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<u>Today</u> : How to find a probabilistic formula for $X_\Gamma(q)$ (Γ : unit interval graph)

§ <u>Notations</u>

$[n] = \{1, 2, \cdots, n\}$

<u>Def</u> $e : [n] \longrightarrow \mathbb{Z}$ is called <u>conjugate Hessenberg fcn</u>

if $\left\{ \begin{array}{l} 0 \leq e(i) < i \\ e(i) \leq e(i+1) \end{array} \right.$ $\forall i \in [n]$

$(\ e(n+1) := n-1\ )$

Γ_e $E_n := \{ e : [n] \to \mathbb{Z} \mid e: \text{conj. Hess. fcn} \}$

UIG $\updownarrow 1:1$ $e \updownarrow h$ $(h(i) = n - e(n+1-i)\ \forall i \in [n])$

Γ_h $H_n := \{ h : [n] \to \mathbb{Z} \mid \begin{array}{l} h(i) \geq i \\ h(i) \leq h(i+1) \end{array} \}$

h

$(h(n+1) = n\ )$

<u>cf</u> $h \in H_n \rightsquigarrow X_h$: Hessenberg var

$H^*(X_h) \circlearrowleft \mathfrak{S}_n$

$\mathrm{Prob}(H^*(X_h)) = \omega(X_{\Gamma_h}(q))$

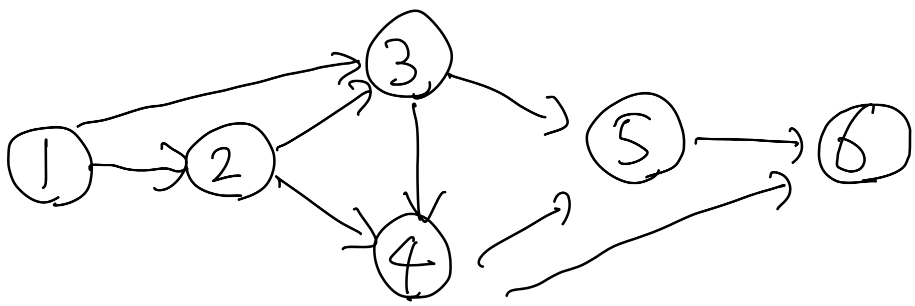
$e_r \leftrightarrow h_r$

(elem sym fcn $e_r(x)$)

$e \in \mathbb{E}_n \rightsquigarrow \Gamma_e$: unit interval graph

$$\begin{cases} \text{vertices} : & [n] \\ \text{edges} : & i \longrightarrow j \text{ iff } e(j) < i < j \end{cases}$$

Ex $e = (0,0,0,1,2,3) \in \mathbb{E}_6$

$\Gamma_e \doteq$ 

Def <u>unit interval graph</u> is a graph Γ_e for some $e \in \mathbb{E}_n$

Ex $e_n \in \mathbb{E}_n$ by $e_n(i) = 0$

$\rightsquigarrow \Gamma_{e_n} =$ complete graph

$\rightsquigarrow X_{\Gamma_{e_n}}(q) = [n]_q! \cdot e_n(x)$

Def $e \in \mathbb{E}_n$, $e' \in \mathbb{E}_{n'} \rightsquigarrow e \cup e' \in \mathbb{E}_{n+n'}$ by

$$e \cup e'(i) = \begin{cases} e(i) & 1 \le i \le n \\ n + e'(i-n) & n+1 \le i \le n+n' \end{cases}$$

$$\rightsquigarrow \Gamma_{e \cup e'} = \Gamma_e \cup \underbrace{\Gamma_{e'}}_{\text{shift the label by } n} \quad \text{(ordered disjoint union)}$$

$\rightsquigarrow \quad X_{\Gamma_{e \vee e'}}(q) = X_{\Gamma_e}(q) \cdot X_{\Gamma_{e'}}(q)$

For $\mu = (\mu_1, \cdots, \mu_\ell)$: composition of n

$$e_\mu := e_{\mu_1} \vee e_{\mu_2} \vee \cdots \vee e_{\mu_\ell}$$
$$= (0, \cdots, 0, \mu_1, \cdots, \mu_1, \mu_1+\mu_2, \cdots \qquad)$$

$$\boxed{X_{\Gamma_{e_\mu}}(q) = \prod [\mu_i]_q! \cdot e_\lambda(x)}$$

$\lambda \vdash n$: rearrangement of μ

$e \in \mathbb{E}_n$, $\quad X_{\Gamma_e}(q) = \sum_{\lambda \vdash n} c_\lambda(\Gamma_e : q) \cdot e_\lambda(x)$

recall

Prop

$$\boxed{\sum_{\lambda \vdash n} q^{|e| - |e_\lambda|} \frac{c_\lambda(\Gamma_e : q)}{\prod_i [\lambda_i]_q!} = 1}$$

$$\left(\begin{aligned} |e| &= e(1) + e(2) + \cdots + e(n) \\ |e_\lambda| &= \sum_{i<j} \lambda_i \lambda_j \end{aligned} \right)$$

Def $\quad P_\lambda(\Gamma_e : q) := q^{|e| - |e_\lambda|} \dfrac{c_\lambda(\Gamma_e : q)}{\prod [\lambda_i]_q!}$

$$V_n := \mathbb{Q}(q) \langle \lambda \mid \lambda \vdash n \rangle$$

$$\boxed{\begin{array}{rcl} \Phi : \mathbb{E}_n & \longrightarrow & V_n \\ \rotatebox{90}{$\in$}\quad & & \quad\rotatebox{90}{$\in$} \\ e & \longmapsto & \displaystyle\sum_{\lambda \vdash n} P_\lambda(T_e ; q) \cdot \lambda \end{array}}$$

- Want to understand Φ

§ First attempt

Try to find a linear map $\Omega_r : V_n \to V_{n+1}$ $(0 \leq r \leq n)$

s.t. $$\Omega_r(\Phi(e)) = \Phi(\underbrace{e + r}_{(e(1), e(2), \cdots, e(n), r)})$$

Ex $n = 0$

$$\Omega_0(\phi) = \Phi(0) = \ydiagram{1}$$

$n = 1$

$$\Omega_0(\ydiagram{1}) = \Phi(0,0) = \ydiagram{2}$$

$$\Omega_1(\ydiagram{1}) = \Phi(0,1) = \ydiagram{1,1}$$

$n = 2$

$$\Omega_0(\ydiagram{2}) = \Phi(0,0,0) = \ydiagram{3}$$

$$\Omega_1(\ydiagram{2}) = \Phi(0,0,1) = \frac{1}{[2]_q}\ydiagram{2,1} + \frac{q}{[2]_q}\ydiagram{3}$$

$$\Omega_2(\ydiagram{2}) = \Phi(0,0,2) = \ydiagram{2,1}$$

$$\Omega_1(\ydiagram{1,1}) = \Phi(0,1,1) = \ydiagram{2,1}$$

$$\Omega_2(\ydiagram{1,2}) = \Phi(0,1,2) = \ydiagram{1,1,1}$$

(do not need $\Omega_0(\ydiagram{1,2})$, but $\underbrace{\Omega_0(\ydiagram{1,2}) = \ydiagram{1,3}}_{\text{natural to take}}$)

$\underline{n=3}$ $$\Omega_r(\ydiagram{3}) = \frac{[r]_q}{[3]_q}\ydiagram{1,3} + \frac{q^r[3-r]_q}{[3]_q}\ydiagram{4}$$

$$\left(\Omega_r(\ydiagram{4}) = \cdots\right)$$

$$\Omega_1(\Phi(0,0,1)) = \Phi(0,0,1,1) = \frac{[2]_q}{[3]_q}\ydiagram{1,3} + \frac{q^2}{[3]_q}\ydiagram{4}$$

$$\shortparallel$$

$$\frac{1}{[2]_q}\Omega_1(\ydiagram{1,2}) + \frac{q}{[2]_q}\Omega_1(\ydiagram{3})$$

$$\Rightarrow \Omega_1(\ydiagram{1,2}) = \ydiagram{1,3}$$

$$\Omega_2(\Phi(0,0,1)) = \Phi(0,0,1,2) = \frac{1}{[2]_q}\ydiagram{2,2} + \frac{q}{[3]_q}\ydiagram{1,3} + \frac{q^3}{[2]_q[3]_q}\ydiagram{4}$$

$$\shortparallel$$

$$\frac{1}{[2]_q}\Omega_2(\ydiagram{1,2}) + \frac{q}{[2]_q}\Omega_2(\ydiagram{3})$$

$$\rightsquigarrow \Omega_2(\ydiagram{1,2}) = \ydiagram{2,2}$$

$$\Omega_2(\Phi(0,0,2)) = \Phi(0,0,2,2) = \ydiagram{2,2}$$

$$\shortparallel$$

$$\Omega_2(\ydiagram{1,2})$$

consistent

But $\Omega_2(\Phi(0,1,1)) \neq \Phi(0,1,1,2) = \frac{1}{[2]_q}\,\yng(1,2) + \frac{q}{[2]_q}\,\yng(1,3)$

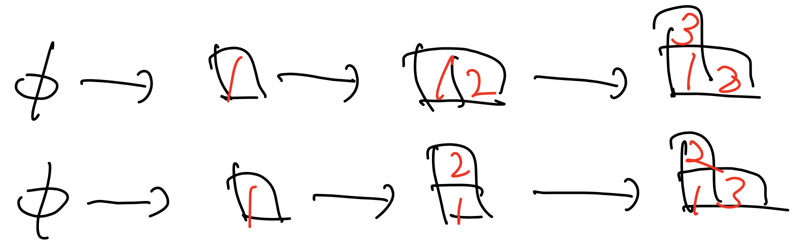
$\qquad\quad \parallel$

$\quad \Omega_2(\yng(1,2))$

inconsistent!

$\Phi(0,0,2)$:

$\Phi(0,1,1)$:



$\rightarrow$ transition probability depends on the past states (not Markov)

§ Markovization

Idea: Use standard Young tableau

$$\tilde{V}_n := \mathbb{Q}(q)\langle T \mid T \in SYT(\lambda),\ \lambda \vdash n \rangle$$

$$\tilde{V}_n \xrightarrow{\pi} V_n \qquad (\pi(T) = \lambda)$$

Prob Find $\tilde{\Omega}_r : \tilde{V}_n \longrightarrow \tilde{V}_{n+1}$ s.t.

(i) $\Phi(e) = \pi\, \Omega_{e(n)} \Omega_{e(n-1)} \cdots \Omega_{e(1)}(\phi)$

(ii) $\Omega_r(T) = \sum \mathbb{Q}(q)\, T'$

T' : adding $\boxed{n+1}$ to T

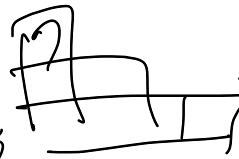
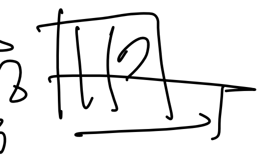
Ex. $\tilde{\Omega}_2\left(\begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$

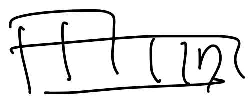
$\tilde{\Omega}_2\left(\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array}\right) = \frac{1}{[2]_q} \begin{array}{|c|c|}\hline 4 & \\ \hline 2 & \\ \hline 1 & 3 \\ \hline\end{array} + \frac{q}{[2]_q} \begin{array}{|c|c|c|}\hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline\end{array}$

<u>One can continue to calculate $\Omega_r(T)$</u> !

(up to $n \leq 5$)

Formulas are very simple except for

$\Omega_4\left(\begin{array}{|c|c|c|c|}\hline 4 & 6 & & \\ \hline 1 & 2 & 3 & 5 \\ \hline\end{array}\right) = \frac{[3]_q}{[2]_q[4]_q}$  $+ \frac{q}{[2]_q^3}$ 

$+ \frac{q^2[3]_q}{[2]_q[4]_q}$ 

<u>Observation</u>

$\Omega_r(T)$ depends only on the places of $\boxed{i}$ with $i > r$ on the top of ~~the~~ columns

<u>Def</u> $T \in SYT_n$, $r \rightsquigarrow \delta(T, r)$: "Maya diagram"

$:= (\delta_i)_{i \in \mathbb{Z}}$ $\left(\text{st } \begin{array}{l} \delta_{<0} = 1 \\ \delta_{>>0} = 0 \end{array}\right)$

$$\delta_i = \begin{cases} 1 & i \leq 0 \text{ or the top box in the } i\text{-th col of } T \text{ has } \boxed{j} > r \\ 0 & i > \lambda_1 \text{ or } \quad \boxed{j} \leq r \end{cases}$$

- transition probability depends only on $\delta(T, r)$

~~this~~ this should be defined on "Maya diagram"

$$\begin{tikzcd} SYT_n \arrow[r, "\delta(-,r)"] \arrow[d, "\widetilde{\Omega}_r"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Maya} \arrow[d, "\overline{\Omega}_r"] \\ SYT_{n+1} \arrow[r, "\delta(-,r)"] & \mathrm{Maya} \end{tikzcd}$$