Combinatorics Associated to Type A Nonsymmetric Macdonald Polynomials

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In 1988 Macdonald [8],[9] introduced symmetric functions $P_\lambda(X;q,t)$ which contain most of the previously studied families of symmetric functions as special cases. The $P_\lambda(X;q,t)$ are multivariate orthogonal polynomials which have become increasingly important in recent years. In 1995 Macdonald [10] introduced a refinement of this theory involving polynomials $E_\alpha(X;q,t)$, now called nonsymmetric Mcadonald polynomials, which also satisfy an orthogonality relation, and which are a basis for the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n](q,t)$ whose coefficients are rational functions in q,t. Here λ is a partition and α a weak composition. There are versions of the P_λ and E_α for arbitrary affine root systems, and Cherednik showed many of the properties of Macdonald polynomials have an interpretation in terms of the representation theory of his double affine Hecke algebra.

The P_λ and E_α have "integral forms" J_λ and $\mathcal{E}_\alpha$ associated to them, which are just scalar multiples of them which clear all denominators, resulting in a polynomial (i.e. an element of $\mathbb{Q}[x_1,\ldots,x_n,q,t]$). A few years ago Haiman, Loehr and the speaker [2] proved a combinatorial formula for the J_λ, and in subsequent work [3] obtained a corresponding combinatorial expression for the $\mathcal{E}_\alpha$. We will mostly use the notational conventions occurring in the discussion of the E_α formula from Appendix C of [1]. It involves nonattacking fillings, which are fillings of the diagram α' whose ith column has height α_i, with positive integers so that no two entries in the same row are equal, and no two entries in successive rows, with the entry in the upper row strictly to the right of the lower entry, are equal. Then

(1)

$$\mathcal{E}_\alpha(X;q,t)=\sum_\sigma x^\sigma q^{\mathrm{maj}}t^{\mathrm{coinv}}\prod_{\substack{s\in\alpha'\\ \sigma(s)\neq\sigma(South(s))}}(1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1})\prod_{\substack{s\in\alpha'\\ \sigma(s)=\sigma(South(s))}}(1-t),$$

where South(s) is the square right below s. The statistic maj is just the sum of the major index of the columns, while the more intricate statistic coinv is a sum, over pairs of squares in the same row, of a generalized concept of coinversion. Arm and leg lengths for composition diagrams are the same as in work of Knop and Sahi on Jack polynomials [6].

In (1) there is also a "basement" consisting of a row of squares below the diagram, which are filled with the numbers $(n,n-1,\ldots,1)$, and which are used in the computation of maj, coinv, and the description of nonattacking. To get the $\mathcal{E}_\alpha$ we need to use the diagram with column heights $(\alpha_n,\ldots,\alpha_1)$. A corresponding formula for J_λ, where λ is the partition rearrangement of α, can be obtained by simply changing the basement to $(2n,2n-1,\ldots,n+1)$. Also, by changing the basement to $(1,2,\ldots,n)$ and letting the ith column have height α_i, we get the version of the nonsymmetric Macdonald polynomial studied by Marshall [11], which we denote $\mathcal{E}'_\alpha$, which are essentially related to the $\mathcal{E}_\alpha$ by reversing the order of the variables, reversing the order of the parts of α, and sending $q \to 1/q$, $t \to 1/t$.

Note that the J_λ version of (1) implies that for $k \in \mathbb{N}$,

$$J_\lambda(X; q, q^k)/(1-q)^n|_{m_\lambda} \mathbb{N}[q], \tag{2}$$

i.e. the coefficient of a monomial symmetric function in (2) is a positive polynomial in q, since when $t = q^k$, each of the factors $(1 - q^{\text{leg}+1} t^{\text{arm}+1})$ or $(1-t)$ becomes $(1 - q^m)$ for some m. There are n of these factors, and combining them with the n powers of $1-q$ in the denominator of (2) we get a product of q-integers. Maple calculations indicate a stronger condition holds, namely Schur positivity.

Conjecture 1 For $k \in \mathbb{N}$,

$$J_\lambda(X; q, q^k)/(1-q)^n|_{s_\lambda} \in \mathbb{N}[q]. \tag{3}$$

During the talk Arun Ram suggested that Conjecture 1 can be embedded in a family of conjectures, where you expand $J_\lambda(X; q, q^m)$ in terms of the basis $J_\mu(X; q, q^{m-1})$, with a positivity condition for each m. Since $P_\mu(X; q, q) = s_\mu$, Ram's conjecture for $m = 2, 3, ..., k$ implies Conjecture 1. (Since the P_μ are not quite the J_μ, some slight modification in the statement of Ram's conjecture is needed.) After the talk Ram described some geometric heuristics involving Macdonald polynomials and quotients of determinants to the speaker which led Ram to his conjecture. These heuristics suggest some version of this phenomenon should hold for the $E_\alpha(X; q, t)$.

There is a lot of interesting combinatorics associated to the case $q = t = 0$ of (1). It is known that the Demazure character, or key polynomial, $\mathcal{K}_\alpha(x_1, \ldots, x_n)$ equals $\mathcal{E}_\alpha(x_1, \ldots, x_n; 0, 0)$, and furthermore the Demazure atom, or standard bases, $\mathcal{A}_\alpha(x_1, \ldots, x_n)$ equals $\mathcal{E}'_\alpha(x_1, \ldots, x_n; 0, 0)$. Standard bases were introduced by Lascoux and Schützenberger [7] in their study of Schubert varieties. They showed that the Schubert polynomial is a positive sum of key polynomials, and that the key polynomial is a positive sum of Demazure atoms. Further results on key polynomials were obtained by Reiner and Shimozono [14]. Now if you start with an identity of Macdonald which expresses P_λ as a sum, over compositions α whose rearrangement α^+ into partition order is λ, of $\mathcal{E}'_\alpha(x_1, \ldots, x_n; q, t)$, and then set $q = t = 0$, we get

$$s_\lambda = \sum_{\substack{\alpha \\ \alpha^+ = \lambda}} \mathcal{A}_\alpha(x_1, \ldots, x_n). \tag{4}$$

S. Mason [12], [13] has given a combinatorial proof of this identity by introducing a generalization of the RSK algorithm.

Recently K. Luoto, S. Mason, S. van Willigenburg and the speaker [4], [5] have introduced a new basis for the ring of quasisymmetric functions called quasisymmetric Schur functions, denoted $\mathrm{QS}_\beta(x_1, \ldots, x_n)$, where β is a (strong) composition of n. It is defined as the sum, over all (weak) compositions α which are shuffles of the parts of β and $n - \ell(\beta)$ zeros, of $\mathcal{A}_\alpha$. Properties of Mason's RSK algorithm are used to show these functions are quasisymmetric, and also to give a decomposition of them into Gessel's fundamental basis F_β. (QS_β equals the sum, over all standard tableaux T of β^+ which get mapped under Mason's RSK to one of the shapes α occurring in the decomposition of QS_β into atoms, of $F_{\mathrm{des}(T)}$.) The QS functions satisfy a refinement of the Littlewood-Richardson rule, as well as many other well-known properties of Schur functions. In a paper to be presented at the FPSAC 2010 conference this summer, A. Lauve and S. Mason have used this refined Littlewood-Richardson rule and other properties of QS functions to obtain an explicit basis of the quotient ring $\mathrm{QSYM}_n/\mathrm{SYM}_n$, where QSYM_n and SYM_n are the rings of quasisymmetric functions and symmetric functions in n variables, thus resolving a conjecture of F. Bergeron and C. Reutenauer.

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