Recall from last time:

\[ P \to \text{Graph} \to \text{DiGraph} \to \text{DiGraph} \]

\[ \text{Shade}(x, y) = \sum_{x \in \text{Vertices}} x_i \quad \text{where} \quad x_i = \sum_{(x, y) \in E(G)} x \]

where \( x_i \) is a shaded path with \( n \) nodes.

How about other graphs instead of just graphs?

Recall the following we learned in previous lectures:

\[ x_i \quad \text{is symmetric, then} \quad \sum_{(x, y) \in E(G)} x \]

Prop: If \( x_i \) is symmetric, then \( x_i \quad \text{shaded} \quad x_i \)

Hence \( x_i \) is path-connected and acc(\( x \)) can be replaced by acc(\( x \)), i.e., \( x_i = \sum_{(x, y) \in E(G)} x \)

Q: When is \( x_i \) symmetric?

We only know "natural" examples:

1. When \( G \) is a unit interval graph \( G(U, P) \) where \( P \) is a unit interval order, then \( x_i \) is symmetric (proven by Steenrod & Stone)

Choose a finite set of closed intervals \( S = \{S_i \} \) of length one on \( R \) and \( x_i \) is the union of these intervals.

2. The associated natural unit interval order \( P \) is a partial order in \( R \) and \( x_i \) is path-connected.

Q: When \( G \) is a naturally labeled cycle, then \( x_i \) is symmetric.

3. When \( G \) is a naturally labeled cycle, then \( x_i \) is symmetric (by Steenrod and Stone).

4. \( G \) has connected components of types (1) or (2), then \( x_i \) is symmetric.

We will focus on connected graphs.

Recall \( \sum_{(x, y) \in E(G)} x_i = \sum_{(x, y) \in E(G)} x \), we have:

- \( x_i \) is e-positive and e-unimodal (i.e., \( \text{flip} x_i \to x_i \): coeff of \( x_i \to \text{coeff} \) of \( x_i \) is e-positive)

- Problem: Characterize representations of \( S_n \) on cohomology of top-dimensional cycles with dual permutation is \( x_i \) (by Poincaré & Steenrod)

Conjecture 1: (Determination of e-positive functions)

If \( G \) is a natural unit interval graph, then \( x_i \) is e-positive and e-unimodal.

Conjecture 2: (Conjecture proposed by Brower, Crowe, and Ewing-Pragiat)

Connection with representation varieties.

Corrections: Conjecture 1 + Schur-positivity (c.f. proof in lecture 3 using P matrices).

Conjecture 2: Main conjecture: Characteristic \( x_i \) Schur-unimodality (Theorem)

Problem: Find a proof of the Schur-unimodality that uses P matrices

Conjecture 3: p-positivity and p-unimodality of \( x_i \) (c.f. paper by symmetric functions)}
Theorem (Shao, Zhang, 1976)

For all graphs $G$, $\chi(H,1) = \sum_{\text{edges } e} \frac{1}{d(e)}$.

$p$-positivity of $\chi(H[1])$.

Let $P$ be a point on $(i_1, i_2)$. A word $a_1a_2..a_n$ over $\{1,2,..,n\}$ has a $P$-accreted at $i_j$ if $a_j = a$, and a left-to-right $P$-none at $i_j$ if $a_j = a$, $a < a_i$. Define $P_{\text{left}} = \{ \forall \forall E : \forall \exists \in P \text{ -- skew or left-to-right P-none (in one-time assignment)} \}$.

Theorem (Shao, Zhang, 1976)

Let $G$ be in $P_{\text{left}}$ where $P$ is a natural unit interval order.

Then coefficient of $\frac{1}{d(e)}$ in the power-series expansion of $\chi(H[1])$ is $\sum_{\text{edges } e} \frac{1}{d(e)}$ where $\chi(H[1]) = \chi(H[1])$.

Know about coeff. $\frac{1}{d(e)}$.

Given a point $P$ on $(i_1, i_2)$ and $\forall \forall E$.

Define $P_{\text{left}}$ as the set of fittings of Young diagram $\lambda$ with $i_1, i_2$ and once such (i.e., standard fitting). It must have an $P_{\text{left}}$ skew and an left-to-right $P_{\text{none}}$.

Theorem (Corollary to Shao, Zhang, proved by K. Thaphaid).

Let $G$ be $P_{\text{left}}$ where $P$ is a natural unit interval order, and $\forall \forall E$.

Then coefficient of $P$ in the power-series expansion of $\chi(H)$ is $\sum_{\text{edges } e} \frac{1}{d(e)}$ where $\chi(H[1])$ is expanded by removing $T$ from left to right starting from $i_1$ to $i_2$.

$p$-unimodularity of $\chi(H[1])$.

Note: This is $\frac{1}{d(e)}$. $\chi(H[1])$ is unimodal ($\forall \forall E$).