Symmetric group characters as symmetric functions

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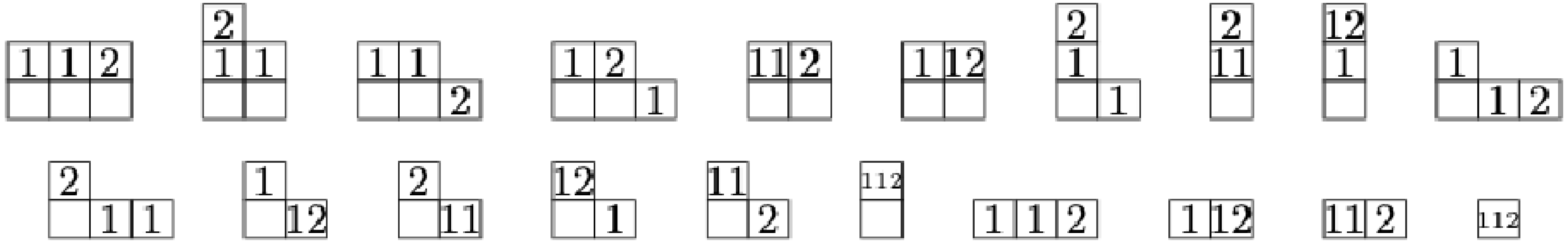
1. Recap: $\tilde{s}_\lambda$ is the preimage of $s_{(n-|\lambda|,\lambda)}$ in the Frobenius map ϕ_n
2. A little data + observations
3. A theorem of Littlewood + Scharf/Thibon
4. A characterization of symmetric functions
5. Better definitions/formulae
   - (a) $\tilde{s} \to p$ via character polynomials
   - (b) $h \to \tilde{h} \to \tilde{s}$ multisets & multiset partitions

$\{\{1|1|2\}\}$ $\{\{11|2\}\}$ $\{\{12|1\}\}$ $\{\{112\}\}$



$$h_{21} = \tilde{s}_3 + \tilde{s}_{21} + 4\tilde{s}_2 + 3\tilde{s}_{11} + 7\tilde{s}_1 + 4\tilde{s}_{()}$$

Recap: $f[\Xi_\mu]$ = evaluation of a symmetric function f (representing a Gl_n character) at the eigenvalues of a permutation matrix $[\delta_{\sigma(i)j}]_{1\le i,j\le n} \in S_n \subseteq Gl_n$ cycle type $\sigma = \mu$

$$p_k[\Xi_\mu] = \sum_{d|k} d m_d$$

$$\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$$

$$p_\lambda[\Xi_\mu] = \prod_{i=1}^{\ell(\lambda)} \sum_{d|\lambda_i} d m_d$$

$$\Phi_n(f) = \sum_{\mu \vdash n} f[\Xi_\mu] \, p_\mu / z_\mu$$

Frobenius character map applied to a Gl_n-character $f(x_1, x_2, \ldots, x_n)$

$$\tilde{S}_\lambda := \Phi_n^{-1}(s_{(n-|\lambda|, \lambda)}) \quad \text{for big } n.$$

From $n=2$ calculate

$\tilde{s}_{()} = 1$ $\quad$ $\tilde{s}_{\cdot}(x_1, x_2) = 1$

$\tilde{s}_{(1)} = s_1 - 1$ $\quad$ $\tilde{s}_{(1)}(x_1, x_2) = x_1 + x_2 - 1$

$n=2$

μ	1,1 (1)(2)	1,−1 (12)
$\tilde{s}_{\cdot}[\Xi_\mu]$	1	1
$\tilde{s}_{(1)}[\Xi_\mu]$	1	−1

$n=3$

	1,1,1 (1)(2)(3)	1,−1,1 (12)(3)	$1, \rho_3, \rho_3^2$ (123)
$\tilde{S}_{()}[\Xi_\mu]$	1	1	1
$\tilde{S}_{(1)}[\Xi_\mu]$	2	0	−1
$\tilde{S}_{11}[\Xi_\mu]$	1	−1	1
$S_{11}[\Xi_\mu]$	3	−1	0

$\phi_3(\tilde{S}_{11}) = S_{111}$

$S_{11}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$

$$\tilde{S}_{11} = S_{11} - \tilde{S}_1$$
$$= S_{11} - S_1 + 1$$

Theorem Littlewood 1958

Scharf/Thibon 1994 $f, g \in \Lambda$ $g \in \Lambda_n$

$$\langle \Phi_n(f), g \rangle = \langle f, g[1 + h_1 + h_2 + h_3 + \cdots] \rangle$$

Proof: $f = p_\lambda$ $g = p_\mu$ $\mu \vdash n$ $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$

$$\langle \Phi_n(p_\lambda), p_\mu \rangle = p_\lambda[\Xi_\mu]$$

$$1 + h_1 + h_2 + \cdots = \prod_{r \geq 1} \exp(p_r / r)$$

$$p_\mu[1 + h_1 + h_2 + \cdots] = p_\mu\Big[\prod_{r \geq 1} \exp(p_r/r)\Big]$$

$$= \prod_{d=1}^{n} p_d\Big[\prod_{r \geq 1} \exp(p_r/r)\Big]^{m_d}$$

$$p_d[p_r] = p_{dr}$$

$$= \prod_{d \geq 1} \prod_{r \geq 1} \exp(p_{dr}/r)^{m_d}$$

$$= \prod_{k \geq 1} \prod_{d | k} \exp\left(m_d p_k / (k/d)\right)$$

take coeff of p_λ

$$\prod_{i=1}^{\ell(\lambda)} \sum_{d | \lambda_i} d m_d = p_\lambda[\Xi_\mu]$$

Lemma: (character polynomials)

For every $f \in \Lambda$, $\deg(f) \le n$, there is a polynomial $q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$

$\forall \mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$

$$f[\Xi_\mu] = q(m_1, m_2, \ldots, m_n)$$

Define $q(x_1, \ldots, x_n) = f\Big|_{p_k \to \sum_{d|k} d x_d}$

$$p_k[\Xi_\mu] = \sum_{d|k} d m_d$$

$$f = q\left(p_1, \frac{p_2 - p_1}{2}, \frac{p_3 - p_1}{3}, \ldots, \frac{1}{n} \sum_{d|n} \mu(n/d) p_d\right)$$

$f, g \in \Lambda \quad \deg(f), \deg(g) \le d$

$\circledast$ $f = g$ iff $f[\Xi_\mu] = g[\Xi_\mu] \quad \forall\ |\mu| \le d$

<u>Proof:</u> If $p(x)$ is a polynomial in one variable $\deg(p(x)) \le d$

then $p(x) = 0$ iff $p(0) = p(1) = \cdots = p(d) = 0$

multivariate version: (prove induction on # variables)

degree

$\deg(x_i) = i$

$q(x_1, x_2, \dots, x_d) = 0$ iff $q(m_1, m_2, \dots, m_d) = 0$

$\forall\ m_1 + 2m_2 + \cdots + d m_d \le d$

$m_i \ge 0$

$h = f - g$ and $q(x_1, \dots, x_n)$ corresponds to h

$q(m_1, \dots, m_n) = h[\Xi_\mu]$

$f, g \in \Lambda \qquad \deg(f)\ \deg(g) \leq d$ for some d

$\circledast \quad f = g$ iff $f[\Xi_\mu] = g[\Xi_\mu] \ \forall \ |\mu| > d$

<u>Proof</u>: reduce to previous case using

$$f[\Xi_{(d+1,\mu)}] = f[\Xi_\mu]$$

apply $\circledast$

Let $b_\mu(x_1,\ldots,x_n) = 1^{a_1}(x_1)_{a_1}\, 2^{a_2}(x_2)_{a_2} \cdots n^{a_n}(x_n)_{a_n}$

$\mu = (1^{a_1} 2^{a_2} \ldots n^{a_n})$

$(x)_k = x(x-1)\cdot\ldots\cdot(x-k+1)$

$$b_\mu(a_1, a_2, \ldots, a_n) = z_\mu$$

$$\tilde{p}_\mu[\Xi_\gamma] = b_\mu(m_1, m_2, \ldots, m_n) = \begin{cases} z_\gamma / z_\tau & m_i \geq a_i \\ 0 & \text{else} \end{cases}$$

$\gamma = (1^{m_1} 2^{m_2} \ldots n^{m_n})$

$p_\gamma = p_\mu \cdot p_\tau$

$\tau = (1^{m_1-a_1} 2^{m_2-a_2} \ldots n^{m_n-a_n})$

$$\tilde{p}_\mu = b_\mu\left(p_1, \frac{p_2 - p_1}{2}, \ldots, \frac{1}{n}\sum_{d|n} \mu(n/d)\, p_d\right)$$

$$\tilde{h}_\mu = \sum_{\gamma \vdash |\mu|} \langle h_\mu, p_\gamma \rangle \frac{\tilde{p}_\gamma}{z_\gamma} \qquad \tilde{x}_\mu = \sum_{\gamma \vdash |\mu|} \langle s_\mu, p_\gamma \rangle \frac{\tilde{p}_\gamma}{z_\gamma}$$

<u>Prop:</u>

$$\Phi_n(\tilde{p}_\mu) = p_\mu h_{n-|\mu|}$$

$$\Phi_n(\tilde{h}_\mu) = h_\mu h_{n-|\mu|}$$

$$\Phi_n(\tilde{x}_\mu) = s_\mu h_{n-|\mu|}$$

<u>Proof:</u> direct calculation for $\tilde{p}_\mu$

then apply Φ_n for n big for $\tilde{h}_\mu$ and $\tilde{x}_\mu$

and deduce that it is right by ⊛

$$\tilde{X}_\lambda = \sum_{\substack{\lambda/\mu \\ \text{h-strip}}} \tilde{S}_\mu \qquad\qquad \tilde{S}_\lambda = \sum_{\substack{\lambda/\mu \\ \text{v-strip}}} (-1)^{|\lambda|-|\mu|} \tilde{X}_\mu$$

<u>Proof:</u> apply $\phi_n^{\ddagger}$ $\circledast$ for n big

then $\phi_n(\tilde{S}_\lambda) = S_{(n-|\lambda|,\lambda)}$ for n big

$$\tilde{h}_\mu = \sum_{|\lambda| \leq |\mu|} K_{(n-|\lambda|,\lambda)\,(n-|\mu|,\mu)} \tilde{S}_\lambda$$

Proof: appy ϕ_n and deduce the idenitity using $\circledast$

$$h_\lambda = \sum_{\substack{\pi \\ \text{multiset} \\ \text{partitions} \\ \text{content } \lambda}} \tilde{h}_{m(\pi)}$$

$$\tilde{h}_\mu = \sum_{\substack{T \\ \text{tableaux}}} \tilde{s}_{w(T)}$$

$m(\pi)$ = partition representing the multiplicities of the parts of π

$\lambda = (2,2)$

$1\|1\|2\|2$	$\tilde{h}_{22}$
$11\|2\|2$	$\tilde{h}_{21}$
$1\|1\|22$	$\tilde{h}_{21}$
$1\|2\|12$	$\tilde{h}_{111}$

$1\|122$	$\tilde{h}_{11}$
$2\|112$	$\tilde{h}_{11}$
$11\|22$	$\tilde{h}_{11}$
$12\|12$	$\tilde{h}_{2}$
1122	$\tilde{h}_{1}$

Sage demo

```
sage: SymmetricFunctions(QQ).inject_shorthands('all')
Defining e as shorthand for Symmetric Functions over Rational Field in the elementary basis
Defining f as shorthand for Symmetric Functions over Rational Field in the forgotten basis
Defining h as shorthand for Symmetric Functions over Rational Field in the homogeneous basis
Defining ht as shorthand for Symmetric Functions over Rational Field in the induced trivial symmetric group character basis
Defining m as shorthand for Symmetric Functions over Rational Field in the monomial basis
Defining o as shorthand for Symmetric Functions over Rational Field in the orthogonal basis
Defining p as shorthand for Symmetric Functions over Rational Field in the powersum basis
Defining s as shorthand for Symmetric Functions over Rational Field in the Schur basis
Defining sp as shorthand for Symmetric Functions over Rational Field in the symplectic basis
Defining st as shorthand for Symmetric Functions over Rational Field in the irreducible symmetric group character basis
Defining w as shorthand for Symmetric Functions over Rational Field in the Witt basis
sage: for d in range(5):
....:     for la in Partitions(d):
....:         print("st"+str(la)," = ", s(st(la)))
....:
st[]  =  s[]
st[1]  =  -s[] + s[1]
st[2]  =  -2*s[1] + s[2]
st[1, 1]  =  s[] - s[1] + s[1, 1]
st[3]  =  s[1] - s[1, 1] - 2*s[2] + s[3]
st[2, 1]  =  3*s[1] - 2*s[1, 1] - 2*s[2] + s[2, 1]
st[1, 1, 1]  =  -s[] + s[1] - s[1, 1] + s[1, 1, 1]
st[4]  =  2*s[1, 1] + s[2] - s[2, 1] - 2*s[3] + s[4]
st[3, 1]  =  -3*s[1] + 3*s[1, 1] - s[1, 1, 1] + 5*s[2] - 3*s[2, 1] - 2*s[3] + s[3, 1]
st[2, 2]  =  -s[1] + 4*s[1, 1] + 2*s[2] - 2*s[2, 1] + s[2, 2] - s[3]
st[2, 1, 1]  =  -4*s[1] + 3*s[1, 1] - 2*s[1, 1, 1] + 3*s[2] - 2*s[2, 1] + s[2, 1, 1]
st[1, 1, 1, 1]  =  s[] - s[1] + s[1, 1] - s[1, 1, 1] + s[1, 1, 1, 1]
```

$$\tilde{S}_\lambda = S_\lambda + \sum_{d \geq 1} (-1)^d \left(\begin{array}{l} \text{terms of degree} \\ |\lambda| - d \end{array} \right)$$

$S_2[\Xi_\mu]$ for $\mu \vdash 4$

```
sage: [s[2].eval_at_permutation_roots(mu) for mu in Partitions(4)]
[0, 1, 2, 4, 10]
```

$\Phi_4(s_2)$

```
sage: s[2].character_to_frobenius_image(4)
s[2, 2] + 2*s[3, 1] + 2*s[4]
```

$$Sym^2(V_4) = S^{(2,2)} \oplus S^{(3,1)} \oplus S^{(3,1)} \oplus S^{(4)} \oplus S^{(4)}$$

$$\tilde{S}_\lambda \cdot \tilde{S}_\mu = \sum_\gamma \bar{g}_{\lambda\mu}^{\gamma} \tilde{S}_\gamma$$

```
sage: st[2]*st[2]
st[] + st[1] + st[1, 1] + st[1, 1, 1] + 2*st[2] + 2*st[2, 1] + st[2, 2] + st[3] + st[3, 1] + st[4]
```

```
sage: s[6,2].kronecker_product(s[6,2])
s[4, 2, 2] + s[4, 3, 1] + s[4, 4] + s[5, 1, 1, 1] + 2*s[5, 2, 1] + s[5, 3] + s[6, 1, 1] + 2*s[6, 2] + s[7, 1] + s[8]
```