Symmetric group characters as symmetric functions

Mike Zabrocki (York University)
Rosa Orellana (Dartmouth College)

1) Recap
2) products of symmetric group characters
3) combinatorial interpretation for $\overline{g}_{(\mu_1)(\mu_2)\cdots(\mu_\ell)\lambda\gamma}$
4) AMA about Kronecker and reduced Kronecker

Day 1: $\mathbb{V}^\lambda \otimes \mathbb{V}^\mu \cong \bigoplus_\gamma (\mathbb{V}^\gamma)^{\oplus c^\gamma_{\lambda\mu}}$ LR-rule

$\mathbb{S}^\lambda \otimes \mathbb{S}^\mu \cong \bigoplus_\gamma (\mathbb{S}^\gamma)^{\oplus g_{\lambda\mu\gamma}}$ Kronecker

SW duality $Gl_n \times S_k$ acts on $(\mathbb{V}^{(1)})^{\otimes k}$ $\quad p_\mu(x_1,\ldots,x_n) = \sum_\lambda \chi^\lambda(\mu) s_\lambda(x_1,\ldots,x_n)$

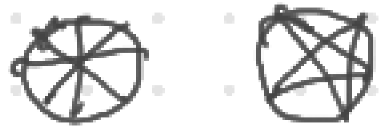
other centralizers $S_n \times P_k(n)$ acts $(\mathbb{V}^{(1)})^{\otimes k}$ $\quad p_\mu(x_1,\ldots,x_n) = \sum_{|\lambda|\le|\mu|} \chi^\lambda_{P_k(n)}(\mu) \tilde{s}_\lambda(x_1,\ldots,x_n)$

Day 2: restriction and Frobenius map $S_n \subseteq Gl_n$

$\tilde{s}_\lambda := \phi_n^{-1}(s_{(n-|\lambda|,\lambda)})$

Day 3: Better definitions in terms of power sums

also $\tilde{h}_\lambda$, $\tilde{x}_\lambda$ and $\tilde{s}_\lambda$

 = a s.f. identity is true if it is true by evaluating it at enough points

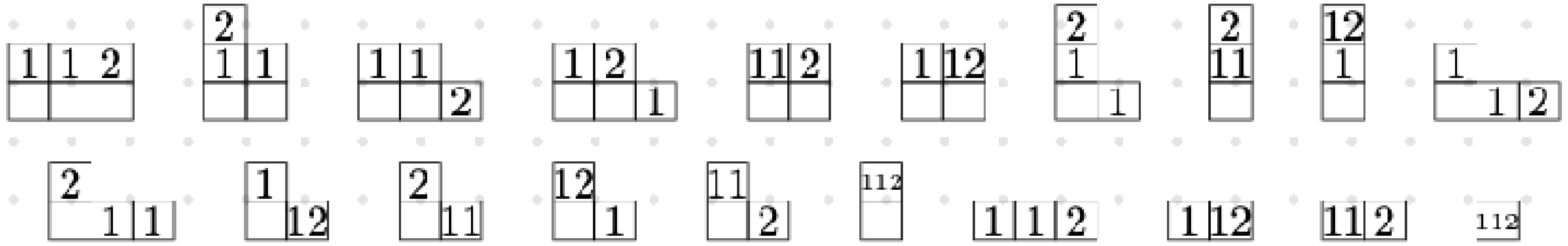
Day 4: multiset tableaux expansion

$$h_\mu = \sum_{\substack{T \in \\ \text{multiset} \\ \text{tableaux} \\ \text{content } \mu}} \tilde{s}_{\text{shape}(T)} \qquad h_\mu = \sum_{\substack{\pi \\ \text{multiset} \\ \text{content } \mu}} \tilde{h}_{m(\pi)}$$

$\{\{1|1|2\}\}$ $\{\{11|2\}\}$ $\{\{1|12\}\}$ $\{\{112\}\}$

$$h_{21} = \tilde{h}_{21} + \tilde{h}_{11} + \tilde{h}_{11} + \tilde{h}_{1}$$

$\pi \Vdash \{\{1^2, 2\}\}$ is a multiset partition
$m(\pi)$ = partition of multiplicities of multisets in π



$$h_{21} = \tilde{s}_3 + \tilde{s}_{21} + 4\tilde{s}_2 + 3\tilde{s}_{11} + 7\tilde{s}_1 + 4\tilde{s}_{\bullet}$$

$$\tilde{s}_\lambda \cdot \tilde{s}_\mu = \sum_\gamma \bar{g}_{\lambda\mu\gamma}\, \tilde{s}_\gamma$$

Proof: Φ_n of this equation

$$s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)} = \sum_\gamma \bar{g}_{\lambda\mu\gamma}\, s_{(n-|\gamma|,\gamma)}$$

$\pi \Vdash T \quad m(\pi)=\lambda \qquad \tau \Vdash S \quad m(\tau)=\mu \qquad T\cap S=\emptyset$

$$\tilde{h}_\lambda \cdot \tilde{h}_\mu = \sum_{\substack{\theta \Vdash T\cup S \\ \theta|_T=\pi \\ \theta|_S=\tau}} \tilde{h}_{m(\theta)}$$

Proof: reduce to $h_{(n-|\lambda|,\lambda)} * h_{(n-|\mu|,\mu)}$

$\mathbb{Z}_{\geq 0}$-matrices

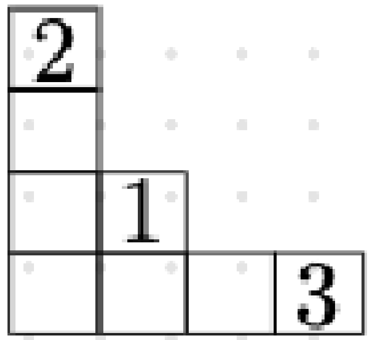
$\pi = \{\{1|1\}\} \qquad \tau = \{\{2|2|3\}\}$

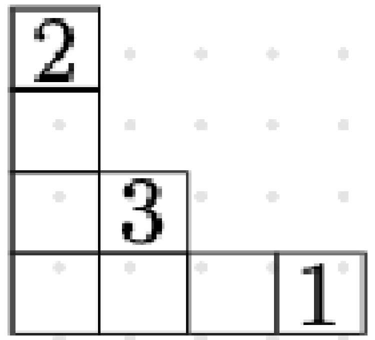
$$\begin{array}{cccccccccc}
\tilde{h}_2 \cdot \tilde{h}_{21} = & \tilde{h}_{221} & + & \tilde{h}_{1111} & + & \tilde{h}_{211} & + & \tilde{h}_{21} & + & \tilde{h}_{111} \\
 & \{\{1|1|2|2|3\}\} & & 1|12|2|3 & & 1|2|2|13 & & 12|12|3 & & 12|2|13
\end{array}$$

$$\tilde{h}_\mu \cdot \tilde{s}_\lambda = \sum_\gamma b^\mu_{\lambda\gamma} \tilde{s}_\gamma$$

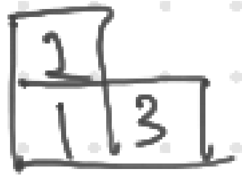
where $$b^\mu_{\lambda\gamma} = \sum_{\substack{\tau^{(*)} \\ \tau^{(i)} \vdash \mu_i}} c^\lambda_{\tau^{(0)}\tau^{(1)}\cdots\tau^{(\ell(\mu))}} \, c^\gamma_{\tau^{(0)}\tau^{(1)}\cdots\tau^{(\ell)}}$$

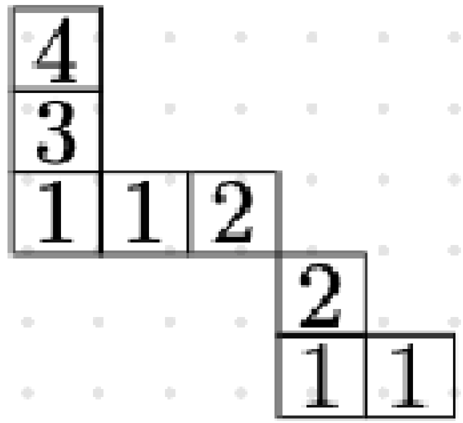
create sequences of tableaux which encode this coefficient.





$C^{(4,2,1,1)}_{(31,21)}$





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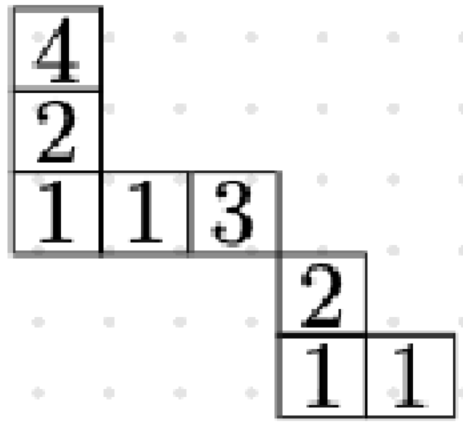

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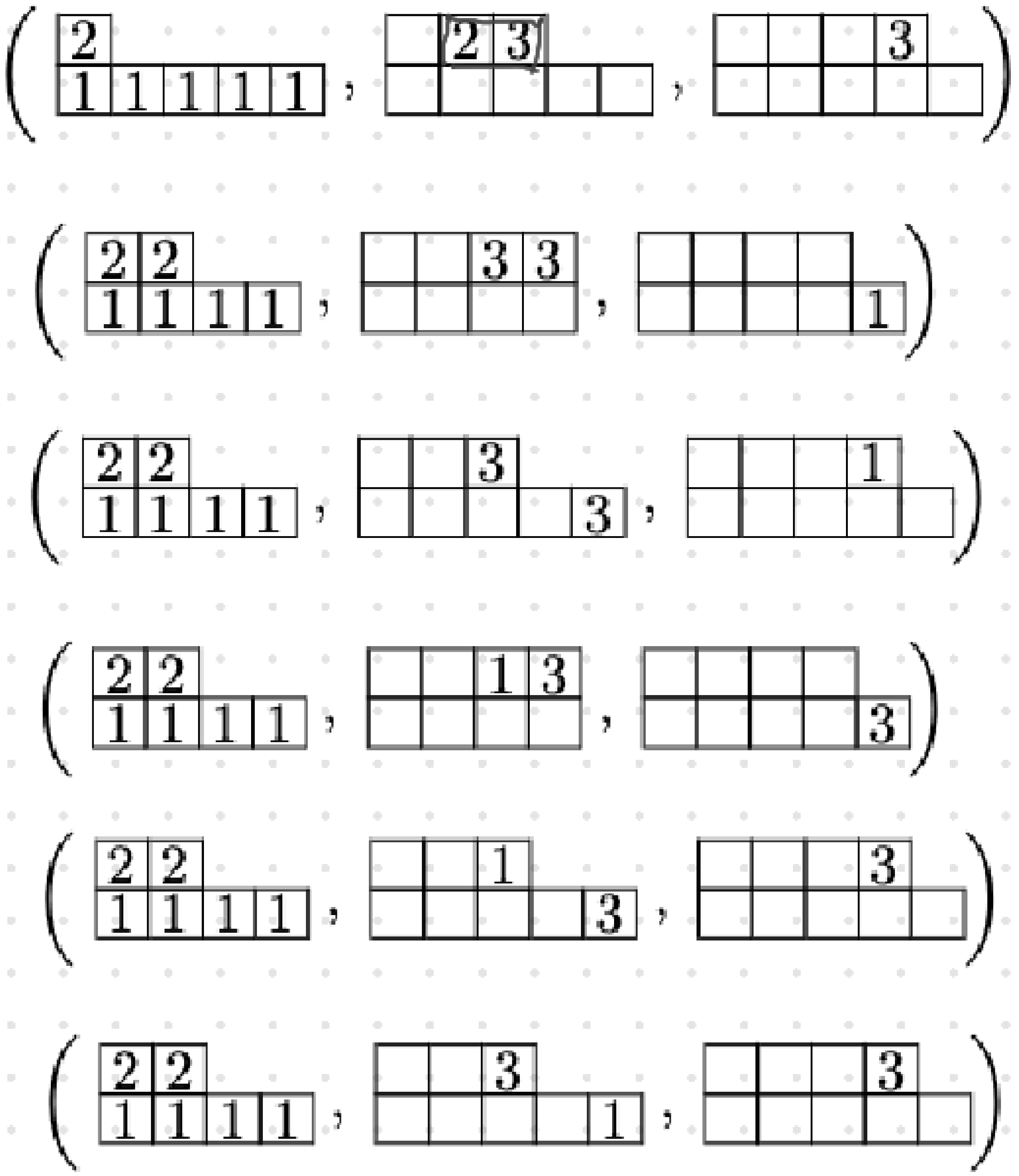
$\tau^{(0)} = (51)$ $\quad \tau^{(1)} = (2)$ $\quad \tau^{(2)} = (1)$



$\tilde{h}_{21} \cdot \tilde{s}_4 = \dots + 6\tilde{s}_{22} + \cdots$

$\tilde{h}_{21} \cdot \tilde{s}_{22} = \dots + 6\tilde{s}_4 + \cdots$

of "these objects"

content all tableaux $= (n-|\gamma|, \gamma)$

outer shape tableaux $= (n-|\lambda|, \lambda)$

sizes of skew tableaux are $\mu_1, \mu_2, \dots, \mu_\ell$

$$\tilde{h}_{\mu_1} \cdot \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_\ell} = \sum_{\theta} \tilde{h}_{m(\theta)}$$

set partitions of the multiset $\{\{1^{\mu_1}, 2^{\mu_2} \ldots \ell^{\mu_\ell}\}\}$

$$h_{\mu_1} \cdots h_{\mu_\ell} = \sum_{\theta \Vdash \{\{1^{\mu_1} \ldots \ell^{\mu_\ell}\}\}} \tilde{h}_{m(\theta)}$$

$$\tilde{s}_{\mu_1} \cdots \tilde{s}_{\mu_\ell} = (\tilde{h}_{\mu_1} - \tilde{h}_{\mu_1 - 1}) \cdots (\tilde{h}_{\mu_\ell} - \tilde{h}_{\mu_\ell - 1}) = \sum_{S \subseteq \{1, \ldots, \ell\}} (-1)^{|S|} \tilde{h}_{\mu_1 - \delta(1 \in S)} \tilde{h}_{\mu_2 - \delta(2 \in S)} \cdots \tilde{h}_{\mu_\ell - \delta(\ell \in S)}$$

$$\tilde{s}_r = \tilde{h}_r - \tilde{h}_{r-1}$$

$$\begin{array}{ccccc} \tilde{s}_{\mu_1} \cdot \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_\ell} \cdot \tilde{s}_\lambda & \leq & \tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_\ell} \tilde{s}_\lambda & \leq & h_{\mu_1} h_{\mu_2} \cdots h_{\mu_\ell} \tilde{s}_\lambda \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \tilde{s}_\mu \cdot \tilde{s}_\lambda & \leq & \tilde{h}_\mu \tilde{s}_\lambda & \leq & s_\mu \cdot \tilde{s}_\lambda \end{array}$$

$\bar{1}$	$\bar{1}1$	$\bar{2}1$	$\bar{2}2$	

$\bar{1}$	$\bar{1}$	$\bar{2}1$	2	
				$\bar{2}1$

$\bar{1}$	$\bar{1}$	1	$\bar{2}1$	
				$\bar{2}2$

$\bar{1}$	$\bar{1}$	1	$\bar{2}2$	
				$\bar{2}1$

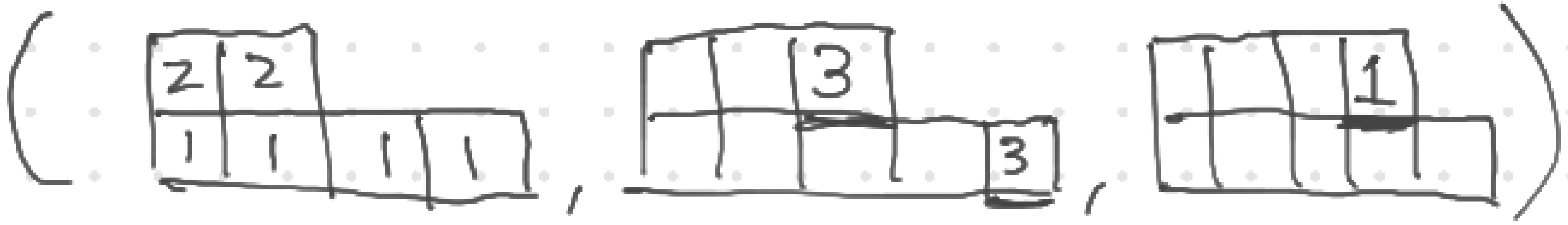
$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}12$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}2$	
				1

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}1$	
				2

$\bar{1}$	$\bar{1}$	$\bar{2}11$	$\bar{2}2$

no unbarred entries



$\tilde{h}_\mu \cdot \tilde{s}_\lambda$