Recall: For some statistics of non-orientability η,

$$F^{(\ell)}(t,\mathbb{p},s_1,s_2,\dots,s_\ell)$$

$$= \sum_{M\in\mathcal{M}^{(\ell)}} \frac{(-t)^{|M|}\, p_{\diamond(M)}\, b^{\eta(M)}}{2^{|V_\bullet(M)|-cc(M)}\, \alpha^{cc(M)}} \prod_{1\le i\le \ell} \frac{(-\alpha s_i)^{|V_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}$$

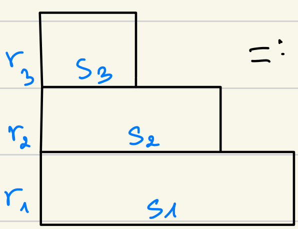
$$= \exp\big(B_\alpha(t,\mathbb{p},-\alpha s_1)\big)\cdots\exp\big(B_\alpha(t,\mathbb{p},-\alpha s_\ell)\big)\cdot 1$$

where $B_\alpha(t,\mathbb{p},u) := \sum_{n\geq 1} \frac{t^n}{n} B_n(\mathbb{p},u)$.

Main Thm: $F^{(\ell)}(t,\mathbb{p},\lambda_1,\lambda_2,\dots,\lambda_\ell) = \sum_\mu t^{|\mu|} p_\mu \theta_\mu(\lambda)$.

Multirectangular coordinates: $\mathbb{r} := (r_1,r_2,\dots)$, $\mathbb{s} := (s_1,s_2,\dots)$

with $s_1 \geq s_2 \geq \dots$



$=: \mathbb{s}^{\mathbb{r}}$

Define: $\tilde{\theta}^{(\alpha)}_\mu(\mathbb{s},\mathbb{r}) := \theta^{(\alpha)}_\mu(\mathbb{s}^{\mathbb{r}})$

Lassalle's conjecture: $(-1)^{|\mu|} z_\mu \tilde{\theta}^{(\alpha)}_\mu(\mathbb{s},\mathbb{r})$ is a polynomial in $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ with non-negative integer coefficients.

Polynomiality + Integrality : Lecture 2. (Łukasiewicz paths).

Goal: Positivity.

$$\mathbb{S} = (s_1, \dots, s_\ell) \ ; \ r = (r_1, \dots, r_\ell)$$

$$\tilde{O}_\mu(\mathbb{S}, r) = O_\mu(\underbrace{s_1, \dots, s_1}_{r_1}, \underbrace{s_2, \dots, s_2}_{r_2}, \dots, \underbrace{s_\ell, \dots, s_\ell}_{r_\ell})$$

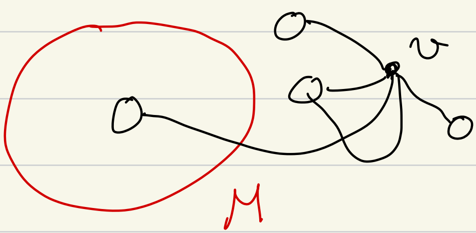
$$= [t^{|\mu|} p_\mu] \exp\left(B_\alpha(-\alpha s_1)\right)^{r_1} \cdots \exp\left(B_\alpha(-\alpha s_\ell)\right)^{r_\ell} \cdot 1$$

$$B_\alpha(u) \equiv B_\alpha(t, \mathbb{p}, u) = \sum_{n \geq 1} \frac{(-t)^n}{n} B_n(\mathbb{p}, u)$$

$$\tilde{O}_\mu(\mathbb{S}, r) = [t^{|\mu|} p_\mu] \exp\left(r_1 B_\alpha(-\alpha s_1)\right) \cdots \exp\left(r_\ell B_\alpha(-\alpha s_\ell)\right) \cdot 1$$

Combinatorial interpretation:

$B_n(\mathbb{p}, u)$ acts "on maps" by adding a black vertex of degree n, possibly new white vertices, with a weight u for each new white vertex.



$\longrightarrow u^3$

$$\tilde{\theta}_\mu(\mathbf{s}, \mathbf{r}) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\alpha)}} \frac{b^{\eta(M)}}{2^{|V_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|V_\circ^{(i)}(M)|} r_i^{|V_\bullet^{(i)}(M)|}}{z_{V_\bullet^{(i)}(M)}}$$

$\to$ $(-1)^{|\mu|} \tilde{\theta}_\mu(\mathbf{s}, \mathbf{r})$ is a polynomial in $b, -s_1, -s_2, \dots, r_1, r_2$ with positive coefficients

we use the fact that for any map M

$$|V_\circ(M)| \geq cc(M)$$

where $\alpha = b + 1$

Creation formula for Jack polynomials:

Thm:

Let $\lambda := [\lambda_1, \dots, \lambda_\ell]$.

$$J_\lambda^{(\alpha)} = B_{\lambda_1}^{(+)} \cdot \dots \cdot B_{\lambda_\ell}^{(+)} \cdot 1$$

where $$B_n^{(+)} := [t^n] \exp\left(B_\alpha(-\alpha n) \right)$$

$$= [t^n] \exp\left(\sum_{k \geq 1} \frac{(-t)^k}{k} B_k(\mathbf{p}, -\alpha n) \right)$$

Proof:
$$J_\lambda^{(\alpha)} = \sum_{\mu \vdash |\lambda|} \theta_\mu^{(\alpha)}(\lambda)\, p_\mu$$

$$= [t^{|\lambda|}] \sum_\nu t^{|\nu|} \theta_\mu^{(\alpha)}(\nu)\, p_\mu$$

$$J_\lambda^{(\alpha)} = [t^{|\lambda|}] \exp\left(B_\infty(-\alpha\lambda_1)\right) \cdots \exp\left(B_\infty(-\alpha\lambda_\ell)\right) \cdot 1$$

$$= \sum_{\substack{n_1, \dots n_\ell \geq 0 \\ n_1 + \dots + n_\ell = |\lambda|}} \left([t^{n_1}] \exp(B_\alpha(-\alpha\lambda_1))\right) \cdots \left([t^{n_\ell}] \exp(B_\alpha(-\alpha\lambda_\ell))\right) \cdot 1$$

If $(n_1, \dots, n_\ell)$ s.t $(n_1, \dots n_\ell) \neq (\lambda_1, \dots, \lambda_\ell)$

then these exists i s.t $n_i > \lambda_i$,

then $[t^{n_i}] \exp(B_\alpha(-\alpha\lambda_i)) \cdots \left([t^{n_\ell}] \exp(B_\alpha(-\alpha\lambda_\ell))\right) \cdot 1$
$= 0$

Link with D operator:

Define D_u on the space of symmetric functions

$$D_u J_\lambda^{(\alpha)}(p) = \prod_{\square \in \lambda} (u + c_\alpha(\square)) \cdot J_\lambda^{(\alpha)}(p)$$

$$= J_\lambda^{(\alpha)}(\underline{u}) \cdot J_\lambda^{(\alpha)}(p)$$

$$J_\lambda^{(\alpha)}(\underline{u}) = J_\lambda^{(\alpha)}(q)\Big|_{q_i = u}$$

where $c_\alpha(\square) = \alpha(i-1) - (j-1)$; $\square = (i, j)$

Prop:
$$B_n(\mathbb{p}, u) = \Delta_u \frac{p_n}{\alpha} \Delta_u^{-1}$$

Proof:
$$\tau(t, \mathbb{p}, \mathbb{q}, \underline{u}) = \sum_\lambda t^{|\lambda|} \frac{J_\lambda(\mathbb{p}) J_\lambda(\mathbb{q}) J_\lambda(\underline{u})}{j_\lambda^{(\alpha)}}$$
$$= \Delta_u \sum_\lambda t^{|\lambda|} \frac{J_\lambda(\mathbb{p}) J_\lambda(\mathbb{q})}{j_\lambda^{(\alpha)}}$$

Thm [Chapuy–Dołęga '22]:

$$\frac{1}{n} B_n(\mathbb{p}, u) \, \tau(t, \mathbb{p}, \mathbb{q}, u) = \frac{\partial}{\partial q_n} \tau(t, \mathbb{p}, \mathbb{q}, u)$$