This lecture

1. Generalized $z$-Eulerian polynomials
2. Digraph version of chromatic quasi-symmetric function

For each $\sigma \in \mathcal{C}(n)$ define

$$\text{inv}_{\leq r}(\sigma) = \sum_{i < j \leq n} \left[ 0 < \sigma(i) - \sigma(j) < r \right]$$

$$\text{DES}_{\geq r}(\sigma) = \{ i \mid \sigma(i) - \sigma(i+1) \geq r \}$$

$$\text{maj}_{\geq r}(\sigma) = \sum_{i \in \text{DES}_{\geq r}(\sigma)} i$$

If $r = 1$, $\text{inv}_{\leq r}(\sigma) = 0$, $\text{maj}_{\geq r} = \text{maj}$

If $r = n$, $\text{inv}_{\leq r}(\sigma) = \text{inv}(\sigma)$, $\text{maj}_{\geq r}(\sigma) = 0$

Rawlings' statistics $r \in \mathbb{Z}$

$$\text{inv}_{\leq r}(\sigma) + \text{maj}_{\geq r}(\sigma)$$

In major index

interpolates between inv and maj
Rawlings shows

\[ \text{inv}_{<} (\sigma) + \text{maj}_{\geq} (\sigma) \]

is Mahonian if \( v \in \mathbb{N} \)

\[ \sum_{\sigma \in \mathfrak{S}_n} \text{inv}_{<} (\sigma) + \text{maj}_{\geq} (\sigma) = [n]^n! \]

Generalization of Foata's

bijection taking \( \text{maj} \) to \( \text{inv} \)

We separate the statistics

\[ \sum_{\sigma \in \mathfrak{S}_n} \text{inv}_{<} (\sigma) \text{ maj}_{\geq} (\sigma) \]

\[ = \sum_{\sigma \in \mathfrak{S}_n} q^{|\text{inv}_{<} (\sigma)|} \text{ maj}_{\geq} (\sigma) = A_n^{(r)} (q, t) \]

\[ A_n^{(1)} (1, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{|\text{inv}_{<} (\sigma)|} \in A_n \mathcal{C}(t) \]

\[ \text{Th (Shareghian + W)} \]

\[ A_n^{(2)} (q, t) = A_n (q, t) \]

\[ \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq} (\sigma)} \text{ des} (\sigma) = \]

\[ \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq} (\sigma)} \text{ des} (\sigma) = \]
Let recall
\[ Q_n(x,t) = W \times (x,t) \]
where
\[ ps(Q_n(x,t)) = \frac{1}{[n]_q!} A_n(q,t) \]
\[ Q_n(x,t) = \sum_{\sigma \in S_n} t^{\text{exc}(\sigma)} F_{\sigma \in S_n} \cdot \text{EX}(\sigma) \]

Hence
\[ ps(W \times_{n-path} (x,t)) = \frac{1}{[n]_q!} A_n(q,t) \]

Now we use
\[ W \times_6 (x,t) = \sum_{\sigma \in S_n} t^{\text{inv}_6(\sigma)} F_{\sigma \in S_n} \cdot \text{EX}(\sigma) \]
where \( G = (\mathbb{H}, (P)) \) and
\[ \text{inv}_6(\sigma) = \left| \left\{ (\sigma(i), \sigma(j)) \in E(6) \mid \sigma(i) > \sigma(j) \right\} \right| \]
\[ DESp(\sigma) = \{ i \in [n-1] \mid \sigma(i) > \sigma(i+1) \} \]

\[ ps(w \times g(x, t)) = \frac{1}{[n]!} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} \text{inv}_g(\sigma) \cdot \mathcal{D} \mathcal{E} \mathcal{S}_n(\sigma) \]

For \( G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)

\[ \text{inv}_g(\sigma) = \text{inv}_{\mathcal{S}_2}(\sigma) \]

\[ \rho = \text{posets on } [n], \text{ with } \rho < \sigma \text{ if } \rho \leq_2 \sigma \]

So \( DESp(\sigma) = DES_{\mathcal{S}_2}(\sigma) \)

\[ \sum DESp(\sigma) = \text{maj}_{\mathcal{S}_2}(\sigma) \]

Thus \( ps(w \times (x, t)) = \frac{1}{[n]!} A_n^{(2)}(g, t) \)

\[ A_n^{(2)}(\alpha_1 t) = A_n(g, t) \]

Is there a bijective proof?

Fonat's bijection taking des to exc
doesn't work

Bigeni found a bijection that does work.

We call $A_{n}^{(r)}(q,t)$ a generalized $q$-Eulerian polynomial.

Recall $A_{n}(q,t)$ is palindromic and $q$-unimodal.

What about $A_{n}^{(r)}(q,t)$?

- Palindromicity - easy
- $q$-Unimodality - not so easy

Even in case $q=1$

Let $P_{n,r} = P((r,r+1, \ldots, n, n, \ldots))$
\[ G_{nr} = G(r, r+1, \ldots, n, n, \ldots, n) \]
\[ G_{nr} = \text{inc}(P_n, r) \]

so \( i < j \) if \( j - i \geq r \)

and \( (i, j, 3) \in E(G_{nr}) \) if \( 0 < j - i < r \)

\[ G_{nj} = n \text{-path } 0 \rightarrow 2 \rightarrow 4 \rightarrow \ldots \rightarrow n \]

\[ G_{nj} \]

\[ p_s(w \times G_{nj}) = \frac{1}{[n/j]!} A_n^{(r)}(q, t) \]

We need to know that \( X_{G_{nj}}(x, t) \) is Schur-unimodal

\[ \Rightarrow A_n^{(r)}(q, t) \text{ is } q \text{-unimodal} \]
Our hope was that Schur-unimodality could be obtained by generalizing the relationship between $X(x,t)$ and the $G_{n,t}$ toric variety associated with the permutohedron and hard Lefschetz theorem.

Stanley, EC1, 1.50 f [4.-]

Prove $\sum_{\sigma \in S_n} \inver(\sigma)$ is unimodal.

Unimodal

DeMuri + Sharman, generalized Eulerian polynomials.

Solution: It's the Poincare polynomial of the regular
Semi-simple Hessenberg variety
Consequently by the hard Lefschetz theorem it's unimodal.

Stanley - is there a more elementary proof?

Tymoczko representation of Sn on cohomology

Our conjecture relating chromatic quasi-symmetric function to Tymochko's representation

Chromatic quasi-symmetric functions
for digraphs

\[ C_n = \]
Consequently, $X_n(X)$ is e-positive.

Ellzey-Watanabe analog Smirnov word

\[
\sum_{n \geq 2} X_n(X, t) u^n = \frac{\sum_{k \geq 2} k(k-1) e_k u^k}{1 - \sum_{k \geq 2} (k-1) e_k u^k}
\]

Consequently, $X_n(X,t)$ is e-positive.
Stanley

Every labeled graph can be viewed as an acyclic directed graph by orienting edges \( (i \rightarrow j) \) as \( (i \rightarrow j) \) if \( i < j \),

as \( \mathcal{C}(G) = \{ (i, j) \in E(G) \mid c(i) < c(j) \} \)

labeled cycle becomes

\[
\begin{align*}
\text{Theorem} & \quad \sum_{n \geq 1} x^n (x+t)^n u^n \\
\text{(Gllzey)} & \quad n \geq 1 \\
& = \sum_{k \geq 0} k! (k-1) e_k u^k \\
& \quad 1 - t \sum_{k \geq 1} (k-1) e_k u^k \\
& \quad 1 - t \sum_{k \geq 0} e_k u^k
\end{align*}
\]
Consequently $X_{G'}(x,t)$ is $e$-positive and $e$-uni-modal when is $X_{G}(x,t)$ symmetric? Th (Ellzey) when $G$ is a cyclic version of unit interval graph, $X_{G'}(x,t)$ is symmetric.
Examples

(i) All natural unit interval graphs correspond to acyclic circular indifference digraphs.

Conjecture

If $G$ is a circular indifference digraph, then $X_G(x, t)$ is $e$-positive and $e$-unimodal.

Th (Ellzey) $\mu X_G(x, t)$ is $p$-positive (coefficients differ such as no notion of $P$-descents).
Open questions for digraph version

- p-unimodality
- Schur-positivity & Schur-unimodality
- Geometric interpretation