Def: An S_n-module is a vector space over a field $\mathbb{F}$ with an S_n-action. (Here we focus on $\mathbb{F} = \mathbb{C}$)

e.g.
- $V_1 := \mathrm{span}_{\mathbb{C}}\{v\}$ (1-dimensional vector space), with $\sigma \cdot v = v \quad \forall \sigma \in S_n$. ($\because \sigma \cdot (cv) = cv \quad \forall c \in \mathbb{C}$) (Trivial module)
  ∴ Trivial action
- $V_2 := \mathrm{span}_{\mathbb{C}}\{v\}$ (1-dimensional vector space), with $\sigma \cdot v = \mathrm{sgn}(\sigma)\, v \quad \forall \sigma \in S_n$. ($\because \sigma \cdot (cv) = \mathrm{sgn}(\sigma)\, cv \quad \forall c \in \mathbb{C}$) (Sign module)
  sign
- $V_3 := \mathrm{span}_{\mathbb{C}}\{v_1, v_2, \dots, v_n\}$ (n-dimensional vector space), with $\sigma \cdot v_i = v_{\sigma(i)} \quad \forall \sigma \in S_n$. ($\because \sigma \cdot (\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i v_{\sigma(i)} = \sum_{j=1}^{n} c_{\sigma^{-1}(j)} v_j \quad \forall c_i \in \mathbb{C}$) (Permutation module)
  permutation
  (GL_n-module : defining representation)
- $V_4 := \mathrm{span}_{\mathbb{C}}\{v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} : a_1 + a_2 + \cdots + a_n = d\} =: \mathrm{Sym}^d(\mathbb{C}^n)$, with $\sigma \cdot (v_1^{a_1} \cdots v_n^{a_n}) = v_{\sigma(1)}^{a_1} \cdots v_{\sigma(n)}^{a_n} \quad \forall \sigma \in S_n$
  $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$

Def: A character for an S_n-module V with a basis $\mathcal{B}_V$ is a map

$$\chi : S_n \longrightarrow \mathbb{C}$$
$$\sigma \longmapsto \sum_{b \in \mathcal{B}_V} \text{coeff. of } b \text{ in } \sigma \cdot b$$

(Like trace of a matrix)

Theorem: Let χ be a character for a S_n-module V.

(1) If $\sigma, \tau \in S_n$ and σ and τ are in the same conjugacy class (i.e. they have the same cycle type μ for some $\mu \in \mathrm{Par}$), then $\chi(\sigma) = \chi(\tau)$.
Hence χ is a class function and we can define $\chi(\mu) := \chi(\sigma)$ for any σ with cycle type μ.

(2) There are irreducible S_n-modules S^λ indexed by $\lambda \vdash n$ (with characters : χ^λ).

(3) There exist integers $m_\lambda(V) \in \mathbb{Z}_{\geq 0}$ for each $\lambda \vdash n$ s.t. $V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus m_\lambda(V)} \iff \chi = \sum_{\lambda \vdash n} m_\lambda(V) \chi^\lambda$.

(4) $s_\lambda = \sum_{\mu \vdash n} \chi^\lambda(\mu) \cdot \frac{p_\mu}{z_\mu}$ and $p_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda$ where $z_\lambda = \prod_{i \geq 1} \underbrace{m_i(\lambda)}_{\#i \text{ in } \lambda}! \cdot i^{m_i(\lambda)}$.

(5) Modules are isomorphic iff their characters are equal. (i.e. Characters "characterize" the modules)

e.g.
$V_1 = \mathrm{span}\{v_1, v_2, v_3\}$ with $\sigma \cdot v_i = \sum_{j=1}^{3} \phi_1(\sigma)_{ij} v_j$
$V_2 = \mathrm{span}\{v_1, v_2, v_3\}$ with $\sigma \cdot v_i = \sum_{j=1}^{3} \phi_2(\sigma)_{ij} v_j$
$V_3 = \mathrm{span}\{v_1, v_2, v_3\}$ with $\sigma \cdot v_i = \sum_{j=1}^{3} \phi_3(\sigma)_{ij} v_j$
where

σ (Cycle notation)	$\phi_1(\sigma)$	$\phi_2(\sigma)$	$\phi_3(\sigma)$
(1)(2)(3)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ tr = 3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ tr = 3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ tr = 3
(1)(23)	$\begin{bmatrix} 1 & -6 & 6 \\ -1 & 2 & -3 \\ -1 & 3 & -4 \end{bmatrix}$ tr = −1	$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 6 & -7 \\ -1 & 5 & -6 \end{bmatrix}$ tr = 1	$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 3 \\ -3 & -1 & -2 \end{bmatrix}$ tr = 1
(2)(13)	$\begin{bmatrix} 0 & -4 & 5 \\ 1 & -5 & 5 \\ 1 & -4 & 4 \end{bmatrix}$ tr = −1	$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$ tr = 1	$\begin{bmatrix} 4 & 2 & 3 \\ -3 & -1 & -3 \\ -3 & -2 & -2 \end{bmatrix}$ tr = 1
(3)(12)	$\begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{bmatrix}$ tr = −1	$\begin{bmatrix} -1 & 7 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ tr = 1	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ tr = 1
(123)	$\begin{bmatrix} 0 & -2 & -1 \\ -1 & 6 & 7 \\ -1 & 5 & -6 \end{bmatrix}$ tr = 0	$\begin{bmatrix} 0 & -2 & -1 \\ -1 & 6 & -7 \\ -1 & 5 & -6 \end{bmatrix}$ tr = 0	$\begin{bmatrix} 4 & 2 & 3 \\ -3 & -2 & -3 \\ -3 & -1 & -2 \end{bmatrix}$ tr = 0
(132)	$\begin{bmatrix} -1 & 7 & -8 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$ tr = 0	$\begin{bmatrix} -1 & 7 & -8 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$ tr = 0	$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 3 \\ -3 & -2 & -2 \end{bmatrix}$ tr = 0

e.g. In V_3, when $\sigma = (1)(23)$, then

$$\sigma \cdot v_2 = \phi_3(\sigma)_{21} v_1 + \phi_3(\sigma)_{22} v_2 + \phi_3(\sigma)_{23} v_3$$
$$= 3v_1 + 2v_2 + 3v_3$$

$$\chi^{V_3}(\sigma) = 1 + 2 + (-2) = 0$$

$\therefore$ $\chi_{V_1}(111) = 3$, $\chi_{V_1}(21) = -1$, $\chi_{V_1}(3) = 0$

$\chi_{V_2}(111) = 3$, $\chi_{V_2}(21) = 1$, $\chi_{V_2}(3) = 0$

$\chi_{V_3}(111) = 3$, $\chi_{V_3}(21) = 1$, $\chi_{V_3}(3) = 0$

$\because \chi_{V_1} \neq \chi_{V_2}$, $\chi_{V_1} \neq \chi_{V_3}$, $\chi_{V_2} = \chi_{V_3}$

$\therefore V_1 \not\cong V_2$, $V_1 \not\cong V_3$, $V_2 \cong V_3$.

Recall the character table of S_3:

	$(1)(2)(3)$	$(1)(23), (2)(13), (3)(12)$	$(123), (132)$
$\chi^{(3)}$	1	1	1
$\chi^{(21)}$	2	0	-1
$\chi^{(111)}$	1	-1	1

$\uparrow$ Irreducible characters

$$\Rightarrow \begin{cases} \chi_{V_1} = \chi^{(21)} + \chi^{(111)} \Leftrightarrow V_1 \cong S^{(21)} \oplus S^{(111)} \\ \chi_{V_2} = \chi^{(21)} + \chi^{(3)} \Leftrightarrow V_2 \cong S^{(21)} \oplus S^{(3)} \\ \chi_{V_3} = \chi^{(21)} + \chi^{(3)} \Leftrightarrow V_3 \cong S^{(21)} + S^{(3)} \end{cases}$$

Def: (Sagan, Macdonald): The character map (Frobenius map) are defined as

$$\mathcal{F}_n(\chi) := \sum_{\mu \vdash n} \chi(\mu) \frac{p_\mu}{z_\mu} \quad \text{for any } S_n\text{-class function } \chi.$$

e.g. For $\lambda \vdash n$, χ^λ is an S_n-class function and $\mathcal{F}_n(\chi^\lambda) = \sum_{\mu \vdash n} \chi^\lambda(\mu) \frac{p_\mu}{z_\mu} = s_\lambda$.

$$\therefore V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus m_\lambda(V)} \Leftrightarrow \chi^V = \sum_{\lambda \vdash n} m_\lambda(V) \chi^\lambda \Leftrightarrow \mathcal{F}_n(\chi^V) = \sum_{\lambda \vdash n} m_\lambda(V) s_\lambda$$

Hence $\mathcal{F}_n$: S_n-class functions $\longrightarrow \Lambda_n :=$ symmetric functions of degree n.

e.g. $Sym^2(\mathbb{C}^2) = span_{\mathbb{C}}\{v_1^a v_2^b : \underbrace{a+b}_{a,b \in \{0,1,2\}} = 2\} = span_{\mathbb{C}}\{v_1^2, v_1v_2, v_2^2\}$

GL_2-character $= s_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$

Check: When $\sigma = (1)(2)$, $\sigma \cdot v_1^2 = v_1^2 \Rightarrow$ coeff $= 1$; $\sigma \cdot v_1v_2 = v_1v_2 \Rightarrow$ coeff $= 1$; $\sigma \cdot v_2^2 = v_2^2 \Rightarrow$ coeff $= 1$ $\}$ $\therefore \chi^V(\sigma) = 1+1+1 = 3$

Corresponding to $\begin{array}{cccc} & v_1^2 & v_1v_2 & v_2^2 \end{array}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore$ Trace $= 3$

When $\sigma = (12)$, $\sigma \cdot v_1^2 = v_2^2 \Rightarrow$ coeff of $v_1^2 = 0$; $\sigma \cdot v_1v_2 = v_1v_2 \Rightarrow$ coeff of $v_1v_2 = 1$; $\sigma \cdot v_2^2 = v_1^2 \Rightarrow$ coeff of $v_2^2 = 0$ $\}$ $\therefore \chi^V(\sigma) = 0+1+0 = 1$

Corresponding to $\begin{array}{cccc} & v_1^2 & v_1v_2 & v_2^2 \end{array}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\therefore$ Trace $= 1$

To see it as an S_2-character: (S_2 viewed as "living" inside GL_2) permutation matrices

$(1)(2)$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues $1, 1 \Rightarrow \chi^V((1)(2)) = s_2(1,1) = 3$

(12) : $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $1, -1 \Rightarrow \chi^V((12)) = s_2(1,-1) = s_2(-1,1) = 1-1+1 = 1$.

$$\mathcal{F}(\chi^V) = 3 \cdot \frac{p_{11}}{z_{11}} + 1 \cdot \frac{p_2}{z_2} = 3 \cdot \frac{p_{11}}{2} + 1 \cdot \frac{p_2}{2} = \frac{3}{2} p_{11} + \frac{p_2}{2} = s_{11} + 2s_2$$

$$\therefore V \cong S^{(11)} \oplus S^{(2)} \oplus S^{(2)}$$

For any d-cycles:

$$d\left\{\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\right.$$

$\leftarrow$ exactly one 1

$\underbrace{\qquad\qquad}_{d}$ $d-1$ "1"

the eigenvalues are $1, \delta_d, \delta_d^2, \dots, \delta_d^{d-1}$ where $\delta_d = e^{\frac{2\pi i}{d}}$.

Define $p_k[\Xi_d] := p_k(1, \delta_d, \delta_d^2, \dots, \delta_d^{d-1}) = 1^k + \delta_d^k + \delta_d^{2k} + \cdots + \delta_d^{k(d-1)} = \begin{cases} d & \text{if } d \mid k \\ 0 & \text{otherwise} \end{cases}$

Then for any $\gamma = 1^{m_1} 2^{m_2} \cdots d^{m_d}$, m_i = multiplicity of i in γ,

$$p_k[\Xi_\gamma] := p_k[\Xi_{\gamma_1}] + \cdots + p_k[\Xi_{\gamma_{\underline{\ell(\gamma)}}}] = m_1 p_k[\Xi_1] + m_2 p_k[\Xi_2] + \cdots + m_d p_k[\Xi_d] = \sum_{r \mid k} m_r p_k[\Xi_r] = \sum_{r \mid k} r m_r.$$

$\ell(\gamma) = m_1 + m_2 + \cdots + m_d$

$\uparrow$ You can apply Mobius Inversion to this formula

Define $p_\mu[\Xi_\gamma] := p_{\mu_1}[\Xi_\gamma] \, p_{\mu_2}[\Xi_\gamma] \cdots p_{\mu_{\ell(\mu)}}[\Xi_\gamma]$.

Define the Frobenius map on symmetric functions :

$$\Phi_n : \Lambda \longrightarrow \Lambda_n \qquad \left(\Lambda = \bigoplus_{n\geq 0} \Lambda_n\right)$$

$$f \mapsto \sum_{\mu \vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu}$$

e.g. $V = Sym^2(\mathbb{C}^3) = span_{\mathbb{C}}\{v_1^a v_2^b v_3^c : a+b+c=3\} = span_{\mathbb{C}}\{v_1^3, v_2^3, v_3^3, v_1^2v_2, v_1^2v_3, v_1v_2^2, v_1v_3^2, v_2^2v_3, v_2v_3^2, v_1v_2v_3\}$ with $\sigma(v_1^a v_2^b v_3^c) = v_{\sigma(1)}^a v_{\sigma(2)}^b v_{\sigma(3)}^c$ $(\sigma \in S_3)$

V is an Gl_3- module and also an S_3-module.

Gl_3 - character : $s_2(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3 = \frac{1}{2}(p_2+p_{11})$

γ	$p_2[\Xi_\gamma]$	$p_{11}[\Xi_\gamma] = p_1[\Xi_\gamma]\cdot p_1[\Xi_\gamma]$	$s_2[\Xi_\gamma] = \frac{1}{2}(p_2[\Xi_\gamma] + p_{11}[\Xi_\gamma])$
111	$p_2[\Xi_1]+p_2[\Xi_1]+p_2[\Xi_1]$ $=3p_2[\Xi_1]$ $=3p_2(1)$ $=3\cdot 1^2 = 3.$	$(p_1[\Xi_{111}])^2$ $=(p_1[\Xi_1]+p_1[\Xi_1]+p_1[\Xi_1])^2$ $=(1+1+1)^2$ $=9$	$\frac{1}{2}(3+9)=6$ (Also the same as $s_2(1,1,1)=6$) Each Ξ_1 has eigenvalue 1
21	$p_2[\Xi_2]+p_2[\Xi_1]$ $=1^2+(-1)^2+1$ $=3$	$(p_1[\Xi_2]+p_1[\Xi_1])^2$ $=[1+(-1)+1]^2$ $=1$	$\frac{1}{2}(3+1)=2$ (Also the same as $s_2(1,-1,1)=2$) Ξ_2 has eigenvalue $1,-1$: $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ Ξ_1 has eigenvalue 1
3	$p_2[\Xi_3]$ $=1^2+\delta_3^2+\delta_3^4$ $=0$	$(p_1[\Xi_3])^2$ $=(1+\delta_3+\delta_3^2)^2$ $=0$	$\frac{1}{2}(0+0)=0$ (Also the same as $s_2(1,\delta_3,\delta_3^2) = 1+\delta_3^2+\delta_3^4+\delta_3+\delta_3^2+\delta_3^3=0$) Ξ_3 has eigenvalues $1,\delta_3,\delta_3^2$: $\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

$$\therefore \Phi_3(s_2) = s_2[\Xi_{111}]\frac{p_{111}}{z_{111}} + s_2[\Xi_{21}]\frac{p_{21}}{z_{21}} + s_2[\Xi_3]\frac{p_3}{z_3}$$

$$= 6\cdot\frac{1}{3!}p_{111} + 2\cdot\frac{1}{2}p_{21} + 0 = p_{111}+p_{21} = 2s_3+2s_{21}$$

$\chi^V(111) = 6$
$\chi^V(21) = 2$
$\chi^V(3) = 0$

Hence $Sym^2(\mathbb{C}^3) \cong \mathcal{S}^{(3)} \oplus \mathcal{S}^{(3)} \oplus \mathcal{S}^{(21)} \oplus \mathcal{S}^{(21)}$ ← Decomposition of an GL_n-module into S_n-modules

Recall the character table of S_3 :

	(1)(2)(3)	(1)(23), (2)(13), (3)(12)	(123), (132)
$\chi^{(3)}$	1	1	1
$\chi^{(21)}$	2	0	-1
$\chi^{(111)}$	1	-1	1

← $\therefore \chi^V = 2\chi^{(3)} + 2\chi^{(21)}$ which also shows that $V \cong \mathcal{S}^{(3)} \oplus \mathcal{S}^{(3)} \oplus \mathcal{S}^{(21)} \oplus \mathcal{S}^{(21)}$

Main Question: How to decompose an GL_n-character into an S_n- character ? (Hard)

Kronecker product on symmetric functions:

Recall the character table of S_4:

χ^λ	(1)(2)(3)(4)	(12)(3)(4), (23)(1)(4), (13)(2)(4), (24)(1)(3), (14)(2)(3), (34)(1)(2)	(12)(34), (13)(24), (14)(23)	(1)(234), (2)(143), (4)(123), (1)(243), (3)(124), (4)(132), (2)(134), (3)(142),	(1234), (1324), (1243), (1423), (1342), (1432)
$\chi^{(4)}$	1	1	1	1	1
$\chi^{(31)}$	3	1	0	0	-1
$\chi^{(22)}$	2	0	2	-1	0
$\chi^{(211)}$	3	-1	0	0	1
$\chi^{(1111)}$	1	-1	1	1	-1

$\therefore\ \chi^{(22)} \otimes \chi^{(22)} = $ " 4 0 4 1 0 " $= \chi^{(4)} \oplus \chi^{(22)} \oplus \chi^{(1111)}$

$\therefore\ S^{(22)} \otimes S^{(22)} = S^{(4)} \oplus S^{(22)} \oplus S^{(1111)}$

More generally, if χ^V and χ^W are characters of V and W, then $\chi^{V\otimes W} = \chi^V \cdot \chi^W$ (i.e. $\chi^{V\otimes W}(\sigma) = \chi^V(\sigma)\cdot\chi^W(\sigma)$.)

Def: The Kronecker product on symmetric functions is defined as:

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \mathbb{1}_{\{\lambda=\mu\}} \frac{p_\lambda}{z_\lambda}$$

(b/c we then get $\mathcal{F}_n(\chi^V) * \mathcal{F}_n(\chi^W) = \mathcal{F}_n(\chi^V \cdot \chi^W) = \mathcal{F}_n(\chi^{V\otimes W})$)

$\therefore\ \Phi_n(f) * \Phi_n(g) = \Phi_n(f\cdot g)$: Check: $\underbrace{\left(\sum_{\mu\vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu}\right)}_{\Phi_n(f)} * \underbrace{\left(\sum_{\gamma\vdash n} g[\Xi_\gamma] \frac{p_\gamma}{z_\gamma}\right)}_{\Phi_n(g)} = \sum_{\mu,\gamma\vdash n} f(\Xi_\mu) g(\Xi_\gamma) \left(\frac{p_\mu}{z_\mu} * \frac{p_\gamma}{z_\gamma}\right) = \underbrace{\sum_{\mu\vdash n} f(\Xi_\mu) g(\Xi_\mu) \frac{p_\mu}{z_\mu}}_{\Phi_n(f\cdot g)}$

For big n, $s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)} = \sum_\gamma \underbrace{\bar{g}^\gamma_{\lambda\mu}}_{} s_{(n-|\gamma|,\gamma)}$

These are called "reduced Kronecker coefficients"

Define, for big n, $\tilde{s}_\lambda := \Phi_n^{-1}(s_{(n-|\lambda|,\lambda)})$ ← Why does this exist?

next lecture

$$\tilde{s}_\lambda \cdot \tilde{s}_\mu = \sum_\gamma \bar{g}^\gamma_{\lambda\mu} \tilde{s}_\gamma \Leftrightarrow \Phi_n(\tilde{s}_\lambda\cdot\tilde{s}_\mu) = \sum_\gamma \bar{g}^\gamma_{\lambda\mu} \Phi_n(\tilde{s}_\gamma)$$

(Sanity check: $\Phi_n(\tilde{s}_\lambda\cdot\tilde{s}_\mu) = \Phi_n(\tilde{s}_\lambda) * \Phi_n(\tilde{s}_\mu)$

$= s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)}$

$= \sum_\gamma \bar{g}^\gamma_{\lambda\mu} s_{(n-|\gamma|,\gamma)}$

$= \sum_\gamma \bar{g}^\gamma_{\lambda\mu} \Phi_n(\tilde{s}_\gamma) = \Phi_n\left(\sum_\gamma \bar{g}^\gamma_{\lambda\mu} \tilde{s}_\gamma\right)$.)

e.g. Recall the character table of S_2:

χ^λ	(1)(2)	(12)
$\chi^{(2)}$	1	1
$\chi^{(11)}$	1	-1

empty partition

$\Phi_2(1) = 1\cdot\frac{p_2}{z_2} + 1\cdot\frac{p_{11}}{z_{11}} = \frac{1}{2}(p_2+p_{11}) = s_2 = s_{2-0,()}$ $\therefore\ \Phi_2^{-1}(s_2) = 1$ i.e. $\tilde{s}_{()} = \Phi^{-1}(s_{2-0,()}) = \Phi^{-1}(s_2) = 1$.

empty partition

$\Phi_2(s_1) = \underset{p_1}{\overset{\shortparallel}{p_1}}\underbrace{[\Xi_2]}_{1+(-1)} \cdot \frac{p_2}{z_2} + p_1\underbrace{[\Xi_{11}]}_{1+1} \cdot \frac{p_{11}}{z_{11}} = 0\cdot\frac{p_2}{2} + 2\cdot\frac{p_{11}}{z_{11}} = p_{11} = s_2 + s_{11}$ $\therefore\ \Phi^{-1}(s_2+s_{11}) = s_1 \Rightarrow 1 + \Phi^{-1}(s_{11}) = s_1$

$\Rightarrow \tilde{s}_1 = \Phi^{-1}(s_{2-1,1}) = \Phi^{-1}(s_{11}) = s_1 - 1$.

(Example added in Lecture 3):

$n=3$: Now we know $\tilde{s}_{()} = 1$ and $\tilde{s}_1 = s_1 - 1 = x_1+x_2+x_3-1$ What is $\underset{\underset{\phi^{-1}(s_{11})}{\parallel}}{\tilde{s}_{11}}$?

$\delta = e^{\frac{2\pi i}{3}}$

	Eigenvalues σ	1 1 1 $\mu=111$ $(1)(2)(3)$	1 −1 1 $(1)(23),(2)(13),(3)(12)$ $\mu=21$	$1,\delta,\delta^2$ $(123),(132)$ $\mu=3$
$\tilde{s}_{()}[\square_\mu]=$	$\chi^{(3)}$	1	1	1
$\tilde{s}_1[\square_\mu]=$	$\chi^{(21)}$	$(1+1+1-1=2)$ 2	$(1+(-1)+1-1=0)$ 0	$(1+\delta+\delta^2-1=-1)$ −1
$\tilde{s}_{11}[\square_\mu]=$	$\chi^{(111)}$	1	−1	0

$\therefore s_{11}(x_1,x_2,x_3) = x_1x_2+x_1x_3+x_2x_3$ ← The reason we look at this is because we want to find $\tilde{s}_{11}$ and it is known that $\underline{\underline{\tilde{s}_\lambda}} = \underline{\underline{s_\lambda}} + \underbrace{\sum_{d\geq 1}(-1)^d (\text{terms of deg } |\lambda|-d)}_{\text{"lower terms"}}$

i.e. $\tilde{s}_{11} = s_{11} + \cdots$

$\therefore$ we look at what this is first

$\therefore s_{11}[\square_\mu]$ gives: " 3 −1 0 " $= (\tilde{s}_1 + \tilde{s}_{11})[\square_\mu]$

$1^2+1^2+1^2 \qquad -1+(-1)+1 \qquad \delta+\delta^2+\delta^3 = \delta(1+\delta+\delta^2)=0$

$\therefore \tilde{s}_{11} = s_{11} - \tilde{s}_1 = s_{11} - (s_1 - 1) = s_{11} - s_1 + 1.$