Geometric Interpretation for the Delta Thm (IV)

Last Time

$$BM_{\vec\lambda, N} := \xi_{\vec\lambda}^{-1}(N) = \{ V_\bullet \in Fl(\alpha) \mid NV_i \subseteq V_i ,\ JT(N \curvearrowleft V_i/V_{i-1}) \leq \lambda^i \}$$

$$\pi_\alpha^{-1}((V_\bullet, N)) \cong \prod_i Sp_{N \curvearrowleft V_i/V_{i-1}}$$

$$\begin{tikzcd}
\widetilde{\mathcal{N}} \arrow[d, "\pi_\alpha"] \arrow[dd, bend right, "p"'] & \\
\widetilde{\mathcal{N}}^\alpha \arrow[d, "\xi"] & \overline{\mathcal{O}}_{\vec\lambda} \arrow[l, hook'] \arrow[dl, "\xi_{\vec\lambda}"] \\
\mathcal{N} &
\end{tikzcd}$$

Thm (Borho-MacPherson 83) Let $d = \sum \underline{n}(\lambda^i)$.
If $\overline{\mathcal{O}}_\lambda$ is rationally smooth at every point of $BM_{\vec\lambda, N}$, then

$$H^i(BM_{\vec\lambda, N}) \otimes H^{2d}(\pi_\alpha^{-1}(V_\bullet, N)) \cong H^{i+2d}(Sp_N)^{V^{\lambda^1} \otimes \cdots \otimes V^{\lambda^\ell}}$$

for any $(V_\bullet, N) \in \mathcal{O}_{\vec\lambda}$.

Remark When $\vec\lambda = (\underbrace{(1), \ldots, (1)}_{n}, \mu)$, then $W_\alpha \cong S_n$ acts on both sides.

Proof of $\underset{\text{Frob}(H^*(\Delta Sp_{n,k});q)}{\underset{\wr\wr}{\Delta'_{e_{k-1}} e_n\big|_{t=0}}} = s^{\perp}_{(k-1)^{n-k}} \underset{\text{Frob}(H^*(Sp_{(n-k+1)^{n-k}});q)}{\underset{\wr\wr}{\underbrace{H_{(n-k+1)^k}(x;q)}}}$

① $\Delta Sp_{n,k} = BM_{(\underbrace{(1),(1),\dots,(1)}_{n},(n-k)^{k-1}),N}$, $JT(N) = (n-k+1)^k$

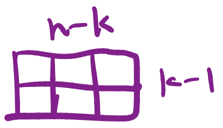
② $\overline{O_{\vec{\lambda}}}$ is rationally smooth on $BM_{\vec{\lambda},N}$

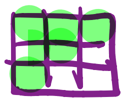
in the case above: $\vec{\lambda} = ((1),\dots,(1),(n-k)^{k-1})$,

$JT(N) = (n-k+1)^k$.

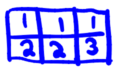
$\rightsquigarrow$ Combinatorial fact about tableaux.

Ex $n=6$, $k=3$

Shape 

$\underset{\mu}{\text{content}} \leq$ 

$\rightsquigarrow$ There is a unique tableau:



③ Translation to skewing formula:

By B-M's thm + rational smoothness:

$$H^i(BM_{((1),\dots,(1),(n-k)^{k-1})}) \otimes H^{2d}(Sp_{(n-k)^{k-1}}) \cong H^{i+2d}(Sp_N)^{V^{(n-k)^{k-1}}}$$

$$H^i(\Delta Sp_{n,k}) \otimes V^{(n-k)^{k-1}} \cong H^{i+2d}(Sp_N)^{V^{(n-k)^{k-1}}}$$

$W_\alpha \cong S_n$ acts on both sides

Exercise If $S_m \circlearrowleft W$, and $|\mu| \le m$,

$$\mathrm{Frob}\big(\underset{S_{m-|\mu|}\circlearrowleft}{\overset{\deg m-|\mu|}{W^{V^\mu}}}\big) = \dim(V^\mu)\cdot \underset{\text{Reduces deg by } |\mu|}{s_\mu^\perp}\, \underset{S_m\circlearrowleft}{\overset{\deg m}{\mathrm{Frob}(W)}}.$$

Combining,

$$\mathrm{Frob}(H^*(\Delta Sp_{n,k});q) = q^{-2d}\, s^\perp_{(n-k)^{k-1}}\, \mathrm{Frob}(H^*(Sp_\nu);q)$$

worevq $\Delta'_{e_{k-1}} e_n\big|_{t=0} = q^{-2d}\, s^\perp_{(n-k)^{k-1}}\, \tilde{H}_{(n-k+1)^k}(x;q)$

$$\boxed{\Delta'_{e_{k-1}} e_n\big|_{t=0} = s^\perp_{(k-1)^{n-k}}\, H_{(n-k+1)^k}(x;q)}$$

$\Rightarrow$ New Schur expansion for LHS

"Battery-powered" tableaux

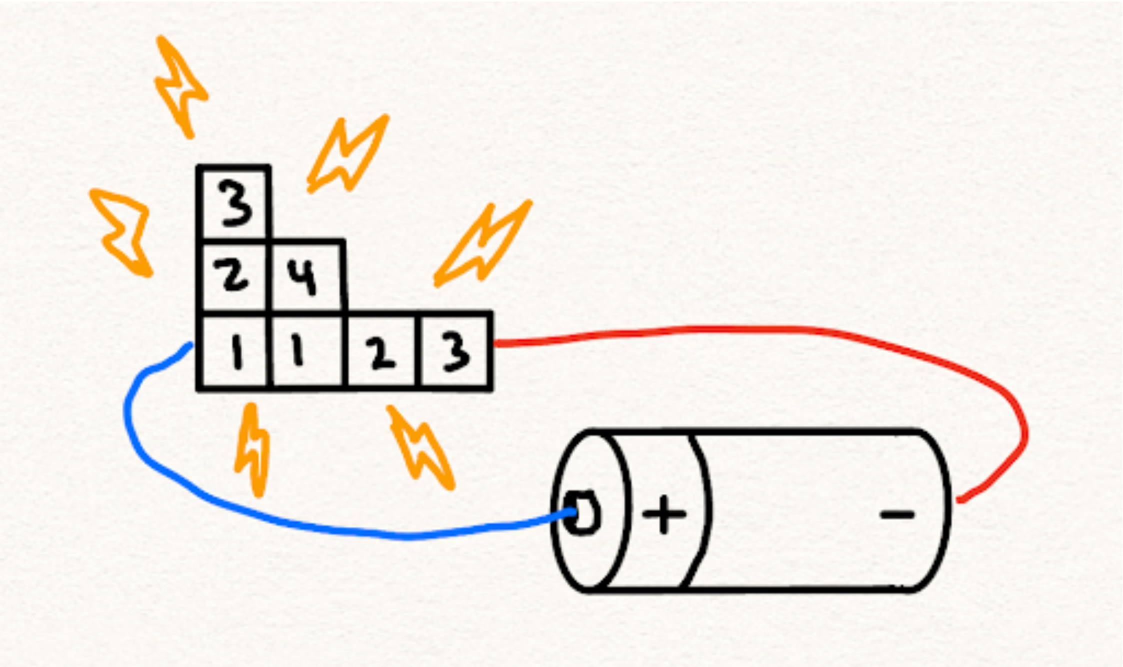


Image created by Maria Gillespie

Affine Springer fibers

Let $\mathcal{O} = \mathbb{C}[[\varepsilon]]$, $K = \mathbb{C}((\varepsilon))$

formal power series $a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots$

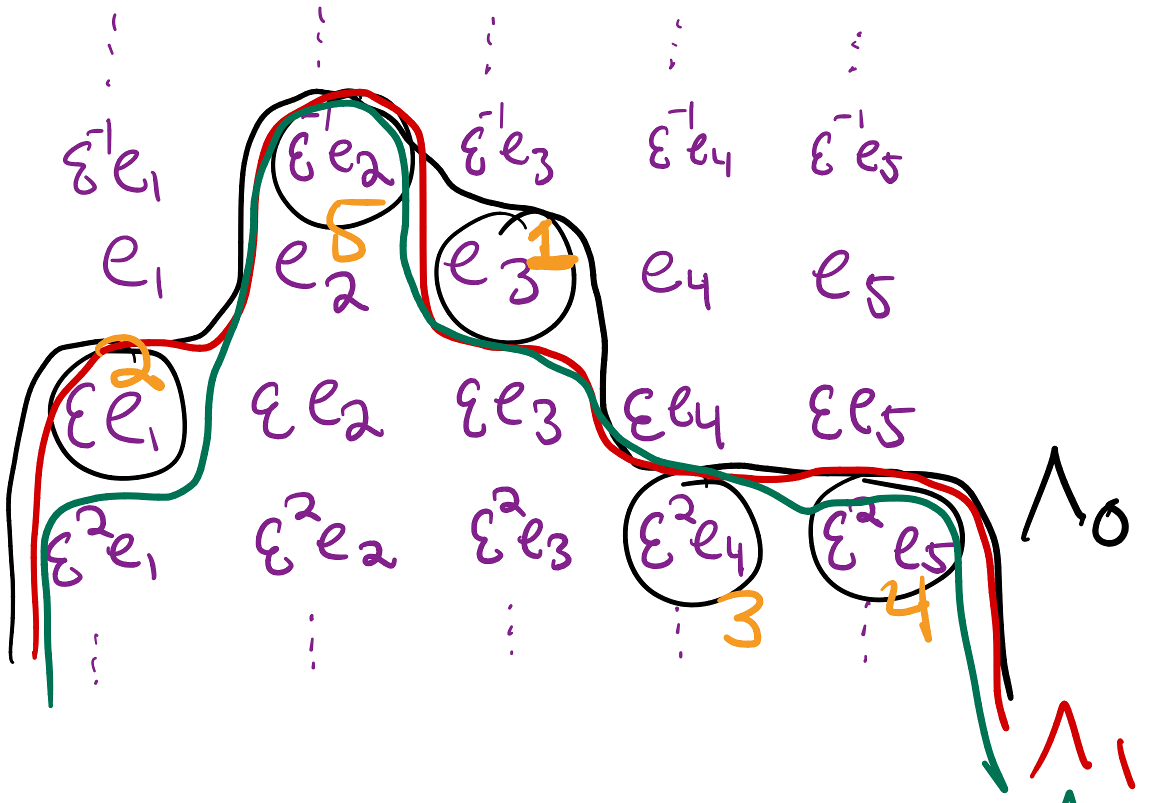
Laurent series $a_{-m}\varepsilon^{-m} + a_{-m+1}\varepsilon^{-m+1} + \dots$ $m \in \mathbb{Z}$

A lattice Λ is an $\mathcal{O}$-submodule of $K^{\oplus n}$ of rank n.

$\hookrightarrow \varepsilon\Lambda \subseteq \Lambda$

Ex $n=5$

$\varepsilon^i e_j$ "span" the j^{th} factor of $K^{\oplus n}$



$$\Lambda = \mathrm{Span}_{\mathcal{O}}\{\varepsilon e_1, \varepsilon^{-1}e_2, e_3, \varepsilon^2 e_4, \varepsilon^2 e_5\}$$

A complete flag of lattices $\Lambda_\bullet$ is

$$\Lambda_0 \supset \Lambda_1 \supset \dots \supset \Lambda_{n-1} \supset \Lambda_n = \overset{\checkmark}{\varepsilon}\Lambda_0$$

s.t. $\dim_{\mathbb{C}}(\Lambda_{i-1}/\Lambda_i) = 1$

Ex

$$\Lambda_\bullet = \left(\begin{array}{c} \mathrm{Span}_{\mathcal{O}}\{\varepsilon e_1 \overset{+3\varepsilon e_4}{,} \varepsilon^{-1}e_2, e_3, \varepsilon^2 e_4, \varepsilon^2 e_5\} = \Lambda_0 \\ \cup \\ \mathrm{Span}_{\mathcal{O}}\{\varepsilon e_1, \varepsilon^{-1}e_2, \varepsilon e_3, \varepsilon^2 e_4, \varepsilon^2 e_5\} = \Lambda_1 \\ \cup \\ \mathrm{Span}_{\mathcal{O}}\{\varepsilon^2 e_1, \varepsilon^{-1}e_2, e_3, \varepsilon^2 e_4, \varepsilon^2 e_5\} = \Lambda_2 \\ \cup \\ \vdots \end{array}\right) = \boxed{\varepsilon\Lambda_0}$$

The affine Grassmannian (for GL_n) is

$$\widetilde{Gr}_n := \{\Lambda \subset K^n \text{ lattice}\} \cong GL_n(K)/GL_n(\mathcal{O})$$

The affine flag variety (for GL_n) is

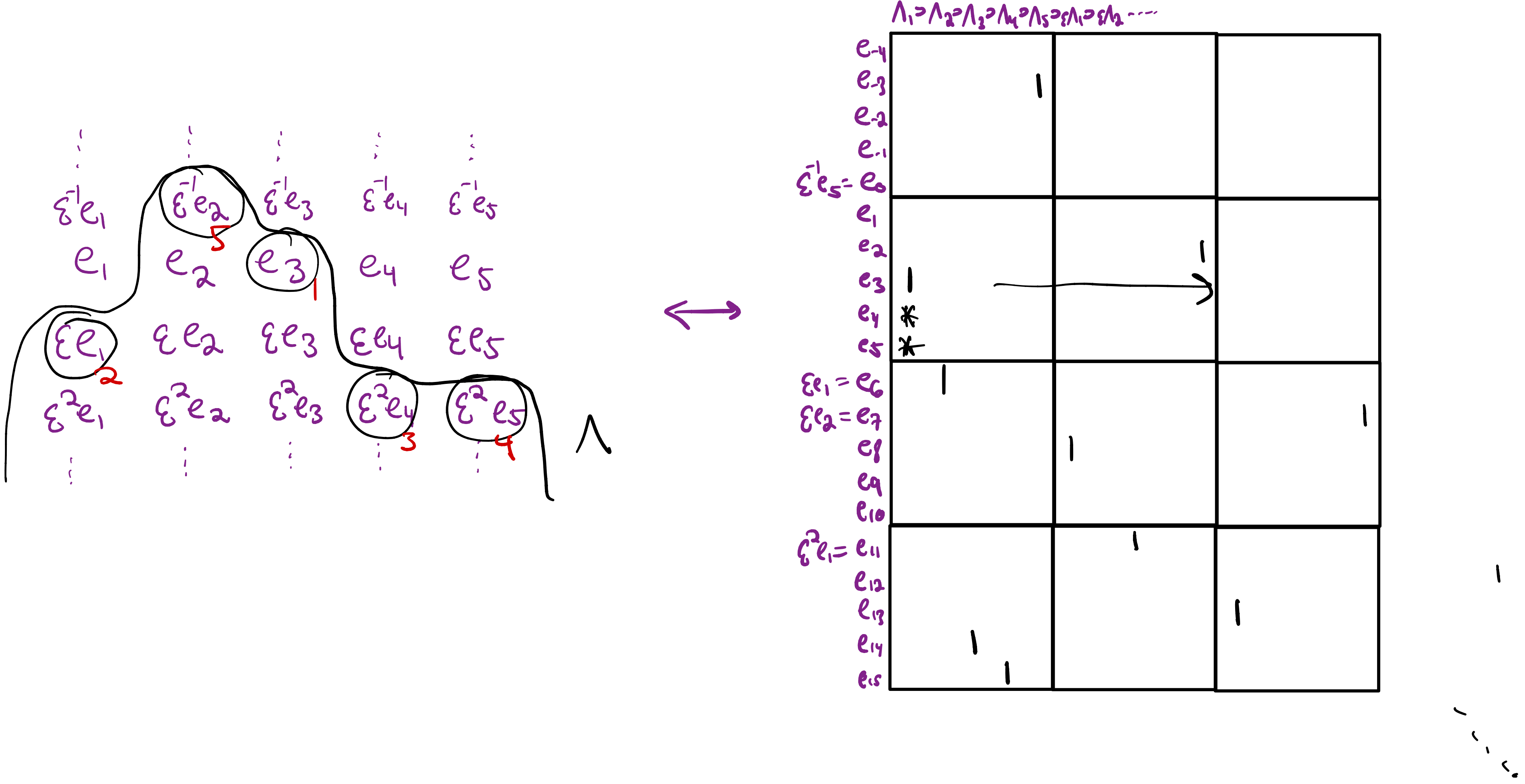
$$\widetilde{Fl}_n := \left\{\Lambda_\bullet \begin{array}{c}\text{complete flags}\\ \text{of lattices}\end{array}\right\} \cong GL_n(K)/I$$

I is an Iwahori subgrp
analog of Borel subgroup

⇝ Infinite periodic matrices up to column operations

Define e_i for $i \in \mathbb{Z}$ by

$$\varepsilon^i e_j = e_{j+ni} \quad (\varepsilon \text{ is shift index by } n)$$



Both $\widetilde{Gr}_n$ and $\widetilde{Fl}_n$ have infinitely many connected components, indexed by $\mathbb{Z} \cong \pi_1(GL_n\mathbb{C})$:

Given $gI \in \widetilde{Fl}_n$, g matrix representative

$$\det(g) = a_m \varepsilon^m + a_{m+1}\varepsilon^{m+1} + \cdots$$

where $a_m \neq 0$

$\Rightarrow$ $m \in \mathbb{Z}$ indexes the connected cmpt

Given $\gamma \in \mathfrak{gl}_n(K)$, the affine Springer fibers are

$$\widetilde{Gr}_\gamma := \{\Lambda \in \widetilde{Gr}_n \mid \gamma\Lambda \subseteq \Lambda\}$$

$$\widetilde{Fl}_\gamma := \{\Lambda_\bullet \in \widetilde{Fl}_n \mid \gamma\Lambda_i \subseteq \Lambda_i\}$$

In this talk, γ is always nilelliptic, $\lim\limits_{m\to\infty} \gamma^m = 0$

There is a projection map

$$\begin{tikzcd}
\widetilde{Fl}_\gamma \arrow[r, "\pi"] & \widetilde{Gr}_\gamma \\
\Lambda_\bullet \arrow[r, mapsto] & \Lambda_0
\end{tikzcd}$$

Each fiber of π is iso to a usual Springer fiber

$$\pi^{-1}(\Lambda) \cong Sp(\gamma \circlearrowright \Lambda/\varepsilon\Lambda)$$

nilpotent

$\longleftrightarrow$ flag in $\Lambda/\varepsilon\Lambda$

Thm (Lusztig) There is an action of S_n on $H_*^{BM}(\widetilde{Fl}_\gamma)$.

(also by the extended affine Weyl grp $\widetilde{S}_n$)

Thm (Hikita)

For $\gamma = \begin{pmatrix} \varepsilon^{0} & & \varepsilon^{2} \\ & \ddots & \\ & \varepsilon & 0 \end{pmatrix}$, letting $\widetilde{Fl}_n = SL_n(K)/I$

$$\mathrm{Frob}\left(H_*(\widetilde{Fl}_\gamma); q, t \right) = \mathrm{worev}_q \left(\underbrace{\sum_{P \in WLD_{n,n}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} x_P}_{\text{Comb. side of } \nabla e_n} \right).$$

$q \leftrightarrow$ homological deg

$t \leftrightarrow$ grading from filtration of $\widetilde{Gr}_\gamma$

Gorsky-Mazin-Vazirani have a nice alternate proof.

For $\gamma = \begin{array}{c} \overbrace{}^{k} \\ \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & & 0 \\ \varepsilon & \ddots & \\ & \varepsilon & 0 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{array}\right) \end{array} \Big\}k \quad \in \mathfrak{gl}_{k(n-k+1)}(\mathcal{O})$

check: γ is nilelliptic ✓

GL version

$$\text{Let } \widetilde{Fl}_\gamma^{+,0} := \{ \Lambda_\bullet \in \widetilde{Fl}_\gamma \mid \Lambda_0 \subseteq \mathcal{O}^{k(n-k+1)}, \Lambda_0 \not\subseteq \varepsilon\mathcal{O} \oplus \mathcal{O}^{k(n-k+1)-1} \}$$

"positive, normalized part of $\widetilde{Fl}_\gamma$"

e_1 appears as a leading term in Λ_0.

Thm (Gillespie-Gorsky-G)

$$\mathrm{Frob}\left(\overset{S_{k(n-k+1)}}{\circlearrowleft} H_*(Fl_\gamma^{+,0}); q, t \right) = (\text{comb. side of})\ \mathrm{worev}_q\left(E_{k(n-k+1),k} \cdot 1\right)$$

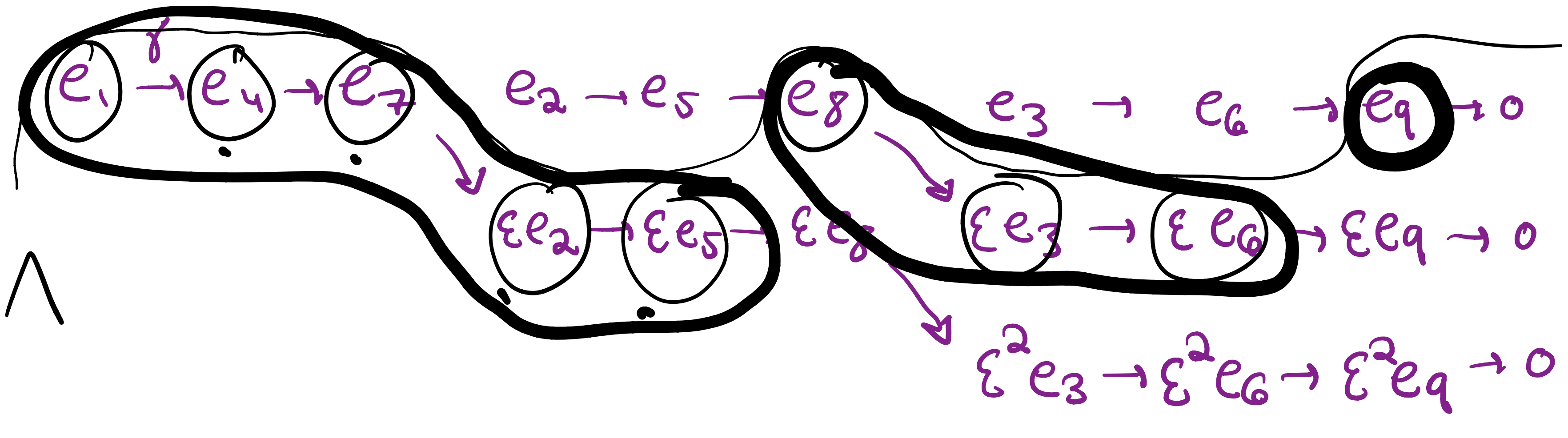
$q \leftrightarrow$ homological grading

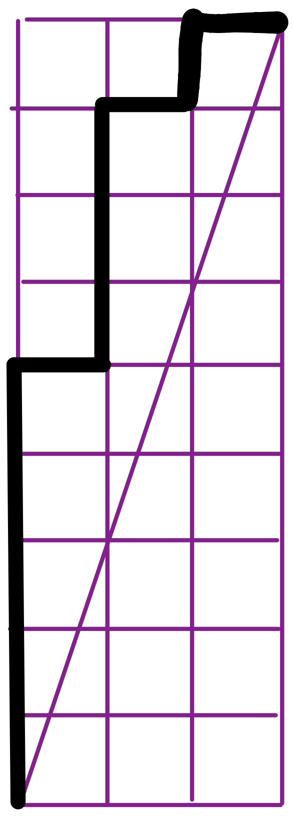
$t \leftrightarrow$ grading by connected cmpt of $\widetilde{Fl}_n$.

"Proof" by picture

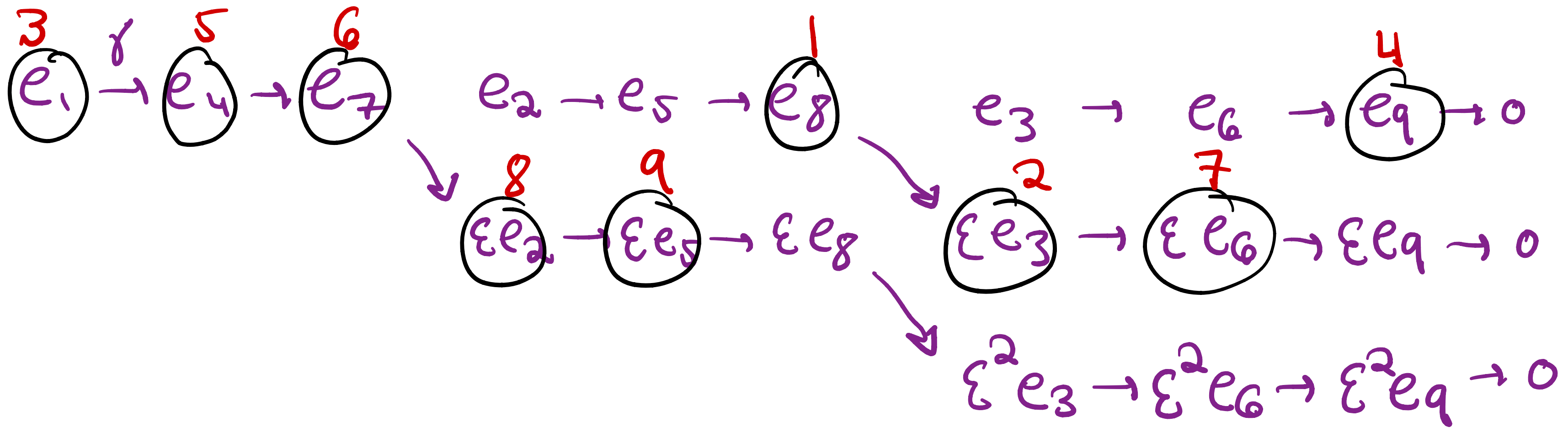
- T-fixed pts of $\widetilde{Gr}_\gamma^{+,0}$ look like...

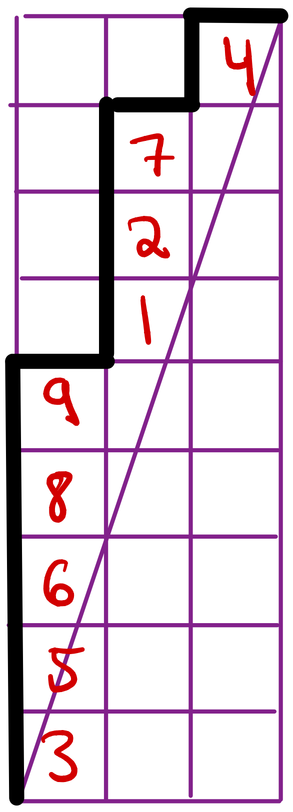
For $n=5, k=3$, γ acts by



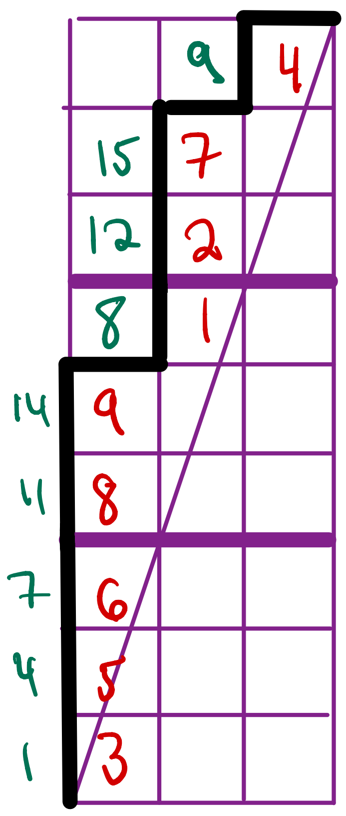


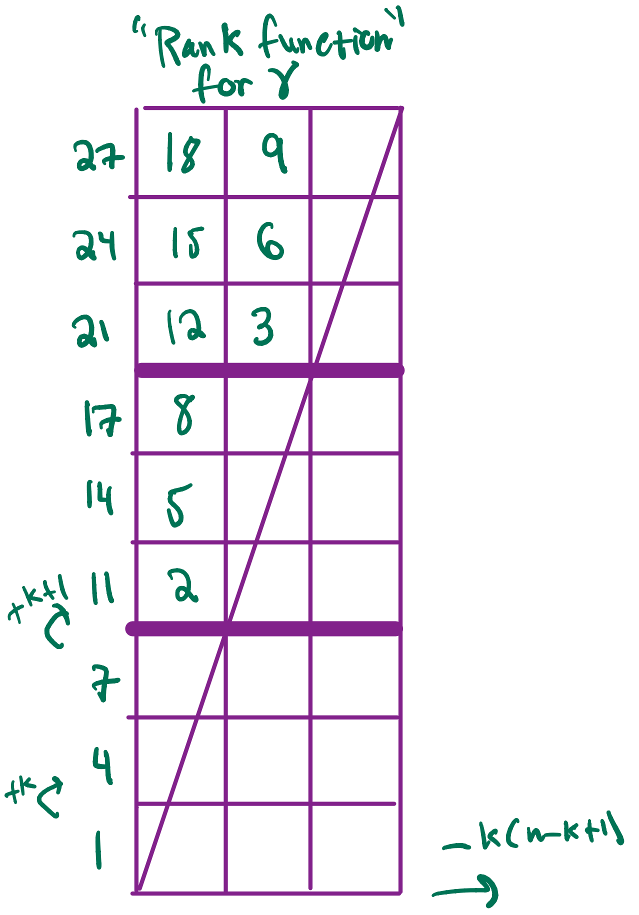
- T-fixed pts of $\widetilde{Fl}_\gamma^{+,0}$ look like...





- There is also bijection between the torus fixed points and <u>γ-strict affine permutations</u>





$\longmapsto$ $[\underset{1}{8},\underset{2}{12},\underset{3}{1},\underset{4}{9},\underset{5}{4},\underset{6}{7},\underset{7}{15},\underset{8}{11},\underset{9}{14}] \in \tilde{S}_9$

window notation

- "Positive" ↔ positive window
- "normalized" ↔ 1 is in window

Affine Borho-MacPherson:

$$\widetilde{BM}_{\vec{\lambda},\gamma} := \{\Lambda_\bullet \in \widetilde{Fl}(\alpha) \mid \gamma\Lambda_i \subseteq \Lambda_i,\ JJ(\gamma \circlearrowright \Lambda_i/\Lambda_{i-1}) \le \lambda^i\}$$

Thm (Gillespie-Gorsky-G)

$$\mathrm{Frob}\big(\underset{S_n \circlearrowright}{H_*}(\widetilde{BM}_{\vec{\lambda},\gamma})\ ; q,t\big) = \overset{\text{(comb. side of)}}{\mathrm{worev}_q(\Delta'_{e_{k-1}} e_n)}$$

where $\vec{\lambda} = \underline{((1),(1),\ldots,(1),(n-k)^{k-1})}$, $\underline{\gamma \text{ as before}}$.

Proof sketch

① Prove

$$\mathrm{Frob}(H_*(\widetilde{Fl}^{+,0}_\gamma)\,; q,t) = \overset{\text{(comb. side of)}}{\mathrm{worev}_q(E_{k(n-k+1),k} \cdot 1)}$$

using an affine paving

② Check a rational smoothness condition for $\underline{\text{all}}$ fibers of $\widetilde{BM}^{+,0}_{\vec{\lambda},\gamma} \to \widetilde{Gr}^{+,0}_\gamma$

$$\Rightarrow \mathrm{Frob}(H_*(\widetilde{BM}^{+,0}_{\vec{\lambda},\gamma})) = s^\perp_{(n-k)^{k-1}} \mathrm{Frob}(H_*(\widetilde{Fl}^{+,0}_\gamma))$$

(up to q shift)

③ Combine:

$$\begin{aligned}\mathrm{Frob}(H_*(\widetilde{BM}^{+,0}_{\vec{\lambda},\gamma})) &= s^\perp_{(n-k)^{k-1}} \mathrm{Frob}(H_*(Fl^{+,0}_\gamma)) \cdot q^\star \\ &= s^\perp_{(n-k)^{k-1}} \mathrm{worev}_q(E_{k(n-k+1),k} \cdot 1) \cdot q^\star \\ &= \mathrm{worev}_q(s^\perp_{(k-1)^{n-k}} E_{k(n-k+1),k} \cdot 1) \\ &= \mathrm{worev}_q(\Delta'_{e_{k-1}} e_n)\end{aligned}$$

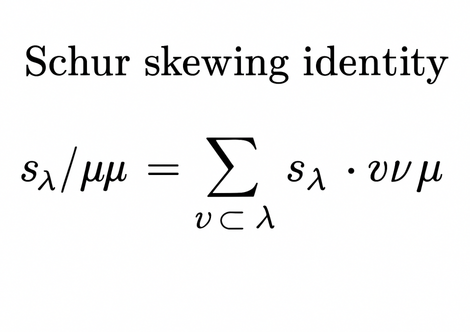
□

- (G, 2025+) A similar construction works for "paths under any line"
- Can the Pawlowski-Rhoades space be used to give an alternate geom construction for $\Delta'_{e_{k-1}} e_n$?
- Connections to recent proof of Fields Conjecture by Murai-Rhoades-Wilson?

Thanks for listening!

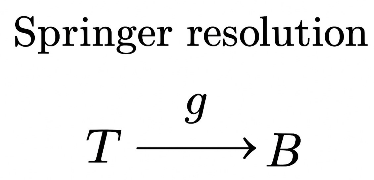
Thanks to the organizers!

Bonus: Springer Theory according to ChatGPT



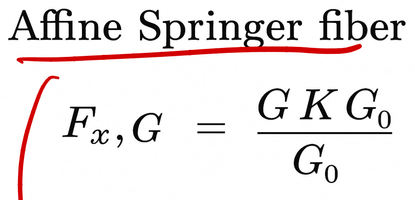
Schur skewing identity

$$s_\lambda/\mu\mu = \sum_{v\subset\lambda} s_\lambda \cdot v\nu\mu$$



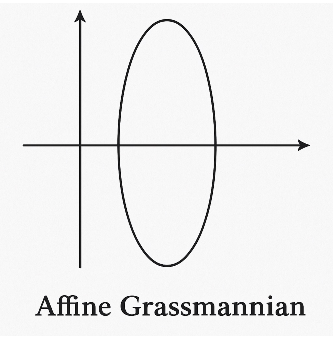
Springer resolution

$$T \xrightarrow{g} B$$

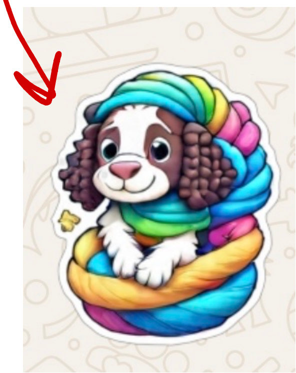


Affine Springer fiber

$$F_x, G = \frac{G\,K\,G_0}{G_0}$$



Affine Grassmannian



WhatsApp AI