Jack characters as generating series of bipartite maps and proof of Lassalle's conjecture (Part 3)

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joint work with

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Plan

- Lecture 1: Introduction and main result
- Lecture 2: Integrality in Lassalle's conjecture (Łukasiewicz paths)
- Lectures 3-5: Positivity in Lassalle's conjecture (maps)

Jack characters

Definition

Fix a partition μ.

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0, & \text{if } |\lambda| < |\mu|. \\ \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)}[p_{\mu,1^{|\lambda|-|\mu|}}]J_\lambda^{(\alpha)}, & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where $m_1(\mu)$ is the number of parts of size 1 in μ.

In particular,

$$J_\lambda^{(\alpha)} = \sum_{|\mu|=|\lambda|} \theta_\mu^{(\alpha)}(\lambda) p_\mu.$$

Goal

Theorem (BD–Dołęga '23+)

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

- $\mathcal{M}_\mu^{(\infty)}$ *is the set of layered maps of face type* μ.
- η *is a statistic on* $\mathcal{M}_\mu^{(\infty)}$.
- $|\mathcal{V}_\bullet(M)|$ *is the number of black vertices of* M,
- $|\mathcal{V}_\circ^{(i)}(M)|$ *is the number of white vertices of* M *labelled by* i,
- $cc(M)$ *is the number of connected components of* M.

- For $\alpha = 1$: Féray–Śniady formula for the characters of the symmetric group.
- For $\alpha = 2$: Féray–Śniady formula for zonal characters.

Plan

1. Recall: Characterization of Jack characters as shifted symmetric functions
2. Maps
3. Statistics of non-orientability
4. Condition 1: Top homogeneous part

Characterization theorem

A function $f(s_1, s_2, \dots)$ is α-shifted symmetric if it is symmetric in the variables $s_1, s_2 - 1/\alpha, s_3 - 2/\alpha \dots$.

$$f(\lambda) := f(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots).$$

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Theorem (Féray '15)

Fix a partition μ. The Jack character $\theta_\mu^{(\alpha)}$ is the unique α-shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)}/z_\mu \cdot p_\mu$, such that $\theta_\mu^{(\alpha)}(\lambda) = 0$ for any partition $|\lambda| < |\mu|$.

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Example

$$\theta_{[2]}^{(\alpha)}(\lambda) = \sum_{i\geq 1} \frac{\alpha}{2}\lambda_i(\lambda_i - 1) - \sum_{i\geq 1}(i-1)\lambda_i.$$

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- Shifted symmetry. Let $s_i' := \lambda_i - \frac{i-1}{\alpha}$.

$$\theta_{[2]}^{(\alpha)}(\lambda) = \frac{\alpha}{2}\sum_{i\geq 1}\left((s_i')^2 - \left(\frac{i-1}{\alpha}\right)^2 - s_i' + \frac{i-1}{\alpha}\right).$$

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- Top homogeneous part

$$\left[\theta_{[2]}^{(\alpha)}\right] = \frac{\alpha}{2}\sum_{i\geq 1}\lambda_i^2 = \frac{\alpha}{2}p_2(\lambda_1, \lambda_2, \ldots).$$

Theorem (Knop–Sahi '96)

For any μ *there exists a unique* α*-shifted symmetric function* J^*_μ *such that*

- $\deg(J^*_\mu) = |\mu|$,
- $J^*_\mu(\lambda) = 0$ *if* $|\lambda| \leq |\mu|$ *and* $|\lambda| \neq |\mu|$,
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Theorem (Lassalle '98)

For any μ *and* λ*, such that* $m := |\lambda| - |\mu| \geq 0$*. Then,*

$$J^*_\mu(\lambda) = \left\langle J^{(\alpha)}_\mu, \frac{\partial^m}{\partial p_1^m} J^{(\alpha)}_\lambda \right\rangle.$$

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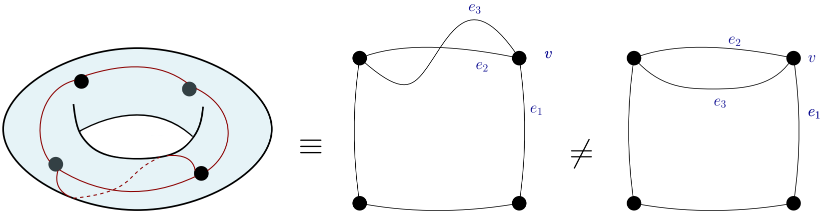
We want to prove that the generating function satisfies the 3 properties of the characterization theorem.

$$F_\mu^{(\infty)}(s_1, s_2 \dots) := \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{(-1)^{|\mu|} b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

Plan

Connected Maps

- A *connected map* is a cellular embedding of a connected graph in a surface, orientable or not.



Orientable connected maps

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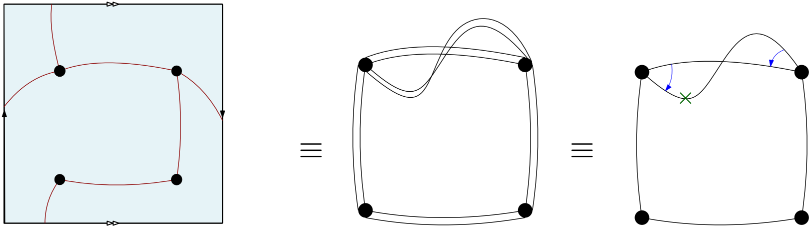


A non orientable connected map

Connected Maps

- A *connected map* is a cellular embedding of a connected graph in a surface, orientable or not.

Such representation of a non-orientable map is not unique.

A non orientable connected map

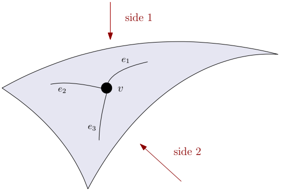
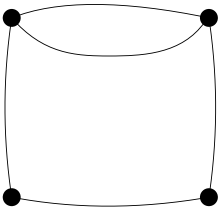
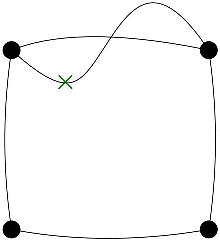


Figure 2: Two possible sides to represent a vertex.

Maps

- A map is a collection of connected maps. A map is *orientable* if its all connected components are embedded into orientable surfaces.





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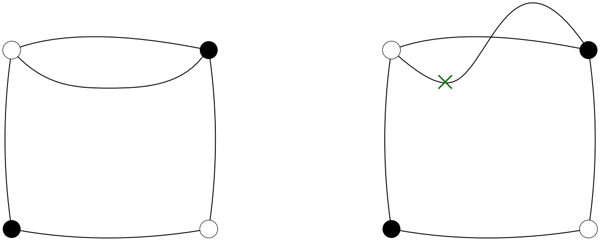
Maps

- A map is a collection of connected maps. A map is *orientable* if its all connected components are embedded into orientable surfaces.
- All maps considered are bipartite.



A non orientable bipartite map

Maps

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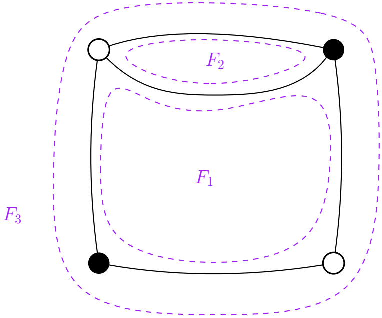
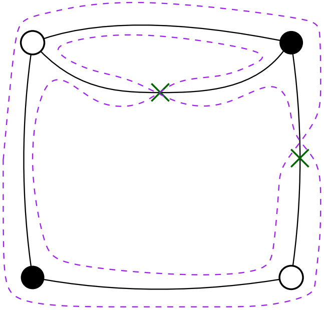
- The face type of a bipartite map M, denoted by $\nu_\diamond(M)$, is the partition given by the face degrees, divided by 2.



A map of size 5 and face type $[2, 2, 1]$.

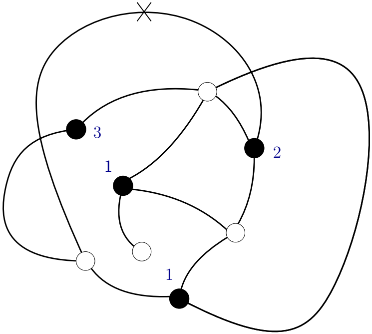


A map of size 5 and face type $[5]$.

Layered maps

Let k be a positive integer. A map M is k-layered if

- each black vertex has a label in $1, 2, \dots, k$.



A 3-layered map

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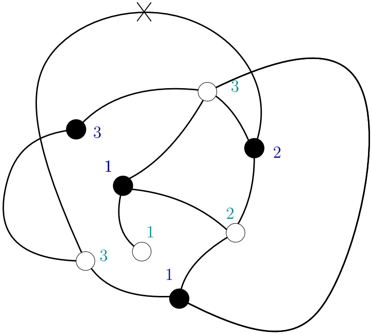

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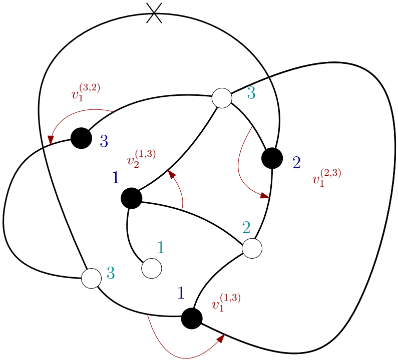
A map if layered if it is k-layered for some $k \geq 0$.

Layered maps

A layered map is labelled if

- each black vertex has a marked oriented corner.
- black vertices in the same layer i and with the same degree j are numbered $v_1^{(i,j)}, v_2^{(i,j)} \dots$.



A labelled 3-layered map

Plan

Recall: main result

Theorem (BD–Dołęga '23+)

$$\theta_\mu^{(\alpha)}(s_1, s_2, \dots) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)|-cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}},$$

- $\mathcal{M}_\mu^{(\infty)}$ *is the set of layered maps of face type* μ.
- η *is a* ***statistic of non-orientability*** *on* $\mathcal{M}_\mu^{(\infty)}$.
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Statistics of non-orientability

Definition (Goulden–Jackson '96)

A statistic of non-orientability (on layered maps) is a statistic which associates to each layered map M a non-negative integer such that $\eta(M) = 0$ if and only if M is orientable.

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When $\alpha = 1$:

$$b^{\eta(M)} = \begin{cases} 1 & \text{if } M \text{ is orientable} \\ 0 & \text{otherwise.} \end{cases}$$

When $\alpha = 2$:

$$b^{\eta(M)} = 1 \quad \text{for any } M.$$

$b = 0$ in the main theorem:

Theorem (Féray–Śniady '11)

$$\theta_\mu^{(1)}(s_1, s_2, \dots) = \sum_{\substack{\text{orientable layered maps } M \\ \text{of face type } \mu}} \frac{(-1)^{|\mu|}}{z_\diamond(M)} \prod_{i \geq 1} (-\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}.$$

$b = 1$ in the main theorem:

Theorem (Féray–Śniady '11)

$$\theta_\mu^{(2)}(s_1, s_2, \dots) = \sum_{\substack{\text{layered maps } M \text{ of face} \\ \text{type } \mu,\ \text{orientable or not}}} \frac{(-1)^{|\mu|}}{2^{\ell(\nu_\diamond(M))} z_\diamond(M)} \prod_{i \geq 1} (-2\lambda_i)^{|\mathcal{V}_\circ^{(i)}(M)|}.$$

How to construct statistics of non-orientability?

General method (La Croix '09, Dołęga–Féray–Śniady '14, Chapuy–Dołęga'22).

1. Fix a map M of size n. We choose **a decomposition algorithm**, by fixing a total order on the edges of M: $e_1, e_2, \ldots, e_n$. We denote $M_i := M \backslash \{e_1, e_2, \ldots, e_{i-1}\}$.

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2. For any map M and edge e, we choose

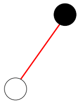
$$\vartheta(M, e) \in \{0, 1\}.$$

We then set

$$\eta(M) = \sum_{1 \leq i \leq n} \vartheta(M_i, e_i).$$

How to obtain a map of size $n+1$ from a map of size n

- We add an isolated edge.



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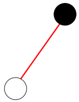

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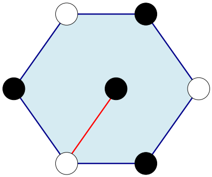

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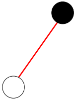

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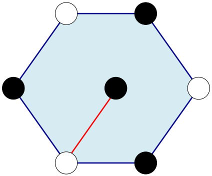

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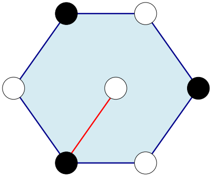

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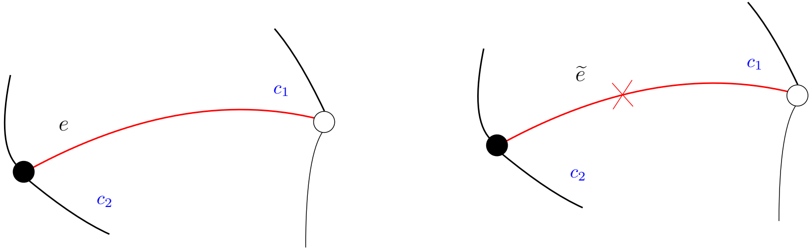

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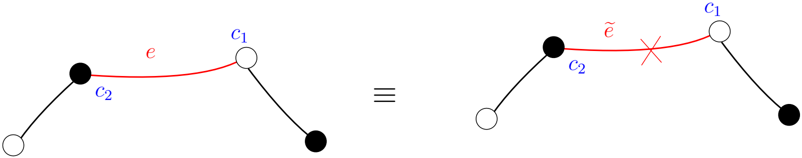

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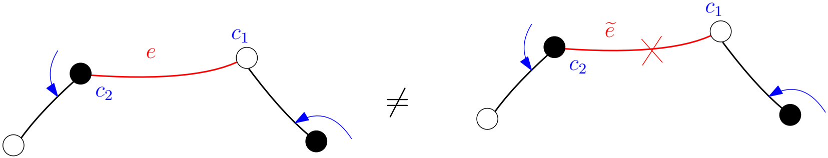
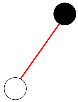


Edge weights

Let M be a map. We fix two corners of M of different colors and we connect them by an edge. We distinguish many cases.

1. the added edge is connected to a new vertex
$\longrightarrow \vartheta(M \cup \{e\}, e) = 0.$



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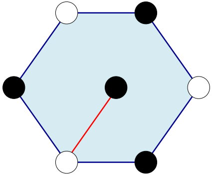
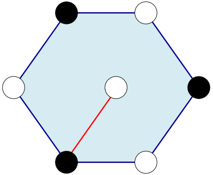


a black leaf

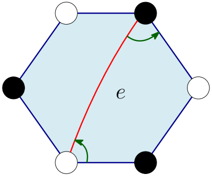


a white leaf

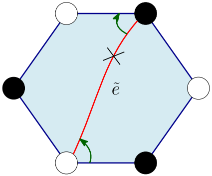
Edge weights

Let M be a map. We fix two corners of M of different colors and we connect them by an edge. We distinguish many cases.

2. The two corners are in the same face of M



a border
$\vartheta(M \cup \{e\}, e) = 0$

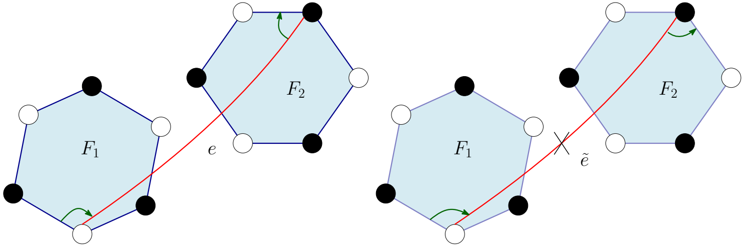


a twist
$\vartheta(M \cup \{e\}, e) = 1$

Edge weights

Let M be a map. We fix two corners of M of different colors and we connect them by an edge. We distinguish many cases.

3. The two corners are in two different faces of the same connected component.

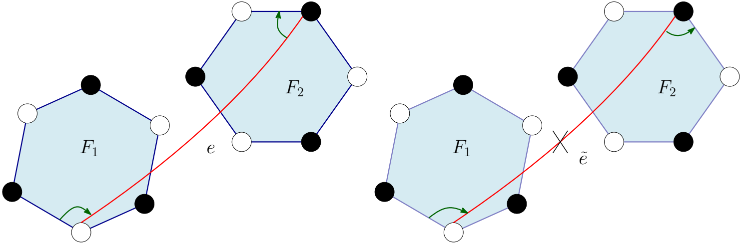


a handle

- $\vartheta(M \cup \{e\}, e) + \vartheta(M \cup \{\widetilde{e}\}, \widetilde{e}) = 1.$
- if M is orientable then exactly one of the maps $(M \cup \{e\}, e)$ and $(M \cup \{\tilde{e}\}, \tilde{e})$ is orientable, we associate to it the weight 0, and to the other map the weight 1.

Edge weights

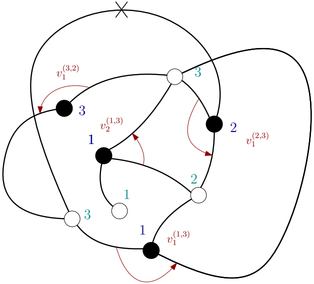
4. The two corners are in two different connected components.



a bridge

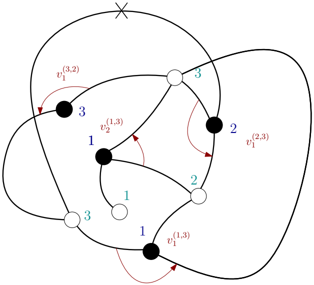
- $\vartheta(M \cup \{e\}, e) = \vartheta(M \cup \{\widetilde{e}\}, \widetilde{e}) = 0.$

How to decompose a layered map?



A labelled 3-layered map

How to decompose a layered map?

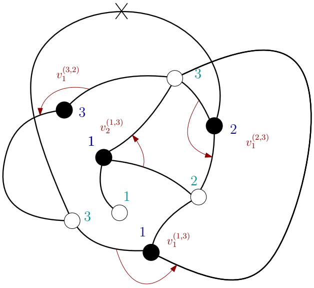


A labelled 3-layered map

Randomly (Dołęga–Féray–Śniady '14);

- works for $k = 1$.
- does not work for $k \geq 3$.

Our decomposition algorithm



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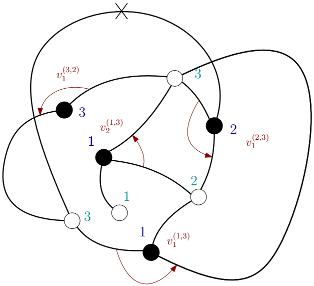
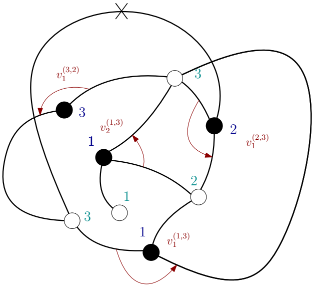


A labelled 3-layered map

- We decompose the map in an increasing order of the layers.

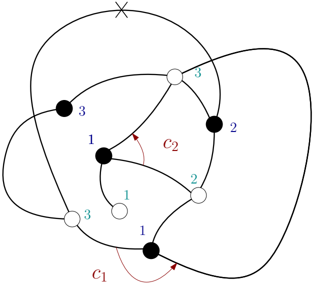
Our decomposition algorithm



A labelled 3-layered map

- We decompose the map in an increasing order of the layers.
- We start by decomposing the vertex of maximal degree and maximal number.

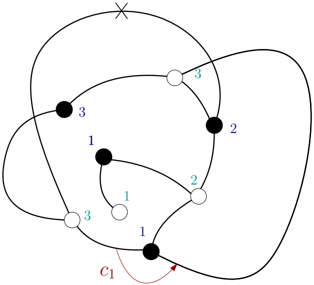
Our decomposition algorithm



Decomposition of a 3-layered map

- We decompose the map in an increasing order of the layers.
- We start by decomposing the vertex of maximal degree and maximal number.
- We delete black vertices in layer 1 with respect to this order, and starting each time at the marked corner.

Our decomposition algorithm



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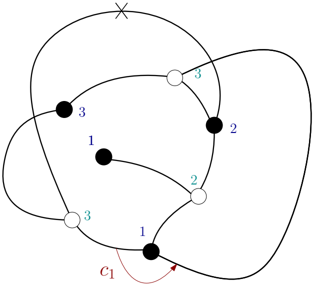

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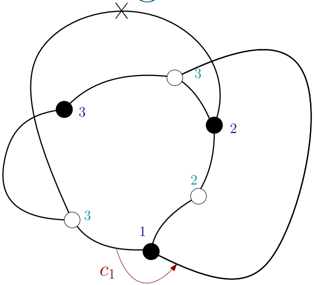

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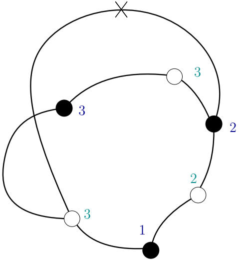

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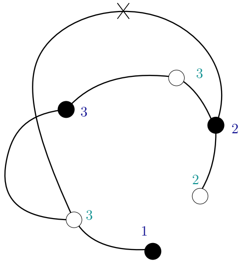

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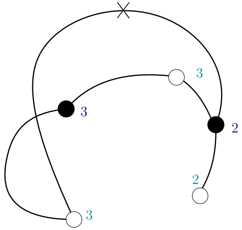

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Plan

Condition 0: Stability property

$$F_{\mu}^{(\infty)}(s_1, s_2 \dots) := \sum_{M \in \mathcal{M}_{\mu}^{(\infty)}} \frac{(-1)^{|\mu|} b^{\eta(M)}}{2^{|\mathcal{V}_{\bullet}(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha s_i)^{|\mathcal{V}_{\circ}^{(i)}(M)|}}{z_{\nu_{\bullet}^{(i)}(M)}}.$$

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$$F_\mu^{(k)}(s_1, s_2 \dots, s_k) := \sum_{M \in \mathcal{M}_\mu^{(k)}} \frac{(-1)^{|\mu|} b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}.$$

$$F_\mu^{(k+1)}(s_1, s_2 \dots, s_k, 0) = F_\mu^{(k)}(s_1, s_2 \dots, s_k)$$

Proof:

Condition 0: Stability property

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Proof:
Left hand-side : $k+1$-layered maps with no white vertices in layer $k+1$.

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Proof:
Left hand-side : $k+1$-layered maps with no white vertices in layer $k+1$.
If v is a black vertex in layer $k+1$, all its neighbours should be in layer $k+1$.
$\longrightarrow$ The layer $k+1$ is empty.

Condition 1: Top homogeneous part

$$F_{\mu}^{(k)}(s_1, s_2 \ldots, s_k) = \sum_{M \in \mathcal{M}_{\mu}^{(k)}} \frac{(-1)^{|\mu|} b^{\eta(M)}}{2^{|\mathcal{V}_{\bullet}(M)| - cc(M)} \alpha^{cc(M)}} \prod_{1 \leq i \leq k} \frac{(-\alpha s_i)^{|\mathcal{V}_{\circ}^{(i)}(M)|}}{z_{\nu_{\bullet}^{(i)}(M)}}.$$

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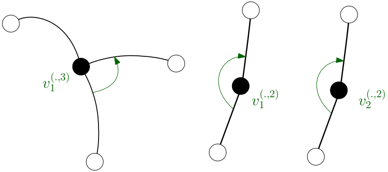
The top homogeneous part corresponds to maps with maximal number of white vertices.



The star map for $\mu = [3, 2, 2]$

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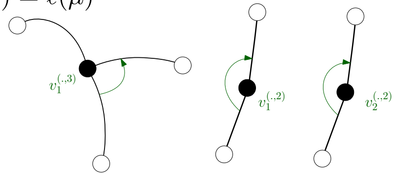
For such maps:

- $\nu_\bullet(M) = \nu_\diamond(M) = \mu.$
- $\eta(M) = 0.$
- $|\mathcal{V}_\bullet(M)| = cc(M) = \ell(\mu)$

$$\longrightarrow \frac{(-1)^{|\mu|} b^{\eta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} = (-1)^{|\mu|} \alpha^{-\ell(\mu)}.$$



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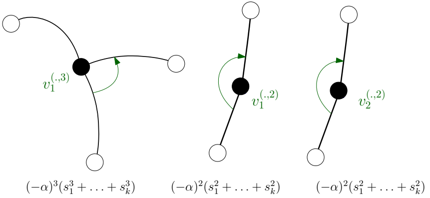
We choose independently a layer for each star; we multiply by $(-\alpha)^{|\mu|} p_\mu(s_1, \dots, s_k)$.



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We multiply by $\frac{1}{z_\mu} \prod_{1 \le i \le k} z_{\nu_\bullet^{(i)}(M)}$ to obtain labelled layered map.

$$\implies \left[F_{\mu}^{(k)}(s_1, \ldots, s_k)\right] = \frac{\alpha^{|\mu| - \ell(\mu)}}{z_\mu} p_\mu(s_1, \ldots, s_k).$$