Jack characters as generating series of bipartite maps and proof of Lassalle's conjecture (Part 4)

Houcine Ben Dali

joint work with

Maciej Dołęga

Plan

- Lecture 3: Statistics of N.O./ Top homogeneous part in $F_\mu^{(k)}$.
- Lecture 4: Construction of $F_\mu^{(k)}$ with differential operators / Vanishing condition.

Example $\mu = [2]$

Theorem (BD–Dołęga '23+)

$$\theta_\mu^{(\alpha)}(s_1, s_2, \dots) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

$$\theta_{[2]}^{(\alpha)}(s_1, s_2 \dots) = \sum_{i \geq 1} \frac{\alpha}{2} s_i(s_i - 1) - \sum_{i \geq 1} (i-1) s_i.$$

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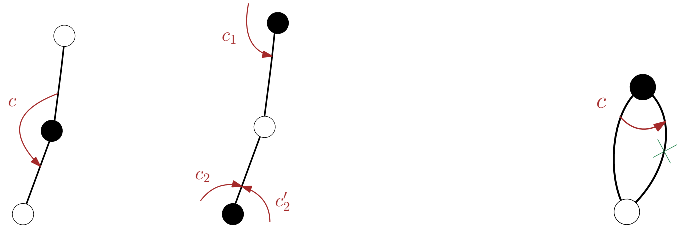

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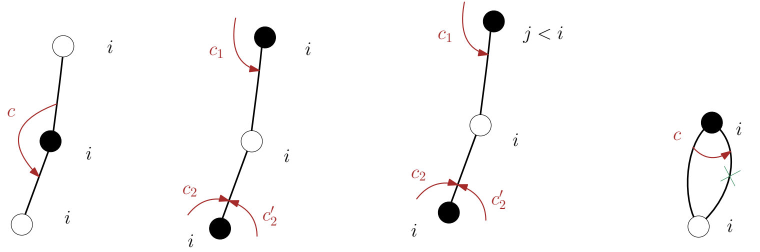

$\alpha^2 s_i^2$ $\quad -\frac{2\alpha s_i}{4\alpha}$ $\quad -\frac{2\alpha s_i \cdot (i-1)}{2\alpha}$ $\quad \frac{-\alpha s_i}{2\alpha}$

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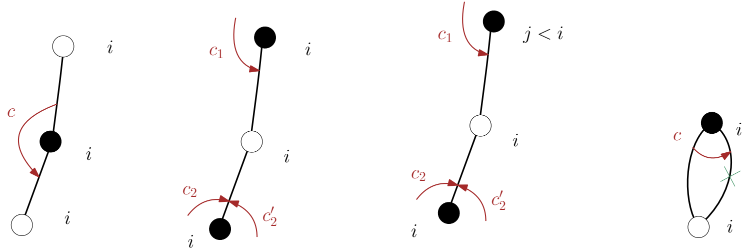

$\alpha^2 s_i^2$	$-\frac{2\alpha s_i}{4\alpha}$	$-\frac{2\alpha s_i \cdot (i-1)}{2\alpha}$	$\frac{-\alpha s_i}{2\alpha}$
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Generating functions

$$F_{\mu}^{(k)}(s_1, s_2 \dots, s_k) := \sum_{M \in \mathcal{M}_{\mu}^{(k)}} \frac{(-1)^{|\mu|} b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

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Let t be a new parameter.

$$F^{(k)}(t, \mathbf{p}, s_1, s_2 \dots, s_k) := \sum_{M \in \mathcal{M}^{(k)}} \frac{(-t)^{|M|} p_{\nu_{\diamond}(M)} b^{\eta(M)}}{2^{|\mathcal{V}_{\bullet}(M)| - cc(M)} \alpha^{cc(M)}} \prod_{1 \le i \le k} \frac{(-\alpha s_i)^{|\mathcal{V}_{\circ}^{(i)}(M)|}}{z_{\nu_{\bullet}^{(i)}(M)}}$$

$$\in \mathbb{Q}(\alpha)[p_1, p_2 \dots, s_1, s_2, \dots][\![t]\!]$$

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We want to prove:

- Vanishing property: $[t^n] F^{(k)}(t, \lambda_1, \dots, \lambda_k) = 0$ if $n > |\lambda|$.
- Shifted symmetry property; $F^{(k)}$ is symmetric in $s_i - i/\alpha$.

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  For $\alpha \in \{1, 2\}$: Combinatorial proof by Féray–Śniady '11.
- Shifted symmetry property; $F^{(k)}$ is symmetric in $s_i - i/\alpha$.

Differential construction

- In order to prove these properties, we use a differential construction of the generating series of layered maps (Tutte decomposition):

$$F^{(k+1)}\left(t,\mathbf{p},s_1,\ldots,s_{k+1}\right)=\exp\left(\sum_{n\geq 1}\frac{(-t)^n}{n}\mathcal{B}_n(\mathbf{p},-\alpha s_1)\right)\cdot F^{(k)}\left(t,\mathbf{p},s_2,\ldots,s_{k+1}\right),$$

where $\mathcal{B}_n(\mathbf{p},-\alpha s_1)$ is an operator which adds a black vertex of degree n with label 1, adding possibly new white vertices (necessarily in layer 1).

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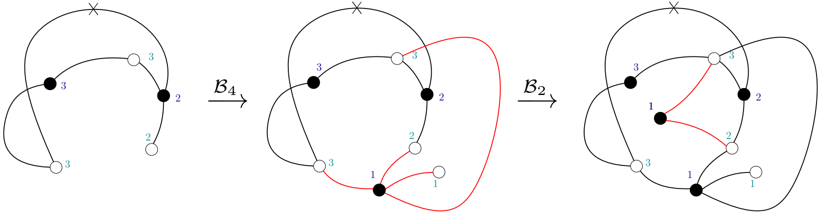


Plan

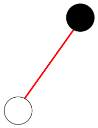
1. Differential operators and construction of maps
2. Tau function in two alphabets
3. Vanishing condition

Adding one edge

We consider bipartite maps (non-layered) counted with the weight $p_{\nu_\diamond(M)}$.

$$p_1 \cdot p_{\nu_\diamond(M)} = p_{\nu_\diamond(N)},$$

N is the map obtained from M by adding an isolated edge.

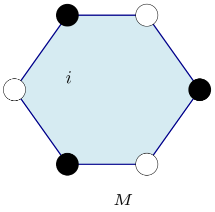


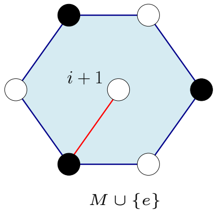
Adding one edge

Fix a bipartite map M.

$$\left(\sum_{i\geq 1} p_{i+1}\frac{i\partial}{\partial p_i}\right) p_{\nu_\diamond(M)} = \sum_e p_{\nu_\diamond(M\cup\{e\})}$$

the sum is taken over all possible ways to add a white leaf e to a black corner of M.





Adding one edge

$$\left(\sum_{i,j\geq 1} p_i p_j \frac{(i+j-1)\partial}{\partial p_{i+j-1}} + \sum_{i\geq 1} p_{i+1}\frac{i\partial}{\partial p_i} + 2\sum_{i,j\geq 1} p_{i+j+1}\frac{i\partial}{\partial p_i}\frac{j\partial}{\partial p_j}\right)\cdot p_{\nu_\diamond(M)} = \sum_e p_{\nu_\diamond(M\cup\{e\})}$$

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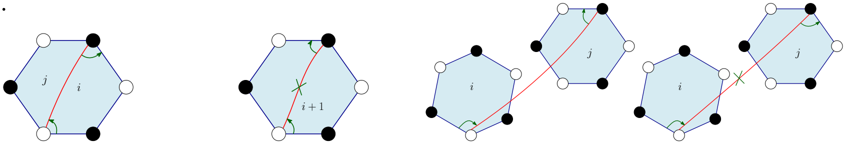
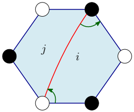


Adding one edge

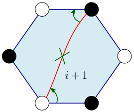
$$\left(\sum_{i,j\geq 1} p_i p_j \frac{(i+j-1)\partial}{\partial p_{i+j-1}} + b \sum_{i\geq 1} p_{i+1} \frac{i\partial}{\partial p_i} + (1+b) \sum_{i,j\geq 1} p_{i+j+1} \frac{i\partial}{\partial p_i} \frac{j\partial}{\partial p_j}\right) \cdot \frac{p_{\nu_\diamond(M)}}{\kappa(M)}$$
$$= \sum_e b^{\vartheta(M\cup\{e\},e)} \frac{p_{\nu_\diamond(M\cup\{e\})}}{\kappa(M\cup\{e\})}$$

the sum is taken over all possible ways to add an edge e between two corners of M, and

$$\kappa(M) := 2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}.$$



$b^{\vartheta(M\cup\{e\},e)} = 1$



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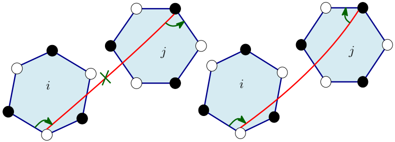

$b^{\vartheta(M\cup\{e\},e)} + b^{\vartheta(M\cup\{\tilde{e}\},\tilde{e})} = 1 + b$
or $b^{\vartheta(M\cup\{e\},e)} = b^{\vartheta(M\cup\{\tilde{e}\},\tilde{e})} = 1$

Adding one edge

$$A_1^{(\alpha)} = \frac{p_1}{\alpha} \qquad \longrightarrow \text{ isolated edge}$$

$$A_2^{(\alpha)} = \left(\sum_{i \geq 1} p_{i+1} \frac{i\partial}{\partial p_i} \right) \qquad \longrightarrow \text{ white leaf edge}$$

$$A_3^{(\alpha)} = \left(\sum_{i,j \geq 1} p_i p_j \frac{(i+j-1)\partial}{\partial p_{i+j-1}} + b \sum_{i \geq 1} p_{i+1} \frac{i\partial}{\partial p_i} + (1+b) \sum_{i,j \geq 1} p_{i+j+1} \frac{i\partial}{\partial p_i} \frac{j\partial}{\partial p_j} \right)$$

$$\longrightarrow \text{edge without new vertices.}$$

$$A_{i+1} = \left[D^{(\alpha)}, A_i \right],$$

where

$$D^{(\alpha)} = \frac{1}{2} \left(\sum_{i,j \geq 1} p_i p_j \frac{(i+j)\partial}{\partial p_{i+j}} + b \cdot \sum_{i \geq 1} p_i \frac{i(i-1)\partial}{\partial p_i} + (1+b) \sum_{i,j \geq 1} p_{i+j} \frac{ij\partial^2}{\partial p_i \partial p_j} \right),$$

is the Laplace-Beltrami operator.

Adding one edge

We can define the operators A_i recursively

$$A_1 := \frac{p_1}{\alpha}, \quad \text{and} \quad A_{i+1} = \left[D^{(\alpha)}, A_i\right].$$

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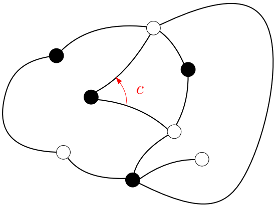
This operator is diagonal on Jack polynomials.

$$D^{(\alpha)} J_\lambda^{(\alpha)} = \left(\frac{\alpha}{2}\sum_{i\geq 1}\lambda_i(\lambda_i - 1) - \sum_{i\geq 1}\lambda_i(i-1)\right)\cdot J_\lambda^{(\alpha)}.$$

Moreover, the action of p_1 on Jack polynomials is given by the Pieri rule.
$\longrightarrow$ We deduce the action of A_i on Jack polynomials.

Chapuy–Dołęga operators '22

A map M is rooted if it has a marked black corner c.



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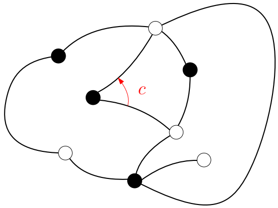

We consider an additional alphabet $Y := (y_0, y_1, y_2, \dots)$. We associate to the root face a weight y_i and to other faces a weight p_i.

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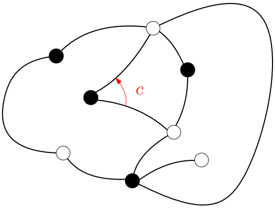

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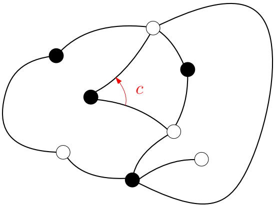

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$\mathcal{P} := \mathrm{Span}_{\mathbb{Q}(b)}\{p_\lambda\}$, is the space spanned by the weights of unrooted maps.

$\mathcal{P}_Y := \mathrm{Span}_{\mathbb{Q}(b)}\{y_i p_\lambda\}$ is the space spanned by the weights of rooted maps.

Chapuy–Dołęga operators

$$\frac{y_0}{1+b} : \mathcal{P} \to \mathcal{P}_Y; \quad \text{adds an isolated root black vertex.}$$

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$$\Gamma_Y = \sum_{i,j\geq 1} y_i p_j \frac{\partial}{\partial y_{i+j-1}} + (1+b) \cdot \sum_{i,j\geq 1} y_{i+j} \frac{i\partial^2}{\partial p_i \partial y_{j-1}} + b \cdot \sum_{i\geq 1} y_{i+1} \frac{i\partial}{\partial y_i} : \mathcal{P}_Y \to \mathcal{P}_Y$$

adds an edge between the root corner and a white corner of the map.

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$$\Theta_Y := \sum_{i \geq 1} p_i \frac{\partial}{\partial y_i} ; \mathcal{P}_Y \to \mathcal{P} \quad \text{"forgets" the root.}$$

$$\mathcal{B}_n(\mathbf{p}, u) := \Theta_Y \left(\Gamma_Y + u Y_+ \right)^n \frac{y_0}{1+b} : \mathcal{P} \to \mathcal{P}.$$

$\longrightarrow$ adds a black vertex of degree n with a weight u for each added white vertex.

Examples

$$\mathcal{B}_n(\mathbf{p}, u) := \Theta_Y \left(\Gamma_Y + uY_+\right)^n \frac{y_0}{1+b} : \mathcal{P} \to \mathcal{P}.$$

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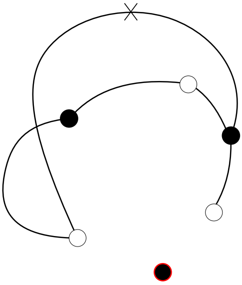

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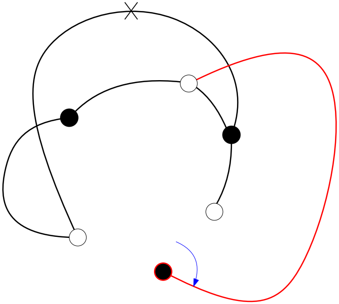

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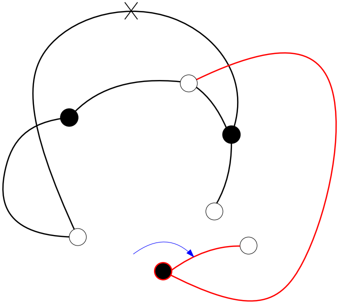

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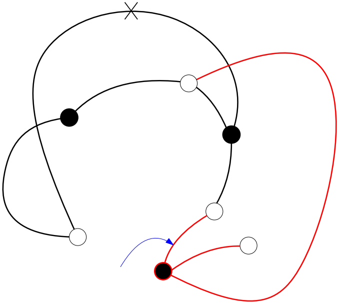

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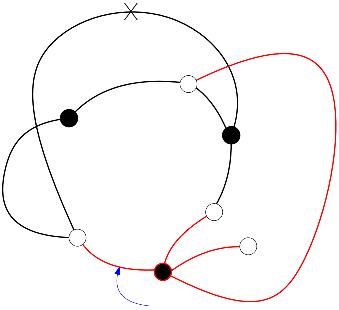

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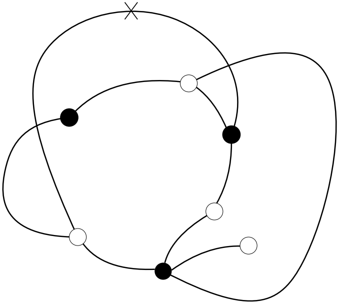
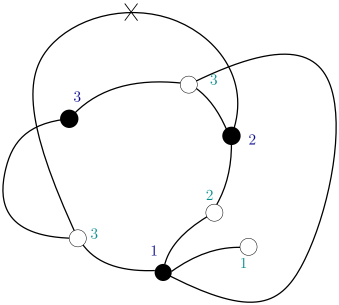


Examples

Layared maps: We act by $(-t)^n \mathcal{B}_n(\mathbf{p}, -\alpha s_1)$ to add a black vertex of degree n in layer 1.



Examples

$$\mathcal{B}_n(\mathbf{p}, u) := \Theta_Y \left(\Gamma_Y + uY_+\right)^n \frac{y_0}{1+b} : \mathcal{P} \to \mathcal{P}.$$

Remark

The variables y_i are catalytic variables.

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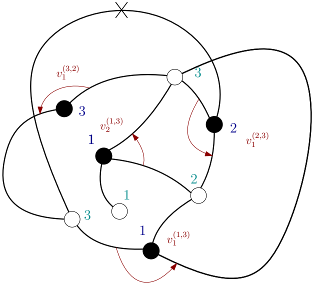
Remark

The variables y_i are catalytic variables.

$$\mathcal{B}_1^{(\alpha)}(\mathbf{p}, u) = \frac{up_1}{\alpha} + \sum_{i \geq 1} p_{i+1} \frac{i\partial}{\partial p_i},$$

$$\mathcal{B}_2^{(\alpha)}(\mathbf{p}, u) = \frac{u^2 p_2}{\alpha} + \sum_{i \geq 1} \left(\big(2u + (i+1)(\alpha - 1)\big) p_{i+2} + \sum_{\substack{j+k=i+2 \\ j,k \geq 1}} p_j p_k \right) \frac{i\partial}{\partial p_i}$$
$$+ \frac{u}{\alpha} \big((\alpha - 1) p_2 + p_{1,1}\big) + \alpha \sum_{i,j \geq 1} p_{i+j+2} \frac{i\partial}{\partial p_i} \frac{j\partial}{\partial p_j}.$$

Recall: decomposition algorithm



A labelled 3-layered map

- We decompose the map in an increasing order of the layers.
- We start by decomposing the vertex of maximal degree and maximal number.
- We delete black vertices in layer 1 with respect to this order, and starting each time at the marked corner.

Differential construction

$$F^{(k+1)}(s_1,\dots,s_{k+1}) = \sum_{\nu} \frac{1}{z_\nu}$$

$$(-t)^{\nu_1}\mathcal{B}_{\nu_1}(\mathbf{p},-\alpha s_1)\dots(-t)^{\nu_{\ell(\nu)}}\mathcal{B}_{\nu_{\ell(\nu)}}(\mathbf{p},-\alpha s_1)F^{(k)}(s_2,\dots,s_{k+1}).$$

Differential construction

$$F^{(k+1)}(s_1,\ldots,s_{k+1}) = \sum_{\nu}\left(\prod_{j\geq 1}\frac{1}{m_j(\nu)!}\right) \frac{(-t)^{\nu_1}\mathcal{B}_{\nu_1}(\mathbf{p},-\alpha s_1)}{\nu_1}\cdots\frac{(-t)^{\nu_{\ell(\nu)}}\mathcal{B}_{\nu_{\ell(\nu)}}(\mathbf{p},-\alpha s_1)}{\nu_{\ell(\nu)}}F^{(k)}(s_2,\ldots,s_{k+1}).$$

Fact: The operators $\mathcal{B}_n$ commute.

$$\begin{aligned}F^{(k+1)}(s_1,\ldots,s_{k+1}) &= \sum_{\ell\geq 1}\sum_{n_1,\ldots n_\ell\geq 1}\left(\prod_{j\geq 1}\frac{1}{\ell!}\right) \frac{(-t)^{n_1}\mathcal{B}_{n_1}(\mathbf{p},-\alpha s_1)}{n_1}\cdots\frac{(-t)^{n_\ell}\mathcal{B}_{n_\ell}(\mathbf{p},-\alpha s_1)}{n_\ell}F^{(k)}(s_2,\ldots,s_{k+1})\\ &= \exp\left(\sum_{n\geq 1}\frac{\mathcal{B}_n(\mathbf{p},-\alpha s_1)}{n}\right)\cdot F^{(k)}(s_2,\ldots,s_{k+1}).\end{aligned}$$

Differential construction

$$F^{(k)}(t, \mathbf{p}, s_1, \ldots, s_k) = \begin{cases} \mathcal{E}(t, \mathbf{p}, -\alpha s_1) \cdot F^{(k-1)}(t, \mathbf{p}, s_2, \ldots, s_k) & \text{if } k \geq 1 \\ 1 & \text{if } k = 0. \end{cases}$$

where

$$\mathcal{E}(t, \mathbf{p}, u) := \exp\left(\sum_{j\geq 1} \frac{(-t)^j}{j} \mathcal{B}_j(\mathbf{p}, u)\right),$$

the operator which adds a layer, with a weight $(-t)$ for each added edge, and a weight u for each new white vertex.

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$$F^{(k)}(t, \mathbf{p}, s_1, \dots, s_k) = \mathcal{E}(t, \mathbf{p}, -\alpha s_1) \cdots \mathcal{E}(t, \mathbf{p}, -\alpha s_k) \cdot 1.$$

Plan

Tau function [Chapuy–Dołęga '22]

$$\tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{u}) := \sum_{\lambda} t^{|\lambda|} \frac{J_{\lambda}^{(\alpha)}(\mathbf{p}) J_{\lambda}^{(\alpha)}(\mathbf{q}) J_{\lambda}^{(\alpha)}(\underline{u})}{j_{\lambda}^{(\alpha)}} \in \mathbb{Q}(\alpha)[\mathbf{p}, \mathbf{q}, u][\![t]\!],$$

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Here

$$J_\lambda^{(\alpha)}(\mathbf{p}) = J_\lambda^{(\alpha)}(x_1, x_2, \dots) \qquad \text{and} \qquad J_\lambda^{(\alpha)}(\mathbf{q}) = J_\lambda^{(\alpha)}(y_1, y_2, \dots),$$

$\mathbf{p} = (p_1, p_2, \dots)$ and $\mathbf{q} = (q_1, q_2, \dots)$ are respectively power-sum symmetric functions in (x_i) and (y_i).

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$\mathbf{p} = (p_1, p_2, \dots)$ and $\mathbf{q} = (q_1, q_2, \dots)$ are respectively power-sum symmetric functions in (x_i) and (y_i). Moreover, $\underline{u} := (u, u, \dots)$

$$J_\lambda^{(\alpha)}(\underline{u}) = J_\lambda^{(\alpha)}(\mathbf{p})_{|p_i = u}.$$

Exmaple: For $\lambda = [2, 2]$

$$J_{[2,2]}^{(\alpha)}(\mathbf{p}) = p_1^4 + 2(\alpha - 1)p_2 p_1^2 - 4\alpha p_3 p_1 + (\alpha^2 + \alpha + 1)p_2 p_2 + (-\alpha^2 + \alpha)p_4.$$

Then

$$J_{[2,2]}^{(\alpha)}(\underline{u}) = u^4 + 2(\alpha - 1)u^3 + (\alpha^2 - 3\alpha + 1)u^2 + (-\alpha^2 + \alpha)u.$$

Tau function

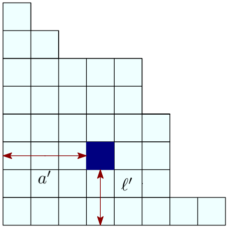
Theorem (Stanley '89)

For any λ,

$$J_\lambda^{(\alpha)}(\underline{u}) = \prod_{\square \in \lambda} \left(u + c_\alpha(\square)\right),$$

with

$$c_\alpha(\square) := \alpha a'(\square) - \ell'(\square).$$



Tau function

Theorem (Chapuy-Dołęga '22)

For any $m \geq 1$,

$$t^m \frac{\mathcal{B}_m(\mathbf{p}, u)}{m} \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{u}) = \frac{\partial}{\partial q_m} \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{u}).$$

$$\tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{u}) = \exp\left(\sum_{m \geq 1} \frac{t^m q_m}{m} \mathcal{B}_m(\mathbf{p}, u)\right) \cdot 1.$$

Consequence : First commutation relation

$$[\mathcal{B}_n(\mathbf{p}, u), \mathcal{B}_m(\mathbf{p}, u)] := \mathcal{B}_n(\mathbf{p}, u) \cdot \mathcal{B}_m(\mathbf{p}, u) - \mathcal{B}_m(\mathbf{p}, u) \cdot \mathcal{B}_n(\mathbf{p}, u) = 0.$$

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Proof:

$$t^{m+n}\frac{\mathcal{B}_n(\mathbf{p},u)}{n}\frac{\mathcal{B}_m(\mathbf{p},u)}{m}\cdot\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\underline{u}) = t^n\mathcal{B}_n(\mathbf{p},u)\frac{\partial}{\partial q_m}\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\underline{u}).$$

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By extracting the coefficient of $J_\lambda^{(\alpha)}(\mathbf{q})$, we get

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$$\frac{\mathcal{B}_n(\mathbf{p},u)}{n}\frac{\mathcal{B}_m(\mathbf{p},u)}{m}\cdot J_\lambda^{(\alpha)}(\mathbf{p}) = \frac{\mathcal{B}_m(\mathbf{p},u)}{m}\frac{\mathcal{B}_n(\mathbf{p},u)}{n}\cdot J_\lambda^{(\alpha)}(\mathbf{p})$$

Plan

Vanishing condition

We want to prove that if $n > |\lambda|$ then $[t^n]F^{(k)}(t, \lambda_1, \ldots, \lambda_k) = 0$.

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Then

$$[t^n]F^{(k)}(t, \lambda_1, \ldots, \lambda_k) = \sum_{\substack{n_1, \ldots, n_k \geq 0 \\ n_1 + \cdots + n_k = n}} \left([t^{n_1}]\mathcal{E}(t, \mathbf{p}, -\alpha\lambda_1)\right) \left([t^{n_2}]\mathcal{E}(t, \mathbf{p}, -\alpha\lambda_2)\right) \ldots \left([t^{n_k}]\mathcal{E}(t, \mathbf{p}, -\alpha\lambda_k)\right) \cdot 1$$

There exists an i for which $n_i > \lambda_i$.

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There exists an i for which $n_i > \lambda_i$. We prove that there exists a sequence of subspaces of $\mathcal{P}$

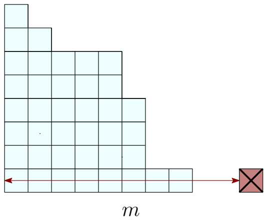
$$\mathbb{Q}(\alpha) = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$$

such that

$$\begin{cases} [t^n]\mathcal{E}(t,\mathbf{p},-\alpha m)\cdot\mathcal{P}_m \subseteq \mathcal{P}_m & \forall n,m; \qquad (\textit{Stability}),\\ [t^n]\mathcal{E}(t,\mathbf{p},-\alpha m)\cdot\mathcal{P}_m = \{0\} & \forall n>m; \qquad (\textit{Annihilation}).\end{cases}$$

The space $\mathcal{P}_m$

Fix a non-negative integer m. Let $\mathcal{P}_m := \mathrm{Span}_{\mathbb{Q}(\alpha)} \left\{ J_\lambda^{(\alpha)}(\mathbf{p}) \right\}_{\lambda_1 \leq m}$.



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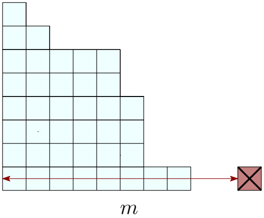

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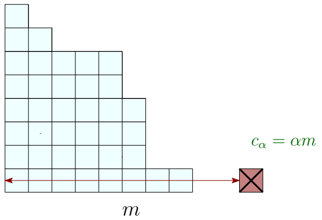
Observation: $\left[J_\lambda^{(\alpha)}(\mathbf{p})\right] \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) \neq 0 \iff \lambda_1 \leq m$

Proof: $$J_\lambda^{(\alpha)}(\underline{-\alpha m}) = \prod_{\square \in \lambda} (c_\alpha(\square) - \alpha m) \neq 0 \iff \lambda \leq m$$



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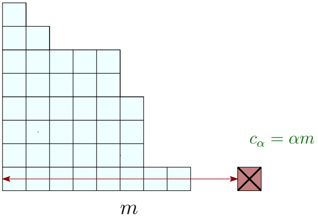
$$\mathcal{O}(\mathbf{p}) \cdot \mathcal{P}_m = 0 \quad \Longleftrightarrow \quad \mathcal{O}(\mathbf{p}) \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) = 0.$$



Annihilation property

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$$\mathcal{E}(z, \mathbf{p}, u) := \exp\left(\sum_{j \geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, u)\right).$$

Fix $n > m$.

$$[z^n]\mathcal{E}(z, \mathbf{p}, \underline{-\alpha m}) \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m})$$

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$$= [z^n] \exp\left(\sum_{j \geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot \exp\left(\sum_{j \geq 1} \frac{(-t)^j}{m} q_j \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot 1$$

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&[z^n]\mathcal{E}(z, \mathbf{p}, \underline{-\alpha m}) \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) \\
&\quad = [z^n] \exp\left(\sum_{j\geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot \exp\left(\sum_{j\geq 1} \frac{(-t)^j}{m} q_j \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot 1 \\
&\quad = \exp\left(\sum_{j\geq 1} \frac{(-t)^j}{m} q_j \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot [z^n] \exp\left(\sum_{j\geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot 1
\end{aligned}$$

Annihilation property

$$\mathcal{O}(\mathbf{p}) \cdot \mathcal{P}_m = 0 \quad \iff \quad \mathcal{O}(\mathbf{p}) \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) = 0.$$

$$\mathcal{E}(z, \mathbf{p}, u) := \exp\left(\sum_{j\geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, u)\right).$$

Fix $n > m$.

$$\begin{aligned}
&[z^n]\mathcal{E}(z, \mathbf{p}, \underline{-\alpha m}) \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) \\
&= [z^n] \exp\left(\sum_{j\geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot \exp\left(\sum_{j\geq 1} \frac{(-t)^j}{j} q_j \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot 1 \\
&= \exp\left(\sum_{j\geq 1} \frac{(-t)^j}{m} q_j \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot [z^n] \exp\left(\sum_{j\geq 1} \frac{(-z)^j}{j} \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot 1 \\
&= \exp\left(\sum_{j\geq 1} \frac{(-t)^j}{j} q_j \mathcal{B}_j(\mathbf{p}, -\alpha m)\right) \cdot [z^n]\tau^{(\alpha)}(z, \mathbf{p}, \underline{1}, \underline{-\alpha m}).
\end{aligned}$$

Annihilation property

$$[z^n]\mathcal{E}(z,\mathbf{p},\underline{-\alpha m})\cdot\tau^{(\alpha)}(t,\mathbf{p},\mathbf{q},\underline{-\alpha m})$$
$$=\exp\left(\sum_{j\geq 1}\frac{(-t)^j}{m}q_j\mathcal{B}_j(\mathbf{p},-\alpha m)\right)\cdot[z^n]\tau^{(\alpha)}(z,\mathbf{p},\underline{1},\underline{-\alpha m}).$$

$$\tau^{(\alpha)}(z,\mathbf{p},\underline{1},\underline{-\alpha m})=\sum_{\lambda}z^{|\lambda|}\frac{J_\lambda^{(\alpha)}(\mathbf{p})J_\lambda^{(\alpha)}(\underline{1})J_\lambda^{(\alpha)}(\underline{-\alpha m})}{j_\lambda^{(\alpha)}},$$

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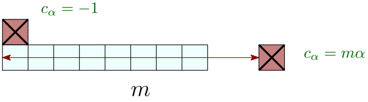
Recall

$$J_\lambda^{(\alpha)}(\underline{u})=\prod_{\square\in\lambda}\left(u+c_\alpha(\square)\right),$$



Stability property

We want to prove that for any $n \geq 0$

$$[z^n]\mathcal{E}(z, \mathbf{p}, -\alpha m) \cdot \mathcal{P}_m \subseteq \mathcal{P}_m,$$

with

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$$t^n \frac{\mathcal{B}_n(\mathbf{p}, -\alpha m)}{n} \cdot \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) = \frac{\partial}{\partial q_n} \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}).$$

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We extract the coefficient of $J_\xi^{(\alpha)}(\mathbf{p}) J_\lambda^{(\alpha)}(\mathbf{q})$

$$[J_\xi^{(\alpha)}(\mathbf{p})] t^n \frac{\mathcal{B}_n(\mathbf{p}, -\alpha m)}{n} \cdot [J_\lambda^{(\alpha)}(\mathbf{q})] \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m})$$
$$= [J_\lambda^{(\alpha)}(\mathbf{q})] \frac{\partial}{\partial q_n} [J_\xi^{(\alpha)}(\mathbf{p})] \tau^{(\alpha)}(t, \mathbf{p}, \mathbf{q}, \underline{-\alpha m}) = 0.$$

Next lecture

$$F^{(k)}(t, \mathbf{p}, s_1, \ldots, s_k) = \mathcal{E}(t, \mathbf{p}, -\alpha s_1) \cdots \mathcal{E}(t, \mathbf{p}, -\alpha s_k) \cdot 1.$$

In order to obtain the shifted symmetry property, we should understand

$$[\mathcal{E}(t, \mathbf{p}, u), \mathcal{E}(t, \mathbf{p}, v)] \neq 0.$$