \tag{45}$$

$$= q^{\binom{k-1}{2}+\binom{n-k}{2}}\frac{1}{[n]_q}\begin{bmatrix} n\\ k-1\end{bmatrix}_q\begin{bmatrix} n\\ n-k\end{bmatrix}_q, \tag{46}$$

giving up to a scaling power of q the q-analog of $N(n,k)$ appearing on the left in (42).

At the Wachfest conference held at the University of Miami in January 2015, Vic Reiner of the University of Minnesota gave a talk involving representation theory and identities involving q-Narayana and q-Kreweras numbers for various types, including (42). In a subsequent private conversation with the second author, Reiner posed the question of finding a q,t-Kreweras number which when t is specialized to $1/q$ reduces to the q-Kreweras number appearing on the right in (42), in the same way that $N(n,k,q,t)$ reduces to the q-Narayana in (46). So far we have been unable to solve this problem, but this study has led to a new and perhaps better way of expressing $N(n,k,q,t)$ as a weighted sum over lattice paths. The expression (43) for $N(n,k,q,t)$ from [19] involves weighted lattice paths from $(0,0)$ to $(n-1,n-1)$ satisfying certain constraints, while in the new formulation below it is expressed as a weighted sum over lattice paths from $(0,0)$ to (n,n) with k columns, a formulation which is compatible (when $q=t=1$) with the definition of the Kreweras numbers and (41).

Using our scalar product identities, we are able to rewrite the q,t-Narayana numbers as follows:

$$\begin{aligned} N(n,k,q,t) &= \langle \Delta_{e_{n-1}} e_{n-1}, h_{k-1}h_{n-k}\rangle \\ &= \langle \Delta_{e_{k-1}e_{n-k}} e_n, s_n\rangle \\ &= \langle \Delta_{e_{k-1}} e_n, e_{n-k}h_k\rangle. \end{aligned}$$

We could have also written this two other ways:

$$\begin{aligned} N(n,k,q,t) &= \langle \Delta_{e_{k-1}e_{n-k}} e_n, s_n\rangle \\ &= \langle \Delta_{e_{n-k}} e_n, e_{k-1}h_{n-k+1}\rangle \end{aligned}$$

and

$$\begin{aligned} N(n,k,q,t) &= \langle \Delta_{e_{k-1}e_{n-k}e_1} \alpha_n p_n, s_n\rangle \\ &= \langle \Delta_{e_{k-1}e_1} \alpha_n p_n, e_{n-k}h_k\rangle \\ &= \langle \Delta_{e_k h_{n-k}} \alpha_n p_n, h_{k-1}e_1\rangle \\ &= \langle \Delta_{e_k h_{n-k}e_1} \alpha_k p_k, s_k\rangle \\ &= \langle \Delta_{h_{n-k}e_1} \alpha_k p_k, e_k\rangle \\ &= \langle \Delta_{h_{n-k}} e_k, e_k\rangle. \end{aligned}$$

The first and second equalities give

$$N(n,k,q,t) = (1+z)(1+w)C_n(q,t,w,z)\Big|_{z^k w^{n-k+1}} = (1+z)(1+w)C_n(q,t,w,z)\Big|_{z^{n-k+1}w^k}.$$

We can then write $N(n,k,q,t)$ as

$$(1+z)(1+w)\sum_{\pi} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi)} \prod_{\substack{b_i>b_{i-1}\\ i\geq 2}} (1+z/q^{b_i}) \prod_{\substack{a_i>a_{i-1}\\ i\geq 2}} (1+w/t^{a_i})\Big|_{z^k w^{n-k+1}}.$$

Recall that if $b_i > b_{i-1}$, then b_i corresponds to the top of a column. Thus, if we are taking the coefficient of z^k, we are asking that our path π contains at least k columns. By taking the coefficient of w^{n-k+1}, we are asking that our path contains at least $n-k$ North steps which are preceded by a North step, i.e. rows i for which $a_i > a_{i-1}$. But our path is from $(0,0)$ to (n,n), leaving only those paths which have precisely k columns. This answers the question of giving $N(n,k,q,t)$ as a sum over Dyck paths with exactly k columns. To be more precise, let $\overline{\mathrm{area}}(\pi)$ be the sum over all a_i for which $a_i \leq a_{i-1}$, and let $\overline{\mathrm{dinv}}(\pi)$ be the sum over all b_i for which $b_i \leq b_{i-1}$. In other words, $\overline{\mathrm{area}}(\pi)$ is the area contributed by rows i whose North step is preceded by an East step. Then,

$$N(n,k,q,t) = \sum_{\pi \text{ with } k \text{ columns}} t^{\overline{\mathrm{area}}(\pi)} q^{\overline{\mathrm{dinv}}(\pi)}. \tag{47}$$

One can then potentially define a q,t-analog of the Kreweras number $\mathrm{Krew}(\lambda)$ by restricting the sum (47) to paths π whose k columns have lengths which rearrange to λ. Unfortunately, examples for small n show that this way of defining a q,t-Kreweras number does not have the desired specialization at $t = 1/q$, and we must leave the question of whether this definition can be modified to obtain the desired $t = 1/q$ specialization as a question for future research.

The equality $N(n,k,q,t) = \langle \Delta_{h_{n-k}} e_k, e_k\rangle$, also found in [1], gives another natural way of decomposing $N(n,k,q,t)$. First note that

$$\Delta_{h_{n-k}} e_k = \sum_{\mu \vdash n-k} \Delta_{m_\mu} e_k.$$

If $\ell(\mu) > k$, we have $\Delta_{m_\mu} e_k = 0$. The partitions of n with k parts are in bijection with set of partitions of $n-k$ with at most k parts, simply by removing the first column. Let $\overline{\lambda}$ be the partition obtained from λ by removing the first column. Then

$$N(n,k,q,t) = \langle \Delta_{h_{n-k}} e_k, e_k \rangle = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \langle \Delta_{m_{\overline{\lambda}}} e_k, e_k \rangle.$$

It can be shown by the methods in [29] that $\langle \Delta_{m_{\overline{\lambda}}} e_k, e_k \rangle|_{q=1}$ gives the number of Dyck paths whose columns have lengths that rearrange to λ. The statistic on t is given by $\overline{\text{area}}$ as is the case in $N(n,k,q,t)$. This means

Remark 3. *For a partition λ of length k, we have*

$$\langle \Delta_{m_{\overline{\lambda}}} e_k, e_k \rangle|_{q=t=1} = \text{Krew}(\lambda),$$

This gives another potential q,t analogue of the Kreweras number. However, at $t = 1/q$, it also does not give the desired q-Kreweras number appearing in (42).

Our last observation is that $N(n,k,q,t)$ also appears in the symmetric function $\Delta_{e_{n-k+1}} \alpha_n p_n$. Recall that (24) gives

$$\Delta_{h_{n-k}} e_k = h_{n-k}^{\perp} \Delta_{e_{n-k+1}} \alpha_n p_n.$$

This means

$$\langle \Delta_{h_{n-k}} e_k, e_k \rangle = \langle \Delta_{e_{n-k+1}} \alpha_n p_n, h_{n-k} e_k \rangle,$$

embedding the q,t-Narayana numbers in the symmetric function $\Delta_{e_{n-k+1}} \alpha_n p_n$. In other words,

$$(-1)^{n-1} \langle \Delta_{e_{n-k+1}} p_n, h_{n-k} e_k \rangle = [n]_q [n]_t N(n,k,q,t).$$

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