All Kronecker coefficients are reduced Kronecker coefficients

Greta Panova
based on joint work with Christian Ikenmeyer: arXiv:2305.03003

University of Southern California

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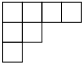
Partitions, SYTs and the Symmetric group

Integer partitions and Young diagrams:

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

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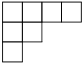
$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

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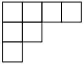
$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

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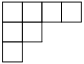
$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

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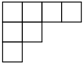
$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots 0$, $\lambda_1 + \lambda_2 + \cdots = n$.  for $\lambda = (4, 2, 1) \vdash 7$.

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Schur functions: characters of V_λ

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}$$

Structure constants

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^{\nu}_{\lambda\mu}}$$

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Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

			1	1	1
	1	2	2		
2	3	3			

			1	1	1
	2	2	2		
1	3	3			

(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

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$$s_\nu(x.y) = \sum_{\mu,\nu} g(\lambda,\mu,\nu) s_\mu(x) s_\lambda(y)$$

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[Murnaghan, 1938]: $c^\lambda_{\mu\nu} = g\left((N-|\lambda|,\lambda),(N-|\mu|,\mu),(N-|\nu|,\nu)\right)$ for $|\lambda| = |\mu| + |\nu|$ and N-large.

A major problem in Algebraic Combinatorics

Problem (Murnaghan 1938, Lascoux, Garsia-Remmel, Stanley 2000)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

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Combinatorial formulas for $g(\lambda, \mu, \nu)$:

Two two-row partitions [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013] ;
One two-row and other restrctions [*Ballantine-Orellana*, 2006]
One hook $\nu = (n-k, 1^k)$ [*Blasiak 2012, Blasiak-Liu 2014, Liu 2015*]
Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Mishna-Rosas-Sundaram, Vallejo].

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Positivity questions:

Tensor square conjecture Haide-Saxl-Tiep-Zalesski (2012): For every $n \geq 9$ there is a $\lambda \vdash n$, such that $g(\lambda, \lambda, \mu) > 0$ for all $\mu \vdash n$.

Saxl conjecture (2012): $g((k, k-1, \ldots, 1), (k, k-1, \ldots, 1), \mu) > 0$ for all $\mu \vdash \binom{k+1}{2}$.

Various partial results: Bessenrodt-Behns (2004), Pak-Panova-Vallejo (2013), Ikenmeyer (2015), Bessenrodt (2017), Luo-Sellke (2016), Li (2020), Harman-Ryba (2021), Zhao (2023+).

Geometric Complexity Theory: $g(n^d, n^d, \mu) > 0$ for certain μ's. [Ikenmeyer-P'2017]

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Computational complexity:

Combinatorial interpretation of $g(\lambda, \mu, \nu) \Longleftrightarrow$ ComputeKron $\in \#$P?

Geometric complexity theory: find inequalities between certain multiplicities (close to $g(\lambda, n^d, n^d)$) and plethysms to show VP $\neq$ VNP. (...long list of works...)

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Asymptotics:

$g(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}) \sim ?$ as n grows for various regimes? (same for $f^{\lambda/\mu}$, $c^{\lambda}_{\mu\nu}$ etc)

The reduced Kronecker coefficients

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

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Relationships between $\overline{g}$ and g (from [BDO]):

$$\overline{g}(\lambda,\mu,\nu) = \sum_{\substack{l_1,l_2 \\ l=l_1+2l_2}} \sum_{\substack{\alpha\vdash r-l_1-l_2 \\ \beta\vdash s-l_1-l_2}} \sum_{\substack{\pi,\rho,\sigma\vdash l_1 \\ \gamma\vdash l_2}} c^{\nu}_{\alpha,\beta,\pi} c^{\lambda}_{\alpha,\rho,\gamma} c^{\mu}_{\gamma,\sigma,\beta} g(\pi,\rho,\sigma)$$

($m = r+s$ and let $\nu \vdash m-l$, $\lambda \vdash r$, $\mu \vdash s$)

Saturation

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Conjecture [Kirillov, Klyachko]: The reduced Kronecker coefficients satisfy the saturation property:

$$\overline{g}(N\alpha, N\beta, N\gamma) > 0 \quad \text{for some } \ N \geq 1 \quad \Longrightarrow \quad \overline{g}(\alpha, \beta, \gamma) > 0\,.$$

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Theorem[Pak-Panova 2020] For all $k \geq 3$, the triple of partitions $(1^{k^2-1}, 1^{k^2-1}, k^{k-1})$ is a counterexample to the Conjecture. For every partition γ s.t. $\gamma_2 \geq 3$, there are infinitely many pairs $(a, b) \in \mathbb{N}^2$ s.t. (a^b, a^b, γ) is a counterexample.

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Proof sketch:
[Dvir]: If $g(\lambda,\mu,\nu) > 0$ then $d(\lambda) \leq 2\, d(\mu)\, d(\nu)$. ($d$– Durfee square size)

Let a, b be s.t. $b \geq \max\{3\ell(\gamma)^{3/2}, |\gamma|/(3\sqrt{2d(\gamma[n])}-6)\}$ and $|\gamma|/(6b) \leq a \leq \sqrt{d(\gamma[n])/2}$. By [Ikenmeyer-P'16] for $N \geq 3\ell^2/a$:

$$\overline{g}(N\alpha, N\alpha, \gamma) \geq g((Na)^{b+1}, (Na)^{b+1}, N\gamma[n]) > 0.$$

Maximal multiplicities

Theorem [Stanley]

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} \; g(\lambda, \mu, \nu) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})},$$

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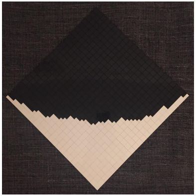
Theorem (Pak-Panova-Yeliussizov'18)

Let $\{\lambda^{(n)}\vdash n\}$, $\{\mu^{(n)}\vdash n\}$, $\{\nu^{(n)}\vdash n\}$ be three partition sequences, such that

$$(*)\qquad\qquad g\big(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\big) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})}\,.$$

Then $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$ are Vershik-Kerov-Logan-Shepp shape. Conversely, for every two VKLS-shape sequences $\{\lambda^{(n)}\vdash n\}$ and $\{\mu^{(n)}\vdash n\}$, there exists a VKLS sequence $\{\nu^{(n)}\vdash n\}$, s.t. (*) holds.

VKLS shape:



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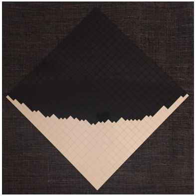

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Theorem (Pak-P'20)

$$\max_{a+b+c\le 3n}\ \max_{\alpha\vdash a}\ \max_{\beta\vdash b}\ \max_{\gamma\vdash c}\ \overline{g}(\alpha,\beta,\gamma) \;=\; \sqrt{n!}\; e^{O(n)}$$

Computational Complexity

KronPos:
`Input:` λ, μ, ν
`Output:` Is $g(\lambda, \mu, \nu) > 0$?

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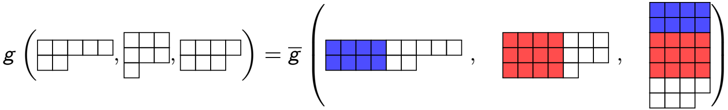
[Pak-P'20] ComputeReducedKron is #P-hard.

Walks like a duck, quacks like a duck....

Theorem (Ikenmeyer-Panova, 2023)

For every $\lambda, \mu, \nu \vdash n$ we have

$$g(\lambda,\mu,\nu) = \overline{g}(\nu_1^{\ell(\lambda)} + \lambda, \nu_1^{\ell(\mu)} + \mu, \nu_1^{\ell(\lambda)+\ell(\mu)} \cup \nu)$$



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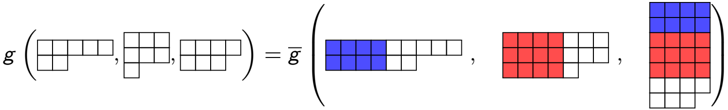

Corollaries: Positivity is NP-hard, Computing is #P-hard.
Saturation fail, asymptotic bounds etc.

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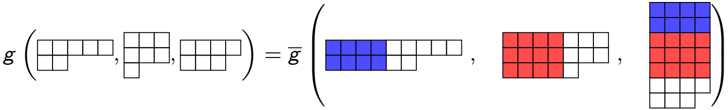

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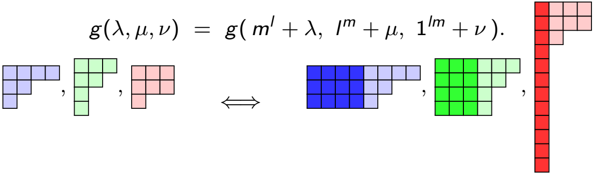
Constructive identities

Lemma

Let λ, μ, ν be partitions with $\ell(\lambda) \le l$, $\ell(\mu) \le m$. Then

$$g(\lambda,\mu,\nu) \ = \ g(\, m^l + \lambda,\ l^m + \mu,\ 1^{lm} + \nu\,).$$



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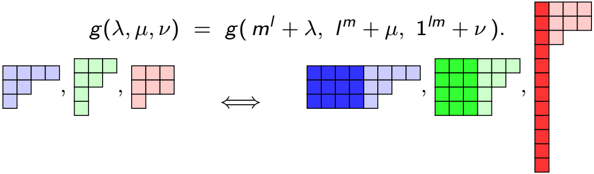

Let $\hat{\nu} = 1^{lm} + \nu$. Variables $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_m$:

$$s_{\hat{\nu}}[x \cdot y] = \sum_{\theta, \tau} g(\hat{\nu}, \theta, \tau) s_\theta(x) s_\tau(y).$$

$$s_{\hat{\nu}}[x \cdot y] = s_\nu[x \cdot y] \prod_{i,j} x_i y_j = (x_1 \ldots x_l)^m (y_1, \ldots, y_m)^l \sum_{\rho, \eta} g(\nu, \rho, \eta) s_\rho(x) s_\eta(y)$$

$$s_{l^m + \mu}(y_1, \ldots, y_m) = (y_1 \ldots y_m)^l s_\mu(y), \ s_{m^l + \lambda}(x_1, \ldots, x_l) = (x_1 \ldots x_l)^m s_\lambda(x).$$

Constructive identities

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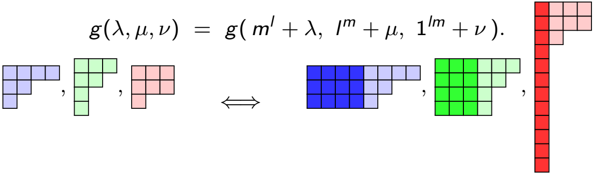
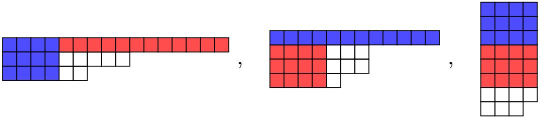


Lemma

Let λ, μ, ν be partitions of the same size, and let $l \geq \ell(\lambda)$, $m \geq \ell(\mu)$ and $c \geq \nu_1$. Let $d = (m+1)c$, $e = (l+1)c$. Then

$$g(\lambda,\mu,\nu) \;=\; g\big(\,(d) \cup (c^l + \lambda),\; (e) \cup (c^m + \mu),\; c^{l+m+1} \cup \nu\,\big).$$



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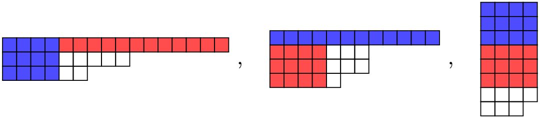


Theorem (Ikenmeyer-P)

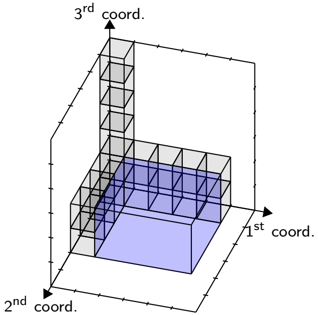
Let λ, μ, ν be partitions of the same size, such that $\lambda_1 \ge \ell(\mu)\cdot\nu_1$ and $\mu_1 \ge \ell(\lambda)\cdot\nu_1$. Then for every $h \ge 0$ we have

$$g(\lambda,\mu,\nu) = g(\,\lambda + h,\; \mu + h,\; \nu + h\,).$$

3d binary contingency arrays

$Q \subseteq \mathbb{N}^3$, $Q : \mathbb{N}^3 \to \{0,1\}$
2d marginals: $Q_{i**} := \sum_{j,k} Q_{i,j,k} = |Q \cap (\{i\} \times \mathbb{N} \times \mathbb{N})|$,
$Q_{*i*} := \sum_{j,k} Q_{j,i,k} = |Q \cap (\mathbb{N} \times \{i\} \times \mathbb{N})|$, $Q_{**i} := \sum_{j,k} Q_{j,k,i} = |Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\})|$.

$$\mathcal{C}(\alpha,\beta,\gamma) := \{Q \subseteq \mathbb{N}^3 \mid Q_{i**} = \alpha_i,\ Q_{*i*} = \beta_i,\ Q_{**i} = \gamma_i \text{ for every } i\}.$$



Lemma:
α,β,γ – compositions with $|\alpha| = |\beta| = |\gamma|$. $a \geq \ell(\alpha)$, $b \geq \ell(\beta)$, $c + h \geq \ell(\gamma)$ and $\sum_{i>c} \gamma_i \leq h$, $\alpha_1 \geq bc + h$, $\beta_1 \geq ac + h$.
Then, for every $Q \in \mathcal{C}(\alpha,\beta,\gamma)$ we have

$$\{1\} \times [b] \times [c] \subseteq Q, [a] \times \{1\} \times [c] \subseteq Q,$$

$$\{1\} \times \{1\} \times [c+h] \subseteq Q,$$

$$Q \cap (\mathbb{N} \times \mathbb{N} \times [c+1, c+h]) = \{1\} \times \{1\} \times [c+1, c+h].$$

In particular, if $\mathcal{C}(\alpha,\beta,\gamma)$ is non-empty, then $a = \ell(\alpha)$, $b = \ell(\beta)$, $\gamma_i = 1$ for all $c+1 \leq i \leq c+h$, and $\alpha_1 = bc + h$, $\beta_1 = ac + h$, $\alpha_2 \leq bc$, and $\beta_2 \leq ac$.

Kronecker coefficients via 3d binary contingency arrays

$$\sum_{\alpha,\beta,\gamma} g(\alpha,\beta,\gamma) s_\alpha(x) s_\beta(y) s_{\gamma'}(z) = \prod_{i,j,k}(1 + x_i y_j z_k),$$

$$g(\alpha,\beta,\gamma) = \sum_{\sigma\in S_{\ell(\alpha)},\pi\in S_{\ell(\beta)},\rho\in S_{|\gamma_1|}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\pi)\,\mathrm{sgn}(\rho) C(\alpha+\sigma-\mathrm{id}, \beta+\pi-\mathrm{id}, \gamma'+\rho-\mathrm{id}).$$

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\end{aligned}$$

Lemma $\Longrightarrow$ $Q_{1**} = bc + h$, $Q_{*1*} = ac + h$, $\sigma_1 = \pi_1 = 1$,
$\{\rho_{c+1},\dots,\rho_{c+h}\} = \{c+1,\dots,c+h\}$, $k \in [c+1, c+h]$: $Q_{i,j,k} = 1$ iff $(i,j) = (1,1)$,
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$$g(\underbrace{\lambda+h}_{\alpha},\underbrace{\mu+h}_{\beta},\underbrace{\nu+h}_{\gamma'}) = \sum_{\sigma\in S_a,\pi\in S_b,\rho\in S_{c+h}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\rho)\, C(\alpha+\sigma-id,\beta+\pi-id,\gamma+\rho-id)$$

$$= \sum_{\sigma\in S_a,\pi\in S_b,\eta\in S_c} \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\operatorname{sgn}(\eta)\, C(\lambda+\sigma-id,\mu+\pi-id,\nu'+\eta-id) = g(\lambda,\mu,\nu),$$

Using Littlewood-Richardson coefficients

Set $\hat{\mu} = \mu + h$, $\hat{\lambda} = \lambda' \cup (1^h) = (\lambda + h)'$ and $\hat{\nu} = \nu' \cup (1^h)(\nu + h)'$

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Multi-LR coefficients $= \#$ certain SSYTs of type $(\alpha^1 \cup \alpha^2 \cup \ldots \cup \alpha^c \cup \ldots)$, shape $\hat{\lambda}$:

1	1	1	1	4	4	6
2	2	2	4	5	7	
3	5	5	6	6		

and

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multi-LR tableaux of shape $\lambda = (7, 6, 5)$ and types $\alpha^1 = (4, 3, 1)$, $\alpha^2 = (3, 3)$, $\alpha^3 = (3, 1)$.

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$\Longrightarrow \alpha^i \subset \hat{\mu}$, $\alpha^i \subset \hat{\lambda}$, so $\ell(\alpha^i) \leq \ell(\mu) = b$, $\alpha^i_1 \leq \hat{\lambda}_1 = a$.

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so α^i are single column partitions, possibly empty.

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$$ac + h = \hat{\mu}_1 \leq \sum_i \alpha^i_1 \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha^i_1.$$

Thus $\alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. $\implies \alpha^i = (1)$ for all $i > c$ and $\sigma_i = i$ for $i = c + 1, \ldots, c + h$.

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$\alpha^i \leq a$, $\alpha^i \subset \hat{\mu}$. Multi-LR of type $(\alpha^1 \cup \alpha^2 \cdots)$ shape $\hat{\mu}$, so

$$ac + h = \hat{\mu}_1 \leq \sum_i \alpha^i_1 \leq \sum_{i=1}^{c} a + \sum_{i=c+1}^{c+h} \alpha^i_1.$$

Thus $\alpha_1^{c+1} + \cdots + \alpha_1^{c+h} \geq h$. $\implies \alpha^i = (1)$ for all $i > c$ and $\sigma_i = i$ for $i = c + 1, \ldots, c + h$.

$$c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots \alpha^{c+h}} = c^{\lambda'}_{\alpha^1 \ldots \alpha^c} \quad \text{and} \quad c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots \alpha^{c+h}} = c^{\mu}_{\alpha^1 \ldots \alpha^c}.$$

$$g(\lambda + h, \mu + h, \nu + h) = \sum_{\sigma \in S_{c+h}} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \hat{\nu}_i - i + \sigma_i} c^{\hat{\lambda}}_{\alpha^1 \alpha^2 \ldots} c^{\hat{\mu}}_{\alpha^1 \alpha^2 \ldots}$$

$$= \sum_{\sigma \in S_c} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \nu'_i - i + \sigma_i} c^{\lambda'}_{\alpha^1 \alpha^2 \ldots} c^{\mu}_{\alpha^1 \alpha^2 \ldots} = g(\nu', \lambda', \mu) = g(\lambda, \mu, \nu),$$

Via the General Linear group

$GL_a \times GL_b \times GL_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha,\beta,\gamma') = \dim \left(HWV_{\alpha,\beta,\gamma} \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) \right),$$

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Raising operators $(E_{i-1,i},0,0)$, $(0,E_{i-1,i},0)$ and $(0,0,E_{i-1,i})$, e.g. $(E_{i,j},0,0)e_{r,1,1} = e_{i,1,1}$ iff $r=j$ and 0 otherwise.

A highest weight vector (HWV) of weight (α,β,γ) is a nonzero weight vector of weight (α,β,γ) that is mapped to zero by all raising operators.

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$e_{i,j,k} := e_i \otimes e_j \otimes e_k$

$$t := e_{1,1,1} \wedge e_{2,1,1} \wedge e_{1,2,2} + e_{1,1,1} \wedge e_{1,2,1} \wedge e_{2,1,2} + e_{1,1,1} \wedge e_{1,1,2} \wedge e_{2,2,1}$$

is a HWV of weight $((2, 1), (2, 1), (2, 1))$ in $\bigwedge^3(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$:

$$(E_{1,2}, 0, 0)t = e_{1,1,1} \wedge e_{1,2,1} \wedge e_{1,1,2} + e_{1,1,1} \wedge e_{1,1,2} \wedge e_{1,2,1} = 0,$$
$$(0, E_{1,2}, 0)t = e_{1,1,1} \wedge e_{2,1,1} \wedge e_{1,1,2} + e_{1,1,1} \wedge e_{1,1,2} \wedge e_{2,1,1} = 0,$$
$$(0, 0, E_{1,2})t = e_{1,1,1} \wedge e_{2,1,1} \wedge e_{1,2,1} + e_{1,1,1} \wedge e_{1,2,1} \wedge e_{2,1,1} = 0.$$

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Raising operators $(E_{i-1,i}, 0, 0)$, $(0, E_{i-1,i}, 0)$ and $(0, 0, E_{i-1,i})$, e.g. $(E_{i,j}, 0, 0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise.

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$$\begin{aligned} \varphi: \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) &\to \bigwedge^{D+h} (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^{c+h}) \\ v &\mapsto v \wedge e_{1,1,c+1} \wedge e_{1,1,c+2} \wedge \cdots \wedge e_{1,1,c+h} \end{aligned}$$

Claim: φ is an isomorphism $HWV_{\lambda,\mu,\gamma} \leftrightarrow HWV_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma}}$.

Via the General Linear group

$GL_a \times GL_b \times GL_c$'s irreducible representations are $V_\alpha \otimes V_\beta \otimes V_\gamma$

$$g(\alpha, \beta, \gamma') = \dim \left(\mathrm{HWV}_{\alpha,\beta,\gamma} \bigwedge^D (\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c) \right),$$

Raising operators $(E_{i-1,i}, 0, 0)$, $(0, E_{i-1,i}, 0)$ and $(0, 0, E_{i-1,i})$, e.g. $(E_{i,j}, 0, 0)e_{r,1,1} = e_{i,1,1}$ iff $r = j$ and 0 otherwise.

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Claim: φ is an isomorphism $\mathrm{HWV}_{\lambda,\mu,\gamma} \leftrightarrow \mathrm{HWV}_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma}}$.

$\to$: check that $R\varphi(v) = 0$ for every raising operator R.

$\leftarrow$: If $u = \sum_{Q^i} a_Q \psi_{Q^i} \in \mathrm{HWV}_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\gamma}}$, where $\psi_Q = e_{q_1} \wedge e_{q_2} \wedge \cdots$ for $Q = \{q_1, q_2, \ldots\} \subset \mathbb{N}^3$, then each Q^i has marginals $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\gamma})$.

3d binary CTs Lemma: $\{1\} \times \{1\} \times [c+1, c+h] \subset Q$ and $Q \cap (\mathbb{N} \times \mathbb{N} \times \{i\}) = \{(1, 1, i)\}$ for all $c+1 \le i \le c+h$, so $\psi_Q = \psi_P \wedge e_{1,1,c+1} \wedge \cdots \wedge e_{1,1,c+h}$

Thank you!