A raising operator formula for Macdonald Polynomials — George H. Seelinger — 6/25/24

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joint with J. Blasiak, M. Haiman, J. Morse, & A. Pun

Motivation For $i<j$ and $\lambda \in \mathbb{Z}^\ell$, define $R_{ij}(\lambda) = \lambda + \varepsilon_i - \varepsilon_j$ "raising operator"

i^{th} coordinate vector

Eg $R_{12}\Big(\underset{(3,2)}{\uparrow}\Big) = \underset{(4,1)}{\uparrow}$ (French notation)

Macdonald (I.3.4'') Schur function: $s_\lambda = \prod_{i<j} (1-R_{ij})\, h_\lambda$ where $R_{ij}(h_\lambda) = h_{R_{ij}(\lambda)}$

is a "raising operator formula" (equivalent to the Jacobi-Trudi identity)

Eg $s_{42} = (1-R_{12})h_{42} = h_{42} - h_{51}$.

However, $\exists$ some confusing subtleties when $R_{ij}(\lambda)$ is not a partition...

I) Virtual GL_ℓ-character series

Def For any $\lambda \in \mathbb{Z}^\ell$, define irreducible GL_ℓ-character $\chi_\lambda(z) \in \Bbbk[z_1^{\pm 1}, \dots, z_\ell^{\pm 1}]^{S_\ell}$ by

$$\chi_\lambda(z) := \sum_{w \in S_\ell} w\left(\frac{z_1^{\lambda_1} \cdots z_\ell^{\lambda_\ell}}{\prod_{i<j} 1 - z_j/z_i} \right) = \sum_{w\in S_\ell} w\left(\frac{z_1^{\lambda_1+\ell-1} \cdots z_\ell^{\lambda_\ell}}{\prod_{i<j}(z_i - z_j)} \right) = \frac{\text{determinant}}{\text{Vandermonde determinant}}$$

("Jacobi Bialternate formula / Weyl character formula.")

Lem For any $\lambda \in \mathbb{Z}^\ell$, $\rho := (\ell-1, \ell-2, \dots, 1, 0)$

$$\chi_\lambda(z) = \begin{cases} \operatorname{sgn}(w_\lambda)\, \chi_{\operatorname{sort}(\lambda+\rho)-\rho}(z) & \text{if } \lambda+\rho \text{ has distinct entries} \\ 0 & \text{otherwise} \end{cases}$$

for $w_\lambda \in S_n$ such that $w_\lambda(\lambda+\rho) = \operatorname{sort}(\lambda+\rho)$.

Pf by manipulating determinants.

Rmk If λ is a partition ($\lambda_\ell \geq 0$), then $\chi_\lambda(z) = s_\lambda(z)$ ← Schur polynomial.

Egs

1) $\chi_{201}(z) = 0$ since $(2,0,1)+(2,1,0) = (4,1,1)$
2) $\chi_{2-11}(z) = -\chi_{200}(z)$ since $(2,-1,1)+(2,1,0) = (4,0,1) \rightsquigarrow (4,1,0)$
   $(4,1,0) - (2,1,0) = (2,0,0)$.
3) $\chi_{1,-1}(z_1,z_2) = \dfrac{z_1^2 z_2^{-1} - z_1^{-1} z_2^2}{z_1 - z_2} = z_1^{-1} z_2^{-1} \left(\dfrac{z_1^3 - z_2^3}{z_1 - z_2} \right) = z_1^{-1} z_2^{-1} (z_1^2 + z_1 z_2 + z_2^2)$.

Rmk $(z_1 \cdots z_\ell)\, \chi_\lambda(z_1, \dots, z_\ell) = \chi_{\lambda + \underbrace{(1,\dots,1)}_{\ell}}(z_1, \dots, z_\ell)$

Def For $f(z) \in \Bbbk[z_1^{\pm 1}, \dots, z_\ell^{\pm 1}]$, set

$$\sigma(f(z)) := \sum_{w \in S_\ell} w\left(\frac{f(z)}{\prod_{1 \leq i<j \leq \ell} (1 - z_j/z_i)} \right) \qquad \text{"Weyl Symmetrization"}$$

So that $\sigma(z^\lambda) = \chi_\lambda(z)$.

Def For $\Lambda := \Lambda(X)$ symmetric functions in formal alphabet X,

set $\mathrm{pol}_X : \Bbbk[z_1^{\pm 1}, \dots, z_\ell^{\pm 1}]^{S_\ell} \longrightarrow \Lambda(X)$ to be the linear extension of

$$\underset{\mu_1 \ge \cdots \ge \mu_\ell}{\chi_\mu(z)} \longmapsto \begin{cases} s_\mu(X) & \text{if } \mu_\ell \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Thm For $\lambda \in \mathbb{Z}^\ell$,

$$h_\lambda(X) = \mathrm{pol}_X \underline{\sigma}\left(\frac{z_1^{\lambda_1} \cdots z_\ell^{\lambda_\ell}}{\prod_{1 \le i < j \le \ell} (1 - z_i/z_j)} \right)$$

where $(1 - z_i/z_j)^{-1} = 1 + z_i/z_j + z_i^2/z_j^2 + \cdots$

Eg
$$h_{42} = \mathrm{pol}_X \underline{\sigma}\left(z_1^4 z_2^2 \left(1 + z_1/z_2 + z_1^2/z_2^2 + z_1^3/z_2^3 + \cdots \right)\right)$$
$$= \mathrm{pol}_X \left(\chi_{42} + \chi_{51} + \chi_{60} + \underbrace{\chi_{7,-1} + \cdots}_{\mapsto 0 \text{ by } \mathrm{pol}_X} \right)$$
$$= s_{42} + s_{51} + s_6$$

Rmk 1) By construction, only a finite number of terms in the series will be nonzero under $\mathrm{pol}_X(-)$.

2) Using traditional raising operator notation

$$\text{“} h_\lambda = \frac{1}{\prod_{i<j} (1 - R_{ij})} s_\lambda \text{”}$$

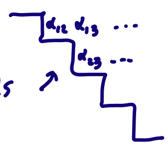
II) Hall-Littlewood Polynomials via Shimozono-Weyman.

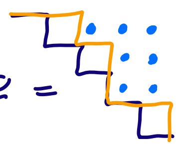
Def Let $R_+ = R_+(GL_\ell) := \{ \alpha_{ij} \mid 1 \le i < j \le \ell \}$ for $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ be the positive roots for GL_ℓ.

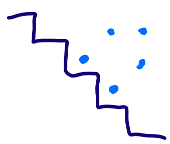
A root ideal $\Psi \subseteq R_+$ is a subset satisfying the condition

$\alpha_{ij} \in \Psi \Rightarrow \alpha_{i'j} \in \Psi$ for all $i' \le i$ and $\alpha_{ij'} \in \Psi$ for all $j' \ge j$.

Picture/Eg $\ell = 5$

Matrix coords ↗ 

$\Psi =$  is a root ideal

 is not a root ideal

Rmk Root ideals of $GL_\ell \xleftrightarrow{1:1}$ Dyck paths in $\ell \times \ell$ grid.

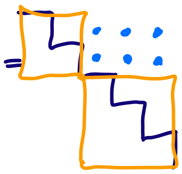
Def For μ a partition of ℓ, write $\{1,\dots,\ell\} = A_{\ell(\mu)} \sqcup \cdots \sqcup A_1$ for A_i an interval with $|A_i| = \mu_i$ and $\max A_j = \min A_{j-1} - 1$.

Eg $\mu = (3,2) \rightsquigarrow \{1,\dots,5\} = [1,2] \sqcup [3,5]$.

Then, the parabolic root ideal B_μ is given by

$$B_\mu := \{\alpha_{ij} \in R_+ \mid i, j \text{ in distinct } A_k\text{'s}\}$$

Eg

$B_{32} =$ 

Thm [Shimozono–Weyman] For any partition μ of ℓ, the modified Hall-Littlewood polynomial $\tilde{H}_\mu(X;t) \in \Lambda_{\mathbb{Q}(t)}$ is given by

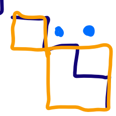
$$\tilde{H}_\mu(X;t) = \omega \, \mathrm{pol}_X \, \underline{\sigma} \left(\frac{z_1 \cdots z_\ell}{\prod_{\alpha_{ij} \in B_\mu} (1 - t\, z_i/z_j)} \right),$$

where $\omega(s_\lambda) = s_{\lambda^*}$

transpose partition

Ad-hoc notation

Rmk Usual modified HL polynomials $H_\mu(X;t) = t^{n(\mu)} \tilde{H}(X;t^{-1})$.

Egs 1) $\mu = (2,1) \Rightarrow B_\mu =$ 

$$\Rightarrow \tilde{H}_{21}(X;t) = \omega \, \mathrm{pol}_X \, \underline{\sigma} \left((1 + t\, z_1/z_2 + t^2 z_1^2/z_2^2 + \cdots)(1 + t\, z_1/z_3 + t^2 z_1^2/z_3^2 + \cdots) z_1 z_2 z_3 \right)$$

$$= \omega \, \mathrm{pol}_X \left(\chi_{111} + t(\chi_{210} + \chi_{201}) + t^2(\chi_{31-1} + \chi_{300} + \chi_{3-11}) + \cdots \right)$$

$$= \omega \left(s_{111} + t\, s_{21} + 0 + 0 + t^2 s_3 - t^2 s_3 \right)$$

$$= s_3 + t s_{21}$$

2) $\mu = (\ell) \Rightarrow B_\mu = \varnothing$

$$\Rightarrow \tilde{H}_\ell(X;t) = \omega \, \mathrm{pol}_X \, \underline{\sigma}(z_1 \cdots z_\ell) = \omega \, s_{\underbrace{1\cdots 1}_{\ell}} = s_\ell = h_\ell$$

3) $\mu = (1^\ell) \Rightarrow B_\mu = R_+$

$$\Rightarrow \tilde{H}_{(1^\ell)}(X;t) = \omega \, \mathrm{pol}_X \, \underline{\sigma} \left(\frac{z_1 \cdots z_\ell}{\prod_{\alpha_{ij} \in R_+} (1 - t\, z_i/z_j)} \right)$$

$$t \to 1 \Rightarrow \tilde{H}_{(1^\ell)} = \omega \, h_{(1^\ell)} = (s_1)^\ell$$

Cor $\tilde{H}_\mu(X;t) = s_\ell(X) + t(\cdots)$