Positive formula for Jack polynomials, Jack characters and proof of Lassalle's conjecture

j.w. Houcine Ben Dali

arXiv: 2305.07966

Plan:

1. Motivation & history of the problem
2. The main idea of the proof
3. Integrality: integrable hierarchy of Nazarov-Sklyanin
4. Positivity: maps & diff. equations

---

1. Jack polynomials

$J_\lambda^{(\alpha)}$ – symm. polynomials, α – parameter

<u>Def:</u> $\exists!$ family of symm. functions $J_\lambda^{(\alpha)}$:

(1) $J_\lambda = \sum_{\mu \leq \lambda} a_\mu^\lambda(\alpha) m_\mu$, $\quad a_\mu^\lambda(\alpha) \in \mathbb{Q}(\alpha)$

(2) $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = \delta_{\lambda,\mu} \cdot j_\lambda^{(\alpha)}$

(3) $a_{1^{|\lambda|}}^\lambda = |\lambda|!$

$$\langle p_\lambda, p_\mu \rangle_\alpha := \delta_{\lambda,\mu} \cdot \alpha^{\ell(\lambda)} \cdot \underbrace{z_\lambda}_{= \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!}$$

Ex:

- $\alpha = 1$ $\quad J_\lambda^{(\alpha)} = \prod_{\square \in \lambda} \mathrm{hook}_\lambda(\square) \cdot s_\lambda$ – Schur
- $\alpha = 2$ $\quad J_\lambda^{(\alpha)} = Z_\lambda$ – zonal polynomial (spherical function for the Gelfand pair (GL_N, O_N))
- $$J_\lambda^{(\alpha)} = \lim_{t\to 1} \frac{J_\lambda(X; q = t^\alpha, t)}{(1-t)^{|\lambda|}}$$

Problem: Find a formula for the expansion of J_λ in the power-sum basis.

$\alpha = 1$: **Th:** (Young's formula)

T – any filling of λ by numbers $1, 2, \dots, |\lambda|$.

$$J_\lambda^{(\alpha=1)} = \sum_{\substack{\sigma_\circ \in CS(T) \\ \sigma_\bullet \in RS(T)}} (-1)^{\mathrm{sgn}(\sigma_\circ)} p_{ct(\sigma_\circ \sigma_\bullet)} \qquad (*)$$

Q: How to generalize $(*)$ to arbitrary α?

Wishful thinking: $\exists$ statistic $st(\sigma_\circ, \sigma_\bullet) \in \mathbb{Z}_{\geq 0}$

s.t. $$J_\lambda^{(\alpha)} = \sum_{\substack{\sigma_\circ \in CS(T) \\ \sigma_\bullet \in RS(T)}} \alpha^{st(\sigma_\circ, \sigma_\bullet)} (-1)^{\mathrm{sgn}(\sigma_\circ)} p_{ct(\sigma_\circ \sigma_\bullet)}$$

Rem: Might be true by coincidence $\rightarrow$ huge number of cancellations in $(*)$.

Change of the parameter: $\alpha \longmapsto b := \alpha - 1$

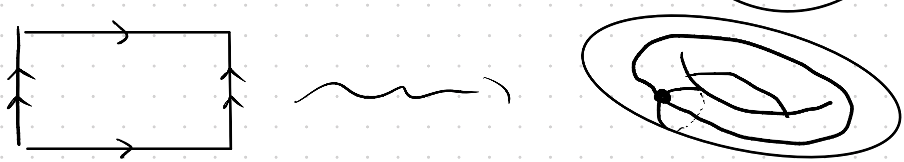
Topological expansion: • in RMT many quantities can be expanded as a gen. ser. involving maps
• top-degree terms correspond to planar objects

The formula $(*)$ can be interpreted as a topological expansion.

Maps: (quick introduction)

Map = graph embedded in a surface s.t. it cuts the surface into polygons

Ex:



A map is bipartite $\rightsquigarrow$ graph is bipartite

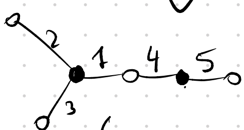
— || — labelled $\rightsquigarrow$ edges are labelled by $1, 2, \dots, n$

<u>Fact:</u> Labeled bipartite orientable maps $\xleftrightarrow{\text{bij.}}$ $(\sigma_\circ, \sigma_\bullet)$

4	5	
1	2	3

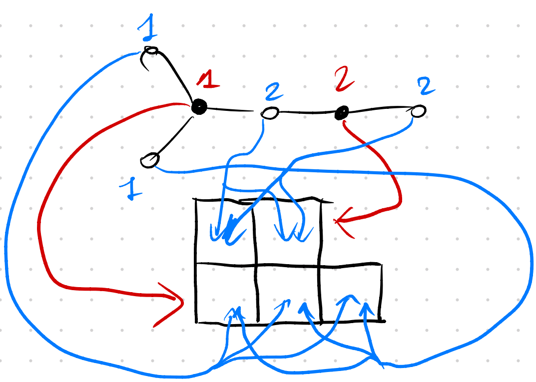
$$\sigma_\circ = (14)(2)(3)(5)$$

$$\sigma_\bullet = (132)(45)$$



$$\sigma_\circ \sigma_\bullet = (13245)$$

embedding of a map into λ



Thm: (Féray, Śniady '11) $\lambda \vdash n$

$$J_\lambda^{(\alpha=1)} = \frac{(-1)^n}{n!} \sum_{M} \sum_{f : V_\circ(M) \to [\ell(\lambda)]} p_{\nu_\diamond(M)} \prod_{1 \le i \le \ell(\lambda)} (-\lambda_i)^{|V_\circ^{(i)}(M)|}$$

M: labeled orientable bipartite

$\diamond$: face type

Lassalle's conjecture:

$$\theta_\mu^{(\alpha)}(\lambda) := \begin{cases} 0 & \text{if } |\lambda| < |\mu| \\ \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} [p_{\mu 1^{|\lambda|-|\mu|}}] J_\lambda^{(\alpha)} & \text{if } |\lambda| \ge |\mu| \end{cases}$$

θ: Jack character

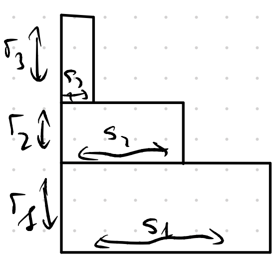
$\alpha = 1$ Féray–Śniady

$\sim$ # embeddings of a bip. map $\nu_\diamond(M) = \mu$ into λ.

Multirectangular coordinates

$$\lambda \longmapsto (s_1 \ge s_2 \ge \dots \ge s_k \ge 1;\ r_1, \dots, r_k \in \mathbb{Z}_{>0})$$

$$\lambda = \mathbf{s}^{\mathbf{r}}$$



NOT UNIQUE!

Ex: $\lambda = (5,5,4,2,1,1) = (5,4,2,1)^{(2,1,1,2)}$

$= (5,5,4,2,1,1)^{(1,1,1,1,1,1)} = (7,5,4,2,1,1)^{(0,2,1,1,1,1)}$

Conjecture: (Lassalle '08)

$(-1)^{|\mu|} z_\mu \theta_\mu^\alpha (\mathbb{S}, \mathbb{r})$ - polynomial in $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ with nonnegative int. coeff.

$b := \alpha - 1$

$\boxed{\alpha = 1}$ Conjectured by Stanley '04, Proved Féray '10, Féray, Śniady '11

$\rightarrow$ degree of a monomial in $\mathbb{S}, \mathbb{r} = \# V(M)$

maximal for planar maps

$\boxed{\alpha = 2}$ Similar formula, but orientable maps

$\downarrow$ Féray, Śniady '11, Dołęga, Féray, Śniady '14

orientable + nonorientable maps

Idea: For general $\alpha \longmapsto$ nonorientable maps counted with $b^{\text{statistic}(M)}$

Def: $k \geq 1$, M - bip. map (orient. or not) is k-layered if:

① $V_\bullet(M) = \dot{\bigcup}_{i=1}^{k} V_\bullet^{(i)}(M)$, $V_\circ(M) = \dot{\bigcup}_{i=1}^{k} V_\circ^{(i)}(M)$,

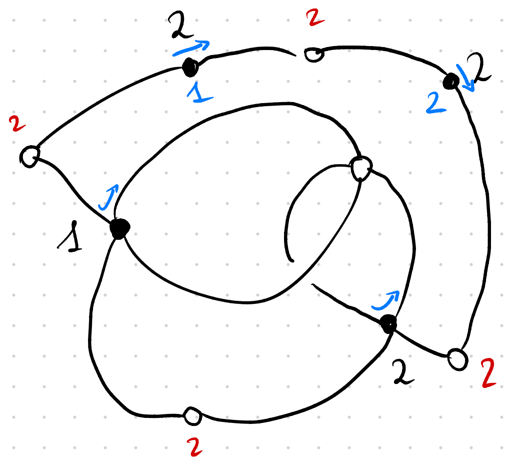
$\forall v_\circ \in V_\circ^{(i)} \ \exists v_\bullet \in V_\bullet^{(i)}$ s.t. $v_\circ - v_\bullet$, $\forall v_\bullet - v_\circ$ $v_\bullet \in V_\bullet^{(j)}(M)$ with $j \leq i$

(B) Let $\nu_\bullet^{(i)}$ – partition of degrees of $v \in V_\bullet^{(i)}(M)$

A k-layered map is labelled if:

(A) $\forall j \geq 1$ $\{v \in V_\bullet^{(i)}(M) \mid \deg(v) = j\}$ is labelled by $1, 2, \dots, m_j(\nu_\bullet^{(i)})$

(B) $\forall v \in V_\bullet(M)$ $\exists$ rooted oriented corner.



$$\nu_\bullet^{(1)} = (4) \qquad \nu_\diamond(M) = (4,4,4)$$
$$\nu_\bullet^{(2)} = (4,2,2)$$

$$\mathcal{M}_\mu^{(k)} \subseteq \mathcal{M}_\mu^{(\infty)}$$

k-layered map with face-type μ.

Theorem 1 (Ben Dali, D. '23)

$$\theta_\mu^{(\alpha)}(\lambda) = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^{(\infty)}} \frac{b^{\eta(M)}}{2^{|V_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|V_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}$$

Theorem 2 (B.D.–D.) $\lambda \vdash n$

$$J_\lambda = (-1)^n \sum_{M \in \mathcal{M}^{(\ell(\lambda))}} p_{\nu_\diamond(M)} \frac{b^{\eta(M)}}{2^{|V_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{(-\alpha\lambda_i)^{|V_\circ^{(i)}(M)|}}{z_{\nu_\bullet^{(i)}(M)}}$$

Theorem 3 (B.D.–D.)

$$(-1)^{|\mu|} z_\mu \, \theta_\mu^{(\alpha)}(s, r) \in \mathbb{Z}_{\geq 0}[b, -s_1, -s_2, \ldots, r_1, r_2, \ldots]$$

Lassalle's conjecture

Idea of the proof:

1. Theorem 1 $\Longrightarrow$ Positivity in Theorem 3

   Integrality? $\longrightarrow$ combinatorics of Nazarov–Sklyanin integrable hierarchy

2. Jack characters are characterized by
   - shifted-symmetric structure
   - vanishing conditions

3. Gen. series of k-layered maps via differential operators

4. 3 satisfies 2