For a box \( (i,j) \) (row \( i \), column \( j \)) of a skew Young diagram, \( C(i,j) \) is a constant.

Fix \( i > j \) on \( \lambda \) such that \( \lambda \) is a skew Young diagram. Then, adjusted content \( C(i,j) = C(i,j) + t \) where the last \( t \) in the \( i,j \) diagram changes.

Reading order: any total ordering of \( \lambda \) in which adjusted content is increasing.

In lexicographic order by content, then by rules of the diagrams. (Note with the same number in the same row diagram can be ordered arbitrarily.)

We write \( \nu = (\nu^1, \ldots, \nu^p) \), a tuple of \( k \) skew Young diagrams.

Let \( a, b \in \nu \) be two boxes in \( \nu \). \( a \) is called an attacking pair with \( b \) if \( a < b \) and \( i < j \).

We say that \( a, b \) form an attacking pair.

\[ \begin{array}{c}
\text{Def: Attacking Inversion} \\
\text{is an attacking pair } (a,b) \text{ with } a \text{ preceding } b \text{ and } \text{entry in box } a \text{ > entry in box } b.
\end{array} \]

\[ \begin{array}{c}
\text{E.g.} \nu = (3^2, 2^2) \quad \text{content of } a \\
C(1,2) = 1 \\
C(2,1) = 1, C(2,2) = 3 \\
C(3,1) = 1, C(3,2) = 3, C(3,3) = 3 \\
\text{in } G_{4}(x, x^2, x^3, x^6) \\
\text{ corresponds to } x^4 + x^5 + x^6 + x^8 + x^{10} + \ldots
\end{array} \]

\[ \begin{array}{c}
\text{Def: The Convoluted LLT polynomial} \text{ indexed by a tuple of skew Young diagrams } \nu \text{ is the generating function} \text{ of } G_{\nu}(x, q) = \sum_{T \in \text{set of } \nu} q^{|T|} x^{\text{content of } T}
\end{array} \]
Let $A = A_+ \cup A_-$ be a "signed" alphabet with $A_+ = \{1, 2, 3, \ldots\}$ and $A_- = \{1, 2, 3, \ldots\}$, and an ordinary total ordering on $A_+$.

1. In this paper, we choose the ordering $\cdots < 3 < 1 < 4 < 2 < \cdots$ (so 2 is the 3\textsuperscript{rd} $\neg$).

Def: A super tableau on a tuple $A$, drawn as shape $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$ is a map

$$T : \nu \rightarrow A \quad \text{(assigning each box in } \nu \text{ a letter in } A)$$

s.t. 1. rows and columns are totally increasing (from left to right along rows, bottom to top along columns)
2. positive letters in columns are strictly increasing (from left to right)
3. negative letters in rows are strictly increasing (from bottom to top)

A SSTT is a super tableau with all entries positive.

Notation: $\text{SSTT}_+(\nu) = \text{set of all super tableaux on } \nu$

Def: An attacking position in a super tableau is an attacking pair $(a, i)$ with $a$ preceding $i$ in reading order, s.t.
1. $T(a) > T(i)$, or
2. $T(a) = T(i)$ and $a \in A_-$ (equal and negative)

Hence, $\text{Inv}(T)$ also defines for super tableau $T$.

E.g. $\nu = (251, 31, 331)$

\[ T = \begin{array}{ccc}
& 1 & 2 \\
3 & & 4 \\
& 5 & 6
\end{array} \]

\[ \text{Inv}(T) = \{123, 145, 234, 356\} \]

Lemma 4.1.2. $\omega_T G_{T(i)}(X, q) = \sum_{T(a) > T(i)} q^{T(a) - T(i)} q^{T(a) - T(i)}$ where $X^T = \prod_{a \in T} T(a)$ and $Y^T = \prod_{i \in T} T(i)$

E.g. $\nu = (251, 31, 331)$

\[ T = \begin{array}{ccc}
& 1 & 2 \\
3 & & 4 \\
& 5 & 6
\end{array} \]

This contributes $q^{2} q^{5} X^{12} Y^{5} Y^{2} Y^{3}$ to $\omega_T G_{T(i)}(X, q)$ with $\nu = (251, 31, 331)$

Corollary 4.1.3. $\omega_T G_{T(i)}(X, q) = \sum_{T(a) > T(i)} q^{T(a) - T(i)} q^{T(a) - T(i)} x^T$ (with any negative terms)

Proof: Set $x = 0$ in Lemma 4.1.2. Then the only surviving terms on RHS are those $T$ with no positive letters.

Hence $\omega_T G_{T(i)}(X, q) = \sum_{T(a) > T(i)} q^{T(a) - T(i)} q^{T(a) - T(i)} x^T = \sum_{T(a) > T(i)} q^{T(a) - T(i)} q^{T(a) - T(i)} y^T$.

Result follows by a change of variables $T \rightarrow x$

$\blacksquare$
Prop. 4.1.4: Given a tuple of skew Young diagrams \( \nu = (\nu^0, \ldots, \nu^m) \), let \( \nu^R = (\nu^m, \ldots, \nu^0) \), where \( (\nu^R)^R \) is the iterated reflection of \( (\nu^R)^R \).

\[
\omega_{\nu^R}(x; q) = q^{2R} \omega_{\nu}(x q^{-1})
\]

where \( R = \text{#attacking pairs in } \nu \).

\[
\nu^R = (32/1, 22/1, \ldots, 21/1)
\]

Proof: Given a positive tableau \( T \) in \( \nu \), let \( T^R \) be the tableau on \( \nu^R \) obtained by reflecting the tableau along with \( x \) and changing negative letters to positive letters. Then \( \nu^R \in \text{SSYT}(\nu^R) \) and \( \nu^R T^R \) is weight preserving, and \( \text{inv}(\nu) = \text{inv}(\nu^R) \).

An attacking pair in \( \nu \) is an inversion in \( T \) if and only if the corresponding attacking pair in \( \nu^R \) is a non-inversion in \( T^R \). (since contents are preserved, attacking pair in \( \nu \) is still an attacking pair in \( \nu^R \)).

Thus, \( \nu^R(T) = \text{inv}(\nu^R) \) if and only if \( \text{inv}(\nu^R) = \text{inv}(\nu) \) if and only if \( \nu^R T^R \in \text{SSYT}(\nu^R) \), equivalently, \( \nu^R T^R \in \nu \) if and only if \( \nu = \nu(T) \).

By Corollary 4.1.3,
\[
\omega_{\nu^R}(x; q) = \sum_{T \in \text{SSYT}(\nu^R)} q^{\text{inv}(T)} x^T = \sum_{T \in \text{SSYT}(\nu^R)} q^{\text{inv}(T^R)} x^{\nu^R T^R} = q^{2R} \omega_{\nu}(x q^{-1})
\]

Lemma 4.1.6: \( G_\lambda(x; q) \) is a linear combination of Schur functions \( s_\mu(x) \) s.t. \( \lambda \preceq \mu \).}

Proof: Let \( r \) be total no. of rows in \( \nu \). Then it is equivalent to show that \( \omega_{\nu^R}(x; q) \) is a linear combination of \( s_\lambda(x) \) s.t. \( \lambda \preceq r \).

By Prop. 4.1.4, \( \omega_{\nu^R}(x; q) \) has a monomial term \( q^{2R} x^\nu \) for \( \nu = (\nu^R)^R \).

Since a term can appear at most once in each column of \( \nu^R \), the exponents of \( x^\nu \) bounded above by \( R \) columns in \( \nu^R \) which is also the number of rows in \( \nu \), i.e., \( r \).