A proof of the Stanley-Stembridge conjecture

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Motivation

geometric rep th

Elliptic world?

Rep th

K-th canonical bases

(microlocal) perverse sheaves

Alg geom $\xleftrightarrow{\text{duality?}}$ Topology

mirror sym

(3d) geom Langlands

e.g. equiv local.

prob?

$t=\infty$

$\leftarrow (q,t) \leftarrow (q,t)$ Kato's formula

chrom sym fcn

e.g. cell decomp

Hessenberg variety

Formula B $\overset{?}{=}$ Formula A

$t=1$

Conj

Example Shuffle Thm (Haglund-Haiman-Loehr-Remmel-Ulyanov

proved Carlsson-Mellit)

$$\nabla e_n = \sum_{\lambda : \text{Dyck path}} D_\lambda$$

Macdonald exp

monomial sym fcn exp

Stanley-Stembridge conj

probabilistic formula = Definition of chromatic quasi sym fcn

§ Introduction

Stanley-Stembridge '93 conj about immanants of Jacobi-Trudi matrices

Stanley '95 chromatic symmetric function

Γ : graph $\rightsquigarrow$ X_Γ : sym fcn

<u>Conj</u> Γ : incomparability graph of $(3+1)$-free poset

$\Rightarrow$ X_Γ : e-positive

<u>Guay-Paquet</u> : enough to prove $(3+1)$-free & $(2+2)$-free case

($\Longleftrightarrow$ unit interval order.)

$\{$unit interval order$\}$ $\overset{1:1}{\longleftrightarrow}$ many combinatorial objects

<u>e.g.</u>
- Dyck path
- Hessenberg function
- 312-avoiding permutations
- ...

<u>Shareshian-Wachs</u> : q-analogue of X_Γ, $X_\Gamma(q)$

"chromatic quasisym fcn"

for Γ : labeled graph

$\Gamma = ([n] = \{1, 2, \ldots, n\},\ \text{Edges})$

$\kappa : [n] \to \mathbb{Z}_{>0}$: proper coloring

$\overset{\text{def}}{\iff}$ $\kappa(i) \neq \kappa(j)$ if $i - j \in \Gamma$

$$\mathrm{asc}(\kappa) := \#\left\{ i - j \in \Gamma \;\middle|\; \begin{array}{l} i<j \\ \kappa(i) < \kappa(j) \end{array} \right\}$$

$$\boxed{X_\Gamma(q) := \sum_{\kappa:\text{ proper coloring}} q^{\mathrm{asc}(\kappa)} \prod_{i \in [n]} x_{\kappa(i)}}$$

<u>Fact</u> (Shareshian-Wachs)
Γ : unit interval graph $\Rightarrow$ $X_\Gamma(q)$: symmetric

<u>Conj</u> (SW) $X_\Gamma(q)$: e-positive

<u>Main Thm</u> $\forall \Gamma$: unit interval graph
$\forall q \in \mathbb{R}_{>0}$, $X_\Gamma(q)$: e-positive
In particular, the Stanley-Stembridge conj holds.

<u>Idea</u> : Find probabilistic formula for $X_\Gamma(q)$

<u>Observation</u>

$$X_\Gamma(q) = \sum_{\lambda \vdash n} \underline{c_\lambda(\Gamma;q)}\, e_\lambda$$

$$\to \boxed{\sum_{\lambda \vdash n} q^{(\lambda_i)} \frac{c_\lambda(\Gamma;q)}{\prod [\lambda_i]_q!} = 1}\ !$$

: $p_\lambda(\Gamma;q)$

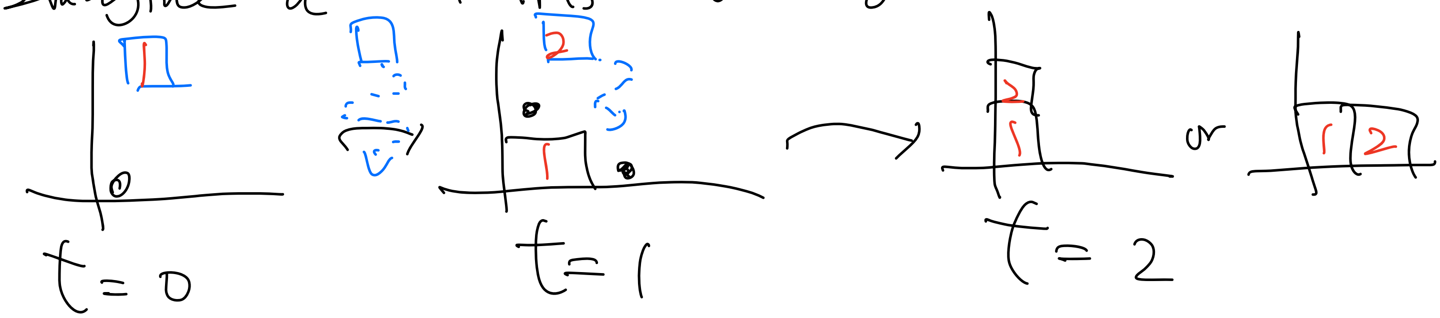
<u>Rem</u> <u>$q=1$</u> : easy (enough to look at coeff of $x_1 x_2 \cdots x_n$)

Proof Both satisfies the modular law (Guay-Paquet ...)
Use Abreu-Nigro's characterization

$$X_P = \frac{X_{P'} + q X_{P''}}{1+q}$$

Strategy of proof

- Imagine a "Tetris-like" game:



$\longrightarrow$ ...

(difficult)

Try to find what transition probability gives P_λ

- Check the modular law (easy) straight-forward

Problem $\nexists$ such transition probability
$\{P_\lambda(P:q)\}$: is not Markov

Idea : "Markovization"
remember all the past states

(Imagine "sheafification")

$\rightsquigarrow$ Markov process on the set of standard Young tableau (of size n)

$$\widetilde{V}_n := \mathbb{Q}(q)\langle T \mid T \in SYT(\lambda)\rangle_{\lambda \vdash n}$$

$$\downarrow \pi \text{ : forget}$$

$$V_n := \mathbb{Q}(q)\langle \lambda \mid \lambda \vdash n\rangle$$

<u>Problem</u> Find $\Omega_r : \widetilde{V}_n \to \widetilde{V}_{n+1}$ $(0 \le r \le n)$

s.t.

(i) $\sum_{\lambda \vdash n} p_\lambda(\Gamma_e ; q)\lambda = \pi\, \Omega_{e(n)} \Omega_{e(n-1)} \cdots \Omega_{e(1)}(\phi)$

e : "conjugate Hessenberg function" $e : [n] \to \mathbb{Z}$, $0 \le e(i) < i$, $e(i) \le e(i+1)$

(ii) $\Omega_r(T)$ is a linear combination of SYT T' obtained by adding $\boxed{n+1}$ on top of some column.

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<u>Today</u>: How to find a probabilistic formula for $X_\Gamma(q)$ (Γ: unit interval graph)

§ Notations

$[n] = \{1, 2, \dots, n\}$

<u>Def</u> $e : [n] \longrightarrow \mathbb{Z}$ is called conjugate Hessenberg fcn

if $\begin{cases} 0 \leq e(i) < i \\ e(i) \leq e(i+1) \end{cases}$ $\forall i \in [n]$

$(e(n+1) := n-1)$

Γ_{e^c} $\mathbb{E}_n := \{ e : [n] \to \mathbb{Z} \mid e: \text{conj. Hess. fcn} \}$

UIG $\quad \updownarrow$ 1:1 $\qquad e \leftrightarrow h \qquad (h(i) = n - e(n+1-i)) \quad \forall i \in [n]$

Γ_h $\mathbb{H}_n := \{ h : [n] \to \mathbb{Z} \mid \begin{array}{l} h(i) \geq i \\ h(i) \leq h(i+1) \end{array} \}$

$\leftrightarrow h$

$(h(n+1) = n)$

<u>cf</u> $h \in \mathbb{H}_n \rightsquigarrow X_h$: Hessenberg var

$H^*(X_h) \circlearrowleft \mathfrak{S}_n$

$\mathrm{Prob}(H^*(X_h)) = \omega(X_{\Gamma_h}(q))$

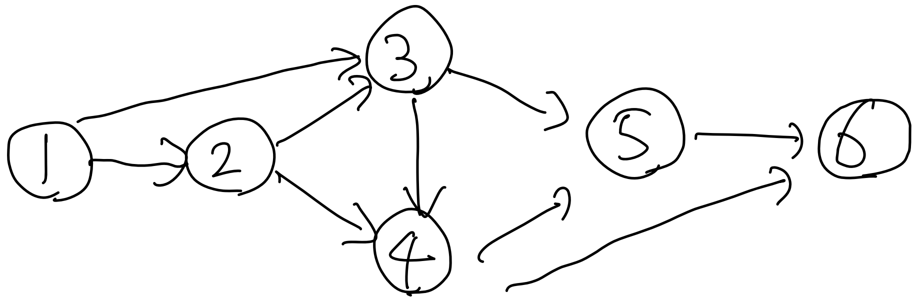
$e_r \leftrightarrow h_r$

(elem sym fn $e_r(x)$)

$e \in \mathbb{E}_n \rightsquigarrow \Gamma_e$: unit interval graph

$$\left(\begin{array}{ll} \text{vertices} : & [n] \\ \text{edges} : & i \longrightarrow j \ \text{ iff } \ e(j) < i < j \end{array}\right.$$

Ex $\quad e = (0,0,0,1,2,3) \in \mathbb{E}_6$

$\Gamma_e =$ 

Def unit interval graph is a graph Γ_e for some $e \in \mathbb{E}_n$

Ex $\quad e_n \in \mathbb{E}_n$ by $e_n(i) = 0$

$\rightsquigarrow \Gamma_{e_n} =$ complete graph

$\rightsquigarrow X_{\Gamma_{e_n}}(q) = [n]_q! \cdot e_n(x)$

Def $\quad e \in \mathbb{E}_n,\ e' \in \mathbb{E}_{n'} \rightsquigarrow e \cup e' \in \mathbb{E}_{n+n'}$ by

$$e \cup e'(i) = \begin{cases} e(i) & 1 \le i \le n \\ n + e'(i-n) & n+1 \le i \le n+n' \end{cases}$$

$\rightsquigarrow \Gamma_{e \cup e'} = \Gamma_e \cup \Gamma_{e'}$ (ordered disjoint union)

shift the label by n

$\rightsquigarrow \quad X_{\Gamma_{e \vee e'}}(q) = X_{\Gamma_e}(q) \cdot X_{\Gamma_{e'}}(q)$

For $\mu = (\mu_1, \cdots, \mu_\ell)$: composition of n

$$e_\mu := e_{\mu_1} \vee e_{\mu_2} \vee \cdots \vee e_{\mu_\ell}$$
$$= (0, \cdots, 0, \mu_1, \cdots, \mu_1, \mu_1 + \mu_2, \cdots)$$

$$\boxed{X_{\Gamma_{e_\mu}}(q) = \prod [\mu_i]_q! \cdot e_\lambda(x)}$$

$\lambda \vdash n$: rearrangement of μ

$e \in \mathbb{E}_n$, $\quad X_{\Gamma_e}(q) = \sum_{\lambda \vdash n} c_\lambda(\Gamma_e : q) \cdot e_\lambda(x)$

recall

Prop
$$\boxed{\sum_{\lambda \vdash n} q^{|e| - |e_\lambda|} \frac{c_\lambda(\Gamma_e : q)}{\prod_i [\lambda_i]_q!} = 1}$$

$$\left(\begin{array}{l} |e| = e(1) + e(2) + \cdots + e(n) \\ |e_\lambda| = \sum_{i<j} \lambda_i \lambda_j \end{array} \right)$$

Def $\quad P_\lambda(\Gamma_e : q) := q^{|e| - |e_\lambda|} \frac{c_\lambda(\Gamma_e : q)}{\prod [\lambda_i]_q!}$

$$V_n := \mathbb{Q}(q) \langle \lambda \mid \lambda \vdash n \rangle$$

$$\boxed{\begin{aligned} \Phi : E_n &\longrightarrow V_n \\ e &\longmapsto \sum_{\lambda \vdash n} \Pr(\Gamma_e ; q)\cdot \lambda \end{aligned}}$$

- Want to understand Φ

§ First attempt

Try to find a linear map $\Omega_r : V_n \to V_{n+1}$ $(0 \le r \le n)$

s.t. $\Omega_r(\Phi(e)) = \Phi(\underbrace{e+r}_{= (e(1), e(2), \dots, e(n), r)})$

Ex $n=0$

$$\Omega_0(\phi) = \Phi(0) = \ydiagram{1}$$

$n=1$

$$\Omega_0(\ydiagram{1}) = \Phi(0,0) = \ydiagram{2}$$

$$\Omega_1(\ydiagram{1}) = \Phi(0,1) = \ydiagram{1,1}$$

$n=2$

$$\Omega_0(\ydiagram{2}) = \Phi(0,0,0) = \ydiagram{3}$$

$$\Omega_1(\ydiagram{2}) = \Phi(0,0,1) = \frac{1}{[2]_q}\ydiagram{2,1} + \frac{q}{[2]_q}\ydiagram{3}$$

$$\Omega_2(\ydiagram{2}) = \Phi(0,0,2) = \ydiagram{2,1}$$

$$\Omega_1(\ydiagram{1,1}) = \Phi(0,1,1) = \ydiagram{2,1}$$

$$\Omega_2(\ydiagram{1,1}) = \Phi(0,1,2) = \ydiagram{1,1,1}$$

(do not need $\Omega_0(\ydiagram{1,1})$, but $\underbrace{\Omega_0(\ydiagram{1,1}) = \ydiagram{2,1}}_{\text{natural to take}}$)

<u>$n=3$</u> $$\Omega_r(\ydiagram{3}) = \frac{[r]_q}{[3]_q}\ydiagram{3,1} + \frac{q^r[3-r]_q}{[3]_q}\ydiagram{4}$$

$$\left(\Omega_r(\ydiagram{4}) = \cdots\right)$$

$$\Omega_1(\Phi(0,0,1)) = \Phi(0,0,1,1) = \frac{[2]_q}{[3]_q}\ydiagram{3,1} + \frac{q^2}{[3]_q}\ydiagram{4}$$

$$\parallel$$

$$\frac{1}{[2]_q}\Omega_1(\ydiagram{2,1}) + \frac{q}{[2]_q}\Omega_1(\ydiagram{3})$$

$$\Rightarrow \Omega_1(\ydiagram{2,1}) = \ydiagram{3,1}$$

$$\Omega_2(\Phi(0,0,1)) = \Phi(0,0,1,2) = \frac{1}{[2]_q}\ydiagram{2,2} + \frac{q}{[3]_q}\ydiagram{3,1} + \frac{q^3}{[2]_q[3]_q}\ydiagram{4}$$

$$\parallel$$

$$\frac{1}{[2]_q}\Omega_2(\ydiagram{2,1}) + \frac{q}{[2]_q}\Omega_2(\ydiagram{3})$$

$$\rightsquigarrow \Omega_2(\ydiagram{2,1}) = \ydiagram{2,2}$$

$$\Omega_2(\Phi(0,0,2)) = \Phi(0,0,2,2) = \ydiagram{2,2}$$

$$\parallel$$

$$\Omega_2(\ydiagram{2,1})$$

consistent

But $\Omega_2(\Phi(0,1,1)) \neq \Phi(0,1,1,2) = \frac{1}{[2]_q}\,\ydiagram{1,1,2} + \frac{q}{[2]_q}\,\ydiagram{1,3}$

$\qquad \| $

$\quad \Omega_2(\ydiagram{1,2})$

inconsistent!

$\Phi(0,0,2): \quad \phi \longrightarrow \begin{ytableau} 1 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 \end{ytableau} \longrightarrow \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$

$\Phi(0,1,1): \quad \phi \longrightarrow \begin{ytableau} 1 \end{ytableau} \longrightarrow \begin{ytableau} 2 \\ 1 \end{ytableau} \longrightarrow \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$

$\rightarrow$ transition probability depends on the past states (not Markov)

§ Markovization

Idea: Use standard Young tableau

$$\tilde{V}_n := \mathbb{Q}(q)\langle T \mid T \in SYT(\lambda),\ \lambda \vdash n \rangle$$

$$\downarrow \pi \qquad (\pi(T) = \lambda)$$

$$V_n$$

Prob Find $\tilde{\Omega}_r : \tilde{V}_n \longrightarrow \tilde{V}_{n+1}$ s.t.

(i) $\Phi(e) = \pi\, \Omega_{e(n)} \Omega_{e(n-1)} \cdots \Omega_{e(1)}(\phi)$

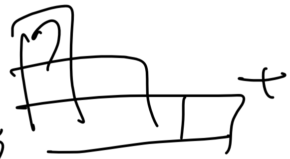
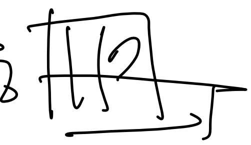
(ii) $\Omega_r(T) = \sum [illegible]\, T'$

T': adding $\boxed{n+1}$ to T

Ex. $\tilde{\Omega}_2\left(\begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$

$\tilde{\Omega}_2\left(\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array}\right) = \frac{1}{[2]_q}\begin{array}{|c|c|}\hline 4 & \\ \hline 2 & \\ \hline 1 & 3 \\ \hline\end{array} + \frac{q}{[2]_q}\begin{array}{|c|c|c|}\hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline\end{array}$

One can continue to calculate $\Omega_r(T)$!
(Up to $n \leq 5$)
Formulas are very simple except for

$\Omega_4\left(\begin{array}{|c|c|c|c|}\hline 4 & 6 & & \\ \hline 1 & 2 & 3 & 5 \\ \hline\end{array}\right) = \frac{[3]_q}{[2]_q[4]_q}$  $+ \frac{q}{[2]_q^3}$ 

$+ \frac{q^2[3]_q}{[2]_q[4]_q}$ 

Observation

$\Omega_r(T)$ depends only on the places of $\boxed{i}$ with $i > r$ on the top of ~~the~~ columns

Def $T \in SYT_n$, $r \rightsquigarrow \delta(T,r)$: "Maya diagram"
$:= (\delta_i)_{i \in \mathbb{Z}}$ $\left(\text{st } \begin{array}{l}\delta_{\ll 0} = 1 \\ \delta_{\gg 0} = 0\end{array}\right)$

$$\delta_i = \begin{cases} 1 & i \leq 0 \text{ or the top box in the } i\text{-th col of } T \text{ has } \boxed{j} > r \\ 0 & i > \lambda_1 \text{ or } \text{''} \ \boxed{j} \leq r \end{cases}$$

transition probability depends only on $\delta(T, r)$

~~this~~ this should be defined on "Maya diagram"

$$
\begin{tikzcd}
SYT_n \arrow[r, "\delta(-,r)"] \arrow[d, "\widetilde{\Omega}_r"'] \arrow[dr, phantom, "\circlearrowright"] & \text{Maya} \arrow[d, "\overline{\Omega}_r"] \\
SYT_{n+1} \arrow[r, "\delta(-,r)"] & \text{Maya}
\end{tikzcd}
$$

Recall

$$T \in SYT_n \xrightarrow{\delta(-,r)} \text{Maya diagram}$$

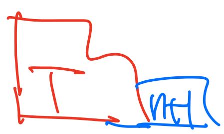
- look at T from above
- replace $i > r$ with 1 & $i \le r$ with 0
- add 1^∞ to the left & 0^∞ to the right

$$T = \begin{array}{|c|c|c|c|}\hline 4 & 6 \\ \hline 1 & 2 & 3 & 5 \\ \hline\end{array} \longrightarrow 4\,6\,3\,5$$

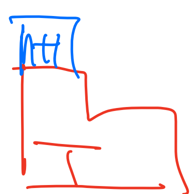
$$r = 4 \qquad \longrightarrow 0\,1\,0\,1 \qquad \delta_1$$

$$\longrightarrow \delta(T,4) = (1^\infty, 0, 1, 0, 1, 0^\infty)$$

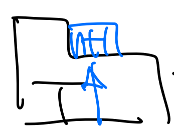
$\underline{r=0}$: $\delta(T,0) = (1^\infty, 1^{\lambda_1}, 0^\infty)$



$\underline{r=n}$: $\delta(T,n) = (1^\infty, 0^{\lambda_1}, 0^\infty)$



Observation

- new box can be added only at the places of leftmost consecutive sequence of 0 in $\delta(T,r)$
- If $\delta(T,r) = (1^\infty, 0^\infty) \rightsquigarrow \widetilde{\Omega}_r(T) =$  $=: f_c(T)$

c-th component

c-th column

$\cdot$ If $\delta(T,r) = (1^\infty 0^a 1^b 0^\infty)$ $\quad a, b \in \mathbb{Z}_{>0}$

$\uparrow$ c-th comp

$\delta = (\cdots 1, \overbrace{0, 0, \cdots 0}^{a}, \overbrace{1 \cdots 1}^{b}, 0, \cdots)$

$$\rightsquigarrow \widetilde{\Omega}_r(T) = \frac{[a]_q}{[a+b]_q} f_c(T) + \frac{q^a [b]_q}{[a+b]_q} f_{a+b+c}(T)$$

$\cdot$ If $\delta(T,r) = (1^\infty 0^a 1^b 0^c 1^d 0^\infty)$, then

(e-th comp at the first 0)

$$\widetilde{\Omega}_r(T) = \frac{[a]_q [a+b+c]_q}{[a+b]_q [a+b+c+d]_q} f_e(T) + \frac{q^a [b]_q [c]_q}{[a+b]_q [c+d]_q} f_{a+b+e}(T)$$

$$+ \frac{q^{a+c} [b+c+d]_q [d]_q}{[a+b+c+d]_q [c+d]_q} f_{a+b+c+d+e}(T)$$

$\rightsquigarrow$ easy to guess the general formula!

<u>Def</u> $\delta = (\delta_i)_{i \in \mathbb{Z}}$: Maya diagram

$$W(\delta) := \{ i \in \mathbb{Z} \mid \delta_i = 0, \delta_{i-1} = 1 \}$$

(leftmost 0)

$$R(\delta) := \{ i \in \mathbb{Z} \mid \delta_i = 1, \delta_{i-1} = 0 \}$$

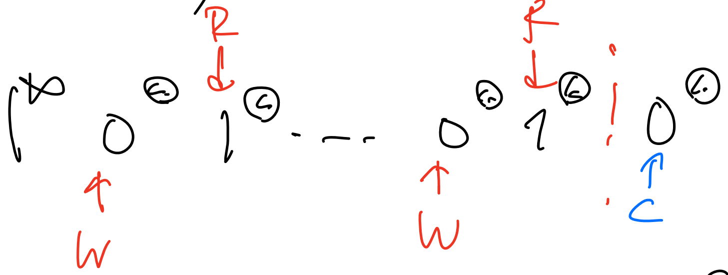
(leftmost 1)

Def (H.. reformulation by Guay-Paquet)

Define a linear map $\widetilde{\Omega}_r : \widetilde{V}_n \longrightarrow \widetilde{V}_{n+1}$ by

$$\boxed{\widetilde{\Omega}_r(T) := \sum_{c \in W(\delta(T,r))} \frac{\prod_{a \in R(\delta(T,r))} [a-c]_q}{\prod_{b \in W(\delta(T,r)) \setminus \{c\}} [b-c]_q} f_c(T)}$$

Rmk . $c \in W(\delta) \Rightarrow |R(\delta)_{<c}| = |W(\delta)_{<c}|$



$\Rightarrow$ (coeff) > 0 (if $q \in \mathbb{R}_{>0}$)

• This process is canonically defined on Maya diagram:

$$\Omega : \mathrm{Span}\langle \mathrm{Maya} \rangle \longrightarrow \mathrm{Span}\langle \mathrm{Maya} \rangle$$

$$\delta \longmapsto \sum_{c \in W(\delta)} \frac{\prod_{a \in R(\delta)} [a-c]_q}{\prod_{b \in W(\delta)} [b-c]_q} f_c(\delta)$$

replace c-th 0 by 1

$m \in \mathbb{Z}$

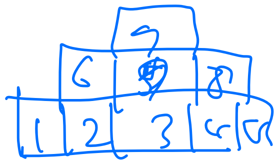
$[m]_q := \frac{1-q^m}{1-q}$

We change "how to read Maya diagram from T" by the data of conjugate Hessenberg fcn e

$[-1]_q = \frac{1-q^{-1}}{1-q} = -q^{-1}$

$[-2]_q = \frac{1-q^{-2}}{1-q} = -q^{-2}[2]_q \cdots$

- The condition $e(i) \leq e(i+1)$ is not used anywhere here, but their meanings are not clear (to me)
- There might be other way to associate Maya diagram from other combinatorial objects



<u>Prop</u>

$$\sum_{c\in W(\delta)} \frac{\prod_{a\in R(\delta)} [a-c]_q}{\prod_{b\in W(\delta)\setminus\{c\}} [b-c]_q} = 1.$$

<u>e.g.</u> $T = \begin{array}{|c|c|c|c|}\hline 4 & 6 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array} \sim \delta = (1^\infty \overset{\delta_1}{0}\,1\,0\,1\,0^\infty)$

$W = \{1, 3, 5\}$

$R = \{2, 4\}$ $\rightarrow$ 8

$$\widetilde{\Omega}_4(T) = \frac{[2-1]_q[4-1]_q}{[3-1]_q[5-1]_q} f_1(T) + \frac{\overset{-q^{-1}[1]_q}{[2-3]_q}\,[4-3]_q}{\underbrace{[1-3]_q}_{-q^{-2}[2]_q}\,[5-3]_q} f_3(T)$$

$$+ \frac{[2-s]_q\,[4-s]_q}{[1-s]_q\,[3-s]_q} f_8(T)$$

<u>Theorem</u> $\forall e \in \mathbb{E}_n$

$$\boxed{\Phi(e) = \pi\, \tilde{\Omega}_{e(n)} \tilde{\Omega}_{e(n-1)} \cdots \tilde{\Omega}_{e(1)}(\emptyset)}$$

$$\Phi(e) = \left(\sum q^{|e|-|e_\lambda|} \frac{G_\lambda(T_e; q)}{\prod [\lambda_i]_q!}\, \lambda \right)$$

<u>Cor</u> The Stanley-Stembridge conj holds.

§ Modular law

<u>Def</u> V : $\mathbb{Q}(q)$-vector space

A map $X : \mathbb{E}_n \longrightarrow V$ satisfies the <u>modular law</u> if

$$X(e) = \frac{X(e') + q X(e'')}{1+q}$$

for any <u>modular triple</u> (e, e', e''), i.e. satisfying one of the following: (Recall $e(n+1) = n-1$)

(I) $1 < {}^{\exists} i \leq n$ s.t. $e(i-1) < e(i) < e(i+1)$

and $\boxed{e(e(i)) = e(e(i)+1)}$

$$e'(j) = \begin{cases} e(j) & j \neq i \\ e(i)-1 & j = i \end{cases}$$

$$e''(j) = \begin{cases} e(j) & j \neq i \\ e(i)+1 & j = i \end{cases}$$

(II) $1 \leq {}^{\exists} i < n$ s.t. $e(i+1) = e(i)+1$

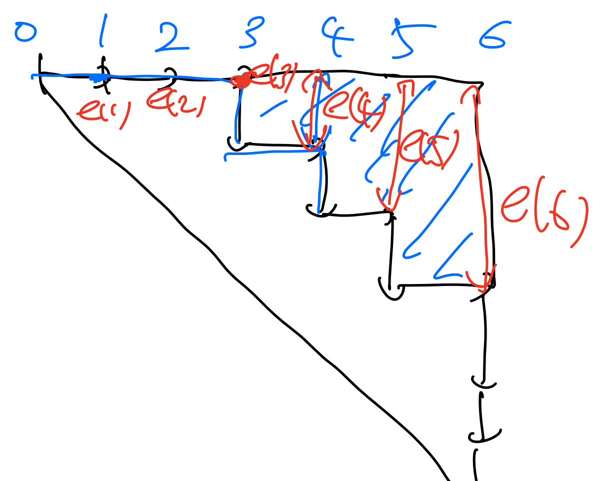
and $\boxed{e^{-1}(i) = \phi}$

$$e'(j) = \begin{cases} e(j) & j \neq i, i+1 \\ e(i) & j = i, i+1 \end{cases}$$

$$e''(j) = \begin{cases} e(j) & j \neq i, i+1 \\ e(i+1) & j = i, i+1 \end{cases}$$

Rmk $e \in \mathbb{E}_n \longleftrightarrow$ Dyck path

$e = (0, 0, 0, 1, 2, 3) \longleftrightarrow$



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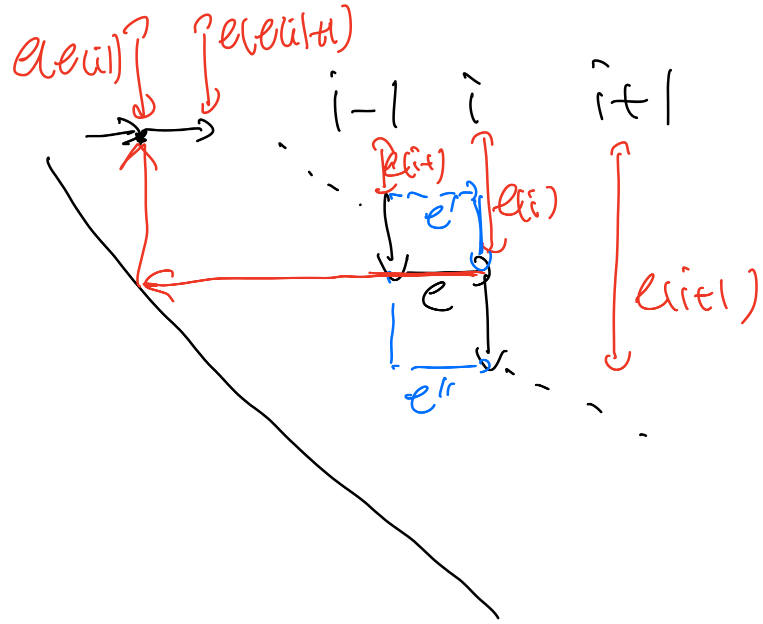

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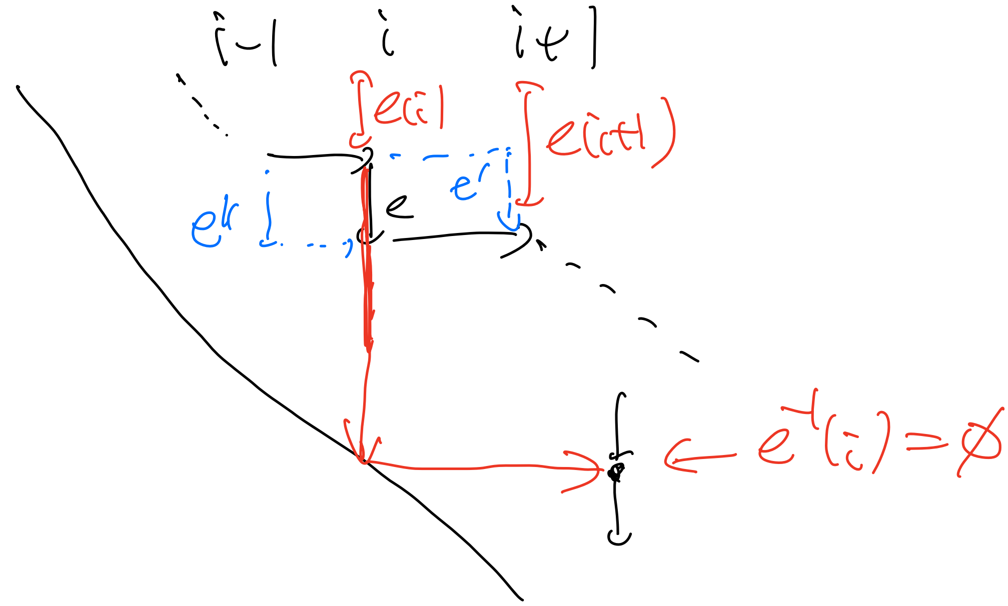


Thm (Abreu-Nigro '21)

If $\chi : \mathbb{E}_n \to V$ satisfies the modular law, then it is uniquely determined by the values $\chi(e_\mu)$ for any composition μ of n.

$\chi(e_\mu)$, $e_\mu = e_{\mu_1} \cup e_{\mu_2} \cup \cdots \cup e_{\mu_\ell}$

$e_n = (0, 0, \dots, 0)$

In particular, the map $e \mapsto X_{P_e}(q)$ is characterized by

- the modular law
- $X_{P_{e_\mu}}(q) = \prod [\mu_i]_q! \, e_\lambda(x)$

($\lambda \vdash n$: rearrangement of μ)

<u>Rmk</u> $\Phi : E_n \longrightarrow V_n$ satisfies

$$\Phi(e) = \frac{q\,\Phi(e') + \Phi(e'')}{1+q}$$

for any modular triple (e, e', e'')

☺ $c_\lambda(P_e : q)$: satisfies the modular law

$|e'| = |e| - 1 \quad |e''| = |e| + 1$

$\left(|e| = e(1) + e(2) + \cdots + e(n) \right)$

$$\rightsquigarrow q^{|e|} c_\lambda(P_e : q) = \frac{q^{|e|} c_\lambda(P_{e'} : q) + q^{|e|+1} c_\lambda(P_{e''} : q)}{1+q}$$

$$= \frac{q \cdot q^{|e'|} c_\lambda(P_{e'} : q) + q^{|e''|} c_\lambda(P_{e''} : q)}{1+q}$$

Cor If $\Phi' : \mathbb{E}_n \longrightarrow V_n$ satisfies

- $\Phi'(e) = \dfrac{q\Phi'(e') + \Phi'(e'')}{1+q}$ $\forall (e, e', e'')$ modular triple

- $\Phi'(e_\mu) = \lambda$

then $\Phi = \Phi'$

<u>Recall</u> δ : Maya diagram

$$c \in W(\delta) := \{ i \mid \delta_{i-1} = 1, \delta_i = 0 \}$$

$$\rightsquigarrow \varphi_c(\delta) := \frac{\prod_{a \in R(\delta)} [a-c]_q}{\prod_{b \in W(\delta) \setminus \{c\}} [b-c]_q}$$

$$\tilde{\Omega}_r(T) := \sum_{c \in W(\delta(T,r))} \varphi_c(\delta(T,r))\, f_c(T)$$

<u>Thm</u> $e \in \mathbb{E}_n$

$$\Phi'(e) := \pi\, \tilde{\Omega}_{e(n)} \tilde{\Omega}_{e(n-1)} \cdots \tilde{\Omega}_{e(1)}(\emptyset)$$

satisfies the modular law (for q^{-1})

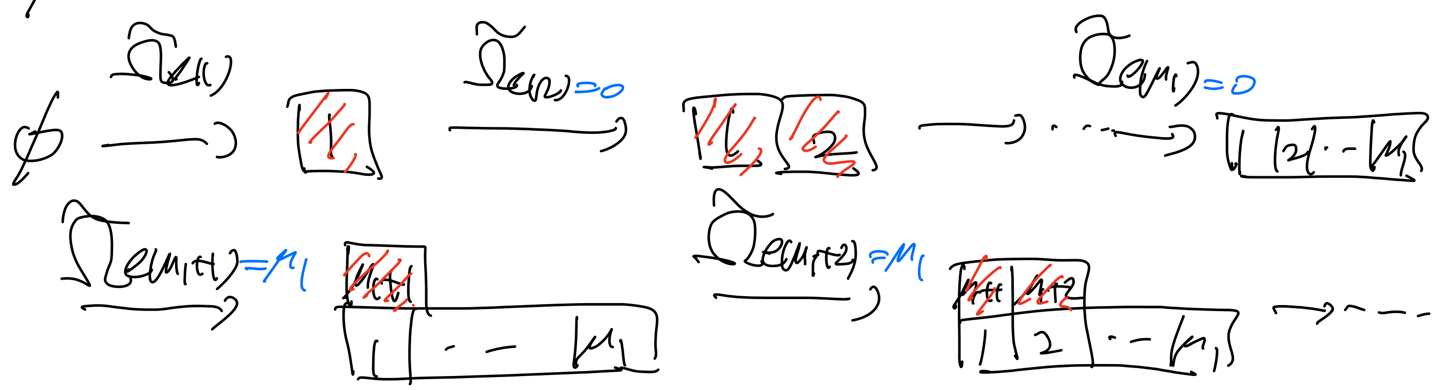
$$\& \quad \Phi'(e_\mu) = \lambda$$

μ : composition

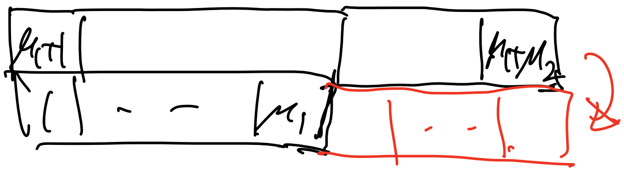
$\lambda \vdash n$ rearrangement of μ

<u>Step 1</u> Check $\Phi'(e_\mu) = \lambda$

$$e_\mu = (0^{\mu_1}, \mu_1^{\mu_2}, (\mu_1+\mu_2)^{\mu_3}, \cdots) \in \mathbb{E}_n$$

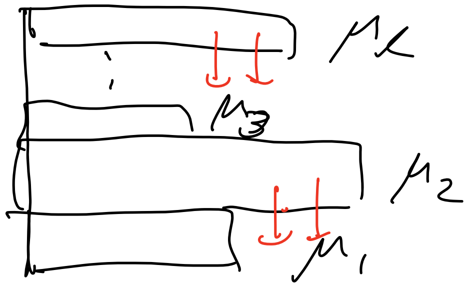


$\widetilde{\Omega}_{e(\mu_1+\mu_2)} \longrightarrow$



$\longrightarrow \cdots$

$\widetilde{\Omega}_{e(n)} \longrightarrow$



$\xrightarrow{\pi} \lambda$

Step 2 Check the modular law of type I

Recall $1 < i \le n$ $\quad e(i-1) < e(i) < e(i+1)$

$$\boxed{e(e(i)) = e(e(i)+1)}$$ not local

$$e'(i) = e(i)-1, \qquad e''(i) = e(i)+1$$

Want to show $\Phi'(e) = \dfrac{q\Phi(e') + \Phi(e'')}{1+q}$

Strategy If $\mathrm{Im}\left(\widetilde{\Omega}_r - \dfrac{q\widetilde{\Omega}_{r-1} + \widetilde{\Omega}_{r+1}}{1+q}\right) \subset \operatorname{Ker}\pi$ would be easy.

& $\widetilde{\Omega}_r$ preserves $\operatorname{Ker}\pi$, then

- The condition $e(e(i)) = e(e(i)+1)$ should restrict the domain of ______ to $\exists M \subset \widetilde{V}$ so that $\widetilde{\Omega}_{e(i-1)} \cdots \widetilde{\Omega}_{e(1)}(\phi) \in M$

- In order for $\tilde{\Omega}_r$ to stay in $\operatorname{Ker}\pi$, we need to prove stronger statement. i.e. find $K \subset \operatorname{Ker}\pi$

s.t. $\quad (M) \subset K$

$\tilde{\Omega}_{e(j)}(K) \subset K$ for $j > i$

<u>Def</u> $T \in SYT_n$ and $m \in [n-1]$

define $\tau_m(T) \in SYT_n$ by swapping

$\boxed{m}$ & $\boxed{m+1}$ in T if they are swappable

and $\tau_m(T) = T$ if not swappable.

<u>Def</u> $m \in [n-1]$, define $M_{m,n}, K_{m,n} \subset \tilde{V}_n$ by

$$M_{m,n} := \operatorname{Span}\left\{ \underbrace{\frac{[b-a+1]_q}{[b-a]_q} T + \frac{[a-b+1]_q}{[a-b]_q} \tau_m(T)}_{=: R_m(T)} \;\middle|\; \begin{array}{l} \boxed{m} \text{ lies in the top of } a\text{-th column in } T \\ \boxed{m+1} \quad \text{〃} \quad b\text{-th} \end{array} \right\}$$

$$K_{m,n} := \operatorname{Span}\{ T - \tau_m(T) \mid T \in SYT_n \}$$

<u>Lemma</u> (1) $1 \leq r < m \Rightarrow \tilde{\Omega}_r(M_{m,n}) \subset M_{m,n+1}$

(2) $r \neq m \Rightarrow \tilde{\Omega}_r(K_{m,n}) \subset K_{m,n+1}$

<u>Proof</u> (1) $m, m+1 > r$

$\Rightarrow$ • no boxes can be added above $\boxed{m}$ & $\boxed{m+1}$

• $\lambda(T, r) = \lambda(\tau_m(T), r) =: \lambda$

$$\Rightarrow \frac{[b-a+1]_q}{[b-a]_q} \widetilde{\Omega}_r(T) + \frac{[a-b+1]_q}{[a-b]_q} \widetilde{\Omega}_r(\tau_m(T))$$

$$= \sum_{c \in W(\lambda)} \varphi_c(\lambda) \left(\underbrace{\frac{[b-a+1]_q}{[b-a]_q} f_c(T) + \frac{[a-b+1]_q}{[a-b]_q} f_c(\tau_m(T))}_{= R_m(f_c(T))} \right)$$

$$\in M_{m, n+1}$$

(2) $r \neq m \Rightarrow \lambda(T, r) = \lambda(\tau_m(T), r) =: \lambda$

$$\Rightarrow \widetilde{\Omega}_r(T) - \widetilde{\Omega}_r(\tau_m(T))$$

$$= \sum_{c \in W(\lambda)} \varphi_c(\lambda) \left(\underbrace{f_c(T) - f_c(\tau_m(T))}_{\in K_{m, n+1}} \right)$$

$$\in K_{m, n+1}$$

//

<u>Key Lemma 1</u> $\forall T \in SYT_n, \ 0 \leq r \leq n$

$$\widetilde{\Omega}_r \widetilde{\Omega}_r (T) \in M_{n+1, n+2}$$

<u>Key Lemma 2</u> $\left(\widetilde{\Omega}_r - \frac{q\widetilde{\Omega}_{r-1} + \widetilde{\Omega}_{r+1}}{1+q} \right) (M_{r,n}) \subset K_{r, n+1}$

Key Lemma $\Rightarrow$ modular law I

$$e(i-1) < e(i) < e(i+1)$$
$$e(e(i)) = e(e(i)+1)$$

Key Lem 1 $\Longrightarrow$ $\widehat{\Omega}_{e(e(i)+1)} \widehat{\Omega}_{e(e(i))} \cdots \widehat{\Omega}_{e(1)}(\phi) \in M_{e(i), e(i)+1}$

$j < i \rightsquigarrow e(j) \leq e(i-1) < e(i)$

Lem (1) $\Longrightarrow$ $\widehat{\Omega}_{e(i-1)} \cdots \widehat{\Omega}_{e(e(i)+1)} \widehat{\Omega}_{e(e(i))} \cdots \widehat{\Omega}_{e(1)}(\phi)$

$\in M_{e(i), i-1}$

Key Lem 2 $\Longrightarrow$ $\left(\widehat{\Omega}_{e(i)} - \dfrac{q\widehat{\Omega}_{e'(i)} + \widehat{\Omega}_{e''(i)}}{1+q}\right) \widehat{\Omega}_{e(i-1)} \cdots \widehat{\Omega}_{e(1)}(\phi)$

$\in K_{e(i), i}$

$j > i \rightsquigarrow e(i) < e(i+1) \leq e(j)$

Lem(2) $\Longrightarrow$ $\widehat{\Omega}_{e(n)} \cdots \widehat{\Omega}_{e(i+1)} \left(\widehat{\Omega}_{e(i)} - \dfrac{q\widehat{\Omega}_{e'(i)} + \widehat{\Omega}_{e''(i)}}{1+q}\right)$

$\widehat{\Omega}_{e(i-1)} \cdots \widehat{\Omega}_{e(1)}(\phi) \in K_{e(i), n}$

$$\Rightarrow \quad \Phi'(e) = \frac{q\Phi'(e') + \Phi'(e'')}{1+q}$$

//

Step 3 Check the modular law of type II

Recall $1 \le i < n$ $\quad e(i+1) = e(i)+1$

$e^{-1}(i) = \phi$

$e'(i) = e(i) \qquad e''(i) = e(i)+1$

$e'(i+1) = e(i) \qquad e''(i+1) = e(i)+1$

Key Lem 3 $\forall T \in SYT_n \quad 0 \le r < n$

$$\left(\widetilde{\Omega}_{r+1} \widetilde{\Omega}_r - \frac{q \widetilde{\Omega}_r \widetilde{\Omega}_r + \widetilde{\Omega}_{r+1} \widetilde{\Omega}_{r+1}}{1+q} \right)(T) \in K_{n+1, n+2}$$

Key Lem 3 $\Longrightarrow$ modular law

$$\left(\widetilde{\Omega}_{e(i+1)} \widetilde{\Omega}_{e(i)} - \frac{q \widetilde{\Omega}_{e'(i+1)} \widetilde{\Omega}_{e'(i)} + \widetilde{\Omega}_{e''(i+1)} \widetilde{\Omega}_{e''(i)}}{1+q} \right)$$

$$\underset{e^{-1}(i)=\phi}{} \widetilde{\Omega}_{e(i-1)} \cdots \widetilde{\Omega}_{e(1)}(\phi) \in K_{i, i+1}$$

$$\xRightarrow{\text{Lem (2)}} \widetilde{\Omega}_{e(n)} \cdots \widetilde{\Omega}_{e(i+2)} \left(\widetilde{\Omega}_{e(i+1)} \widetilde{\Omega}_{e(i)} - \cdots \right)$$

$$\widetilde{\Omega}_{e(i-1)} \cdots \widetilde{\Omega}_{e(1)}(\phi) \in K_{i, n}$$

$$\Rightarrow \overline{\Phi}(e) = \frac{q \overline{\Phi}(e') + \overline{\Phi}(e'')}{1+q}$$

§ Proof of the Key Lemmas

Lem δ : Maya diagram

$c \in W(\delta)$, $\delta_d = 1$

$e_d(\delta)$: replace $\delta_d = 1$ by 0

$$\Rightarrow \boxed{\varphi_c(e_d(\delta)) = \frac{[d+1-c]_q}{[d-c]_q}\varphi_c(\delta)}$$

Rem If $c = d+1 \rightsquigarrow$

$$\begin{array}{rl} & \quad\;\; d \;\; d+1=c \\ \delta = & (\ldots 1\ 0 \ldots\ ) \\ & \qquad\quad\; c \\ e_d(\delta) = & (\ldots 0\ 0 \ldots\ ) \end{array}$$

$\Rightarrow c \notin W(e_d(\delta))$

Proof Recall $\varphi_c(\delta) = \dfrac{\prod_{a \in R(\delta)} [a-c]_q}{\prod_{b \in W(\delta)\setminus\{c\}} [b-c]_q}$

Assume $c \neq d+1$

Case 1

$$\begin{array}{rl} & \qquad\; d-1\ \ d\ \ d+1 \\ \delta = & (\ldots \quad 0 \quad \underset{\in R}{1} \quad \underset{\in W}{0} \ldots\ ) \\ e_d(\delta) = & (\ldots \quad 0 \quad \underset{\notin R}{0} \quad \underset{\notin W}{0} \ldots\ ) \end{array}$$

$\Rightarrow W(e_d(\delta)) = W(\delta) \setminus \{d+1\}$

$R(e_d(\delta)) = R(\delta) \setminus \{d\}$

$$\Rightarrow \quad \varphi_c(e_d(\delta)) = \frac{[d+1-c]_q}{[d-c]_q}\varphi_c(\delta)$$

<u>Case 2</u>

$$\begin{array}{cccccccc} & & & d-1 & d & d+1 & & \\ \delta = & (& \ldots & 1, & 1, & 0, & \ldots &) \\ & & & & \notin W & \in W & & \end{array}$$

$$\begin{array}{cccccccc} e_d(\delta) = & (& \ldots & 1, & 0, & 0, & \ldots &) \\ & & & & \in W & \notin W & & \end{array}$$

$$\Rightarrow \quad W(e_d(\delta)) = W(\delta) \setminus \{d+1\} \cup \{d\}$$

$$R(e_d(\delta)) = R(\delta)$$

$$\Rightarrow \quad \varphi_c(e_d(\delta)) = \frac{[d+1-c]_q}{[d-c]_q}\varphi_c(\delta)$$

<u>Case 3</u>

$$\begin{array}{cccccccc} & & & d-1 & d & d+1 & & \\ \delta = & (& \ldots & 0, & 1, & 1, & \ldots &) \\ & & & & \in R & \notin R & & \end{array}$$

$$\begin{array}{cccccccc} e_d(\delta) = & (& \ldots & 0, & 0, & 1, & \ldots &) \\ & & & & \notin R & \in R & & \end{array}$$

$$\Rightarrow \quad W(e_d(\delta)) = W(\delta)$$

$$R(e_d(\delta)) = R(\delta) \setminus \{d\} \cup \{d+1\}$$

$$\Rightarrow \quad \varphi_c(e_d(\delta)) = \frac{[d+1-c]_q}{[d-c]_q}\varphi_c(\delta)$$

<u>Case 4</u>

$$\begin{array}{cccccccc} & & & d-1 & d & d+1 & & \\ \delta = & (& \ldots & 1, & 1, & 1, & \ldots &) \\ & & & & \notin W & \notin R & & \end{array}$$

$$\begin{array}{cccccccc} e_d(\delta) = & (& \ldots & 1 & 0 & 1 & \ldots &) \\ & & & & \in W & \notin R & & \end{array}$$

$$\Rightarrow \quad W(e_d(\delta)) = W(\delta) \cup \{d\}$$

$$R(e_d(\delta)) = R(\delta) \cup \{d+1\}$$

$$\Rightarrow \varphi_c(e_d(\delta)) = \frac{[d+1-c]_q}{[d-c]_q} \varphi_c(\delta) \quad //$$

<u>KeyLem 1</u> $\quad \widetilde{\Omega}_r \widetilde{\Omega}_r(T) \in M_{n+1,n+2}$

$$c-1 \; c \; c+1$$
$$\delta = (\ldots 1, 0, 0, \ldots)$$
$$\curvearrowright$$
$$(\ldots 1, 0, 1, \ldots)$$

<u>Proof</u> $\quad \delta = \delta(T, r) \qquad c \in W(\delta)$

$$\rightarrow f_c(\delta) = \delta(f_c(T), r)$$

$$W(f_c(\delta)) = \begin{cases} W(\delta) \setminus \{c\} \cup \{c+1\} & \text{if } \delta_{c+1} = 0 \\ W(\delta) \setminus \{c\} & \text{if } \delta_{c+1} = 1 \end{cases}$$

$$\therefore \widetilde{\Omega}_r \widetilde{\Omega}_r(T) = \widetilde{\Omega}_r \Big(\sum_{c \in W(\delta)} \varphi_c(\delta) f_c(T) \Big)$$

$$= \sum_{\substack{c \in W(\delta) \\ \delta_{c+1} = 0}} \varphi_c(\delta) \sum_{d \in W(\delta) \setminus \{c\} \cup \{c+1\}} \varphi_d(f_c(\delta)) f_d f_c(T)$$

$$+ \sum_{\substack{c \in W(\delta) \\ \delta_{c+1} = 1}} \varphi_c(\delta) \sum_{d \in W(\delta) \setminus \{c\}} \varphi_d(f_c(\delta)) f_d f_c(T)$$

$$\in M_{n+1, n+2}$$

$$= \sum_{\substack{c \in W(\delta) \\ \delta_{c+1} = 0}} \varphi_c(\delta) \varphi_{c+1}(f_c(\delta)) f_{c+1} f_c(T)$$

$$e_d(f_d(\delta)) \qquad R_{n+1}(f_{c+1} f_c(T)) = [2]_q f_{c+1} f_c(T)$$

$$+ \sum_{\{c,d\} \subset W(\delta)} \Big(\varphi_c(\delta)\, \varphi_d(f_c(\delta))\, f_d f_c(\tau)$$

$$+ \varphi_d(\delta)\, \varphi_c(f_d(\delta))\, f_c f_d(\tau) \Big)$$

$$\varphi_c(f_c(\delta))$$

$$\frac{[d+1-c]_q}{[d-c]_q}\, \varphi_c(f_d(\delta))\, \varphi_d(f_c(\delta))\, f_d f_c(\tau)$$

$$\|$$

$$\frac{[c+1-d]_q}{[c-d]_q}\, \varphi_d(f_c(\delta))\, \varphi_c(f_d(\delta))\, f_c f_d(\tau)$$

$$\tau_{n+1}(f_d f_c(\tau))$$

$$= \varphi_c(f_d(\delta))\, \varphi_d(f_c(\delta))\, R_{n+1}(f_d f_c(\tau))$$

$$\in M_{n+1,\, n+2}$$