Def: A SE lattice path $X$ from $(m,n)$ to $(0,0)$ $(m,n \neq (0,0))$ is admissible if it starts with a S-step and ends with an E-step.

Notation: 
- $\tilde{S}(i) = (h_i, b_i, d_i)$ where $h_i$ = # S-step on $h_{i-1}$, $b_i$ = $b_{i-1}$, $d_i$ = $d_{i-1}$
- $\tilde{E}(i) = (g_i, a_i, c_i)$ where $g_i$ = # E-step on $g_{i-1}$, $a_i$ = $a_{i-1}$, $c_i$ = $c_{i-1}$
- $D_{x} = D_{x_e} \cdot E_{x_e} = E_{x_e} \cdot D_{x_e}$
- $\tilde{Y}$ = transpose of $Y$ (S-step ↔ E-step) and hence also admissible
  - $\tilde{S}(Y) = (h_{1}, a_{2}, d_{2})$
  - $\tilde{E}(Y) = (g_{1}, b_{2}, c_{2})$
  - $E_{x} = \tilde{S}(Y)(x)$ (前进 $E_{x_{1}}, x_{2}, \ldots, x_{n} \rightarrow Y(x_{1}, x_{2}, \ldots, x_{n})$)

Prop 4.3.3: If $X$ is an admissible path, then $D_{x} = E_{x}$.

Proof: Let $X$ be an admissible path from $(m,n)$ to $(0,0)$, $m,n \neq (0,0)$.

When $m_i > 0$,

$$D_{x} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{I}) & \text{Adp}(X^{II}) \end{pmatrix} = \begin{pmatrix} \text{Adp}(X^{I}) \end{pmatrix} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{I}) \\
\text{Reg} \cdot \text{Adp}(X^{II}) \end{pmatrix} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{I}) \end{pmatrix} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{II}) \end{pmatrix} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{I}) \end{pmatrix}$$

When $m_i = 0$,

$$D_{x} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{I}) \end{pmatrix} = \begin{pmatrix} \text{Reg} \cdot \text{Adp}(X^{II}) \end{pmatrix}$$

Recall Prop 4.2.1:

$$[D_{x_{1}} D_{x_{2}} \ldots] = -M \frac{\partial}{\partial x_{1}} D_{x_{2}} \ldots$$

Hence, only consider cases $m_{i} \neq 0$.

$$\therefore \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots = \text{Adp}(X^{I}) D_{x_{1}} D_{x_{2}} \ldots$$
When $m = 1$, recall $E_1 = E$ and $r_1 = \text{nothing}$.

$$D_1 = D_1 = \frac{r_1}{E} = \text{nothing} = E_1 = E \quad \forall \text{ and } \forall$$

When $m, n > 1$, we proceed by induction.

**Assume $D_y = E_y$ for all admissible paths $x$ from $(m, n)$ to $(m', n')$ where $m' \leq m$ and $n' \leq n$ and $(m', n') \neq (m, n)$.

Denote $x_0$ path with 0 steps.

Suppose $y \neq x_0$. Then $y$ contains an $E$-step from $(m_0, n_0)$ to $(m, n)$ and a $S$-step from $(m, n)$ to $(m_1, n_1)$ where $m_1 > m_0$ and $n > n_0$. Set $m_0, m_1, n_0, n_1$.

Then $x = y \cdot x_0$ for shorter admissible paths $x$ and $y$, where $y \cdot x_0$ is defined to be the unique path obtained by `sandwiching` end of $x_0$ with start of $y$.

Here $x_0$ is an admissible path from $(m_0, n_0)$ to $(m, n)$ and $y$ is an admissible path from $(m, n)$ to $(m_1, n_1)$, and we form $y \cdot x_0$ by putting $x_0$ in $y$ at $(m, n)$.

$$\begin{align*}
E_{m_0, n_0} & = y_0, \\
E_{m, n} & = E_0, \\
E_{m_1, n_1} & = x_0(0)
\end{align*}$$

Define $y \cdot x_0$ to be the path obtained by `sandwiching` the last $E$-step of $x_0$ to a $S$-step and the first $S$-step of $y$ to an $E$-step and combine the new end of $x_0$ and new start of $y$ together.

Denote $y = y \cdot x_0$.

Recall $D_{m_0, n_0} = D_{m_0, n_0} - \frac{q_0}{E_0} E_{m_0, n_0} x_0(0) = \text{nothing} = D_{m, n}$ (cf. Eq. 1)

$$E_{m, n} E = E_{m, n} - \frac{q_0}{E_0} E_{m_0, n_0} x_0(0)$$

$$\therefore D_x D_y = D_x = \frac{q_0}{E_0} D_y$$

$$E_0 E \frac{q_0}{E_0} = E_0 - \frac{q_0}{E_0} E_0$$

By induction, $D_x = E_x$, $D_y = E_y$. Hence $D_x = \frac{q_0}{E_0} D_y = E_0 - \frac{q_0}{E_0} E_0 \Rightarrow D_x E_0 = \frac{q_0}{E_0} (D_y - E_0) = q_0 (D_y - E_0) = \cdots = (q_0)^{1} (D_y - E_0)$.

Hence it suffices to prove $D_y = E_y$, i.e. $D_y = E_y$ for $y \neq x_0$.
We can assume $D_{n+1}^{E_{m-1}} = E_{m-1,n-1}$, because the corresponding admissible paths start from $(0, m)$ and end at $(n, 0)$. 

\[
[D_{n+1}^{E_{m-1}}] = [\rho_{n+1}^{x(m-n)} D_{n+1}^{E_{m-1}}] = -M_{n+1}(Ad\, k(x^n)) D_{n+1}^{E_{m-1}} = -M(Ad\, k(x^n)) E_{m-1,n-1}
\]

Recall Prop. 4.2.4:

\[
D_{n+1}^{E_{m-1}} = -M(Ad\, k(x^n)) D_{n+1}^{E_{m-1}}\text{, hence only consider } m < n.
\]

Recall Lemma 4.1.2:

\[
(Adk(x^n)) E_{m-1,n-1}^n = \rho^n_{n-1} \left( \frac{\omega^n_{n-1, n-1} x^n}{n} \right)
\]

Hence:

\[
\sum_{g=1}^{n-1} \delta^{\rho^n_{n-1, n-1}}_{\rho^n_{g-1, n-1}} = \sum_{g=1}^{n-1} \delta^{\rho^n_{n-1, n-1}}_{\rho^n_{g-1, n-1}} E_{m-1,n-1}^n
\]

\[
\Rightarrow \sum_{g=1}^{n-1} \delta^{\rho^n_{n-1, n-1}}_{\rho^n_{g-1, n-1}} E_{m-1,n-1}^n = 0
\]

For $1 \leq g < n-1$:

\[
\delta^{\rho^n_{n-1, n-1}}_{\rho^n_{g-1, n-1}} = \text{corresponding } D \text{ and } E,
\]

Thus,

\[
D_{g-1} + \sum_{g=1}^{n-1} \delta^{\rho^n_{n-1, n-1}}_{\rho^n_{g-1, n-1}} E_{m-1,n-1}^n = 0
\]

Corollary 4.3.4: For any $a_\ldots a_2 \in E$, we have

\[
E_{a_\ldots a_2} \cdot 1 = E_{a_\ldots a_2} \cdot 0 \cdot 1 \quad \text{(independent of } a_2)\]

**Proof:** Suppose $a_2 \cdot 0 \cdot 1$. Then $a_{2 \ldots 2}$ corresponds to an admissible path from $(a, 0)$ to $(a_2, a_2)$.

Consider $D_{a_2}$. By using Lemma 4.2 in Prop. 4.2 (i.e., $a_2 = a$) we have $D_{a_2} E_{a_\ldots a_2} = 0$, independent of path $a_2 \rightarrow a_2$. Thus, the

\[
E_{a_\ldots a_2} \text{ is independent on the number of } E \text{ steps on } a_{2 \ldots 2} \text{ as long as there is one } E \text{ step on } a \text{ (as long as } a_2 \text{) to keep } E \text{ admissible.}
\]
Known. The symmetry of \( \mathbb{R}^n \): \( f(x') = \hat{f}(x'^{\text{min}}) \) sends \( E_{a_{1}, \ldots, a_{n}} \) to \( E_{a_{1}, \ldots, a_{n}} \cdot -1 \), where \( \hat{x} \mapsto \hat{x}^{\text{min}} \) and \( \hat{x}^{\text{min}} \mapsto \hat{x} \). This corresponds to

By Lemma 3.4 (Corollary 1.1), \( \forall k \in \mathbb{Z}, \forall a \in \mathbb{R}^n \), \( \forall \hat{x} \in \mathbb{R}^n \), \( \forall \hat{x}' \in \mathbb{R}^n \)

\[
v_k(F_{a_{1}, \ldots, a_{n}} \cdot \hat{x}) = F_{a_{1}, \ldots, a_{n}} (v_k \hat{x}')
\]

\[
\Rightarrow v_k(F_{a_{1}, \ldots, a_{n}} \cdot \hat{x}) = v_k(F_{a_{1}, \ldots, a_{n}} (v_k \hat{x}'))
\]

\[
\Rightarrow v_k(F_{a_{1}, \ldots, a_{n}} \cdot \hat{x}) = E_{a_{1}, \ldots, a_{n}} (v_k \hat{x}')
\]

\[
\Rightarrow (v_k (F_{a_{1}, \ldots, a_{n}} \cdot \hat{x}))' = (E_{a_{1}, \ldots, a_{n}} \cdot v_k \hat{x})'
\]

\[
\Rightarrow v_k (F_{a_{1}, \ldots, a_{n}} \cdot \hat{x} \cdot \cdot \cdot \hat{x}) = E_{a_{1}, \ldots, a_{n}} (v_k \hat{x})
\]

\[
\Rightarrow v_k (F_{a_{1}, \ldots, a_{n}} \cdot \hat{x}) = E_{a_{1}, \ldots, a_{n}} (v_k \hat{x})
\]

\[
\Rightarrow v_k = 1 \Rightarrow \forall a \in \mathbb{R}^n.
\]

For any \( a \in \mathbb{R}^n \), we can find a large enough \( k \) such that \( a + k > 0 \) \( \forall a \in \mathbb{R}^n \). For any \( a \in \mathbb{R}^n \), the result above states \( E_{a_{1}, \ldots, a_{n}} \cdot -1 = E_{a_{1}, \ldots, a_{n}} \cdot v_k \hat{x} \), \( a \in \mathbb{R}^n \cdot -1 \) \( = E_{a_{1}, \ldots, a_{n}} \cdot -1 \).

Then by the result above, \( E_{a_{1}, \ldots, a_{n}} \cdot -1 = E_{a_{1}, \ldots, a_{n}} \cdot v_k \hat{x} \Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot -1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x}) \)

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot -1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot \hat{x}
\]

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot 1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot 1
\]

\[
= \hat{x}
\]

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot 1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot 1
\]

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot 1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot 1
\]

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot 1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot 1
\]

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot 1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot 1
\]

\[
\Rightarrow E_{a_{1}, \ldots, a_{n}} \cdot 1 = v_k (E_{a_{1}, \ldots, a_{n}} \cdot \hat{x})
\]

\[
= E_{a_{1}, \ldots, a_{n}} \cdot 1
\]