Geometric Interpretation for the Delta Thm (II)

Last Time

Goal today

$$\Delta'_{e_{k-1}} e_n = S^{\perp}_{(k-1)^{n-k}} \overbrace{E_{k(n-k+1),k}\cdot 1}^{H_{(n-k+1)^k}(x;q)}$$

at $\underline{\underline{t=0}}$ geometrically.

$$\begin{tikzcd} \Delta'_{e_{k-1}} e_n \arrow[d] & S^{\perp}_{(k-1)^{n-k}} E_{k(n-k+1),k}\cdot 1 \arrow[d, squiggly] \\ H^*(\Delta Sp_{n,k}) & H^*(Sp_N) \end{tikzcd}$$

Thursday:

Explain the identity (w/out $t=0$) using affine Springer fibers

$$\widetilde{gl}_n = \{(V_\bullet, x) \in Fl(\mathbb{C}^n) \times gl_n \mid xV_i \subseteq V_i\}$$

$$gl_n^{rs} = \{x \in gl_n \text{ regular semisimple}\}$$

$\lambda_i \neq \lambda_j$

$$x \sim \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\widetilde{gl}_n^{rs} = \{(V_\bullet, x) \in Fl(\mathbb{C}^n) \times gl_n^{rs} \mid xV_i \subseteq V_i\}$$

$$\begin{tikzcd} \widetilde{gl}_n^{rs} \arrow[r, hook] \arrow[d, "p^{rs}"] & \widetilde{gl}_n \arrow[d, "p'"] & \tilde{N} \arrow[l, hook'] \arrow[d, "p"] & (V_\bullet, N) \arrow[d, mapsto] \\ gl_n^{rs} \arrow[r, hook, "\text{open dense}"'] & gl_n & N \arrow[l, hook', "\text{closed}"'] & N \end{tikzcd}$$

$$S_n \circlearrowright \text{Fibers of } p^{rs} \overset{\text{Perverse sheaves}}{\Longrightarrow} S_n \circlearrowright \text{Cohomology of fibers of } p' \text{ and } p$$

$$H^*(Sp_N)$$

$S_n \cong$ Weyl group of GL_n

Thm (Springer) Let $JT(N) = \mu$, $\underline{n}(\mu) = \sum \binom{\mu_i'}{2} = \dim(Sp_N)$,

$$H^{2\underline{n}(\mu)}(Sp_N) \underset{S_n}{\cong} V^{\mu} \quad \text{(irreducible Specht module)}$$

Thm (Lusztig-Procesi)

$$\text{Frob}(H^*(Sp_N); q) = \tilde{H}_\mu(x;q)$$

Further recall

$$\Delta Sp_{n,k} := \{V_\bullet = (V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{k(n-k+1)}) \mid NV_i \subseteq V_i,\ \ker N \subseteq V_n\}$$

where $JT(N) = $ $k\underbrace{\boxed{}}_{}$ with width $n-k+1$

$$JT(N) = \overset{n-k+1}{k\,\boxed{}}$$

Borho and MacPherson:

For each α, there is a "partial resolution" of $\mathcal{N}$

$$\tilde{\mathcal{N}}^\alpha = \{ (V_\bullet, N) \in Fl(\alpha) \times \mathcal{N} \mid NV_i \subseteq V_i \}$$

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[d] \arrow[dd, bend right=50, "p"'] & \text{smooth} \arrow[l] \\
\tilde{\mathcal{N}}^\alpha \arrow[d, "\xi"] & = \bigsqcup \tilde{\mathcal{O}}_{\lambda} \\
\mathcal{N} &
\end{tikzcd}$$

Geometric properties:

- All maps are <u>semismall</u>
  (dims of fibers are "small")
- $\tilde{\mathcal{N}}^\alpha$ is not smooth, but it is rationally smooth
  (locally cohomologies are the same as those of a smooth space)

For $\tilde{\mathcal{N}}^\alpha$, there is no S_n action, but...

$$\begin{tikzcd}
\tilde{\mathfrak{gl}}^{\alpha,rs} \arrow[r, hook] \arrow[d] & \tilde{\mathfrak{gl}}^{\alpha} \arrow[d] & \tilde{\mathcal{N}}^\alpha \arrow[l] \arrow[d, "\xi"] \\
\mathfrak{gl}^{rs} \arrow[r, hook] & \mathfrak{gl} & \mathcal{N} \arrow[l]
\end{tikzcd}$$

$$\tilde{\mathfrak{gl}}^{\alpha,rs} = \{ (V_\bullet, x) \in Fl(\alpha) \times \mathfrak{gl}^{rs} \mid xV_i \subseteq V_i \}$$

Each V_i/V_{i-1} is dim α_i is a direct sum of α_i eigenlines of x

$\rightsquigarrow$ Action of the <u>relative Weyl group</u>

$$W_\alpha = \{\text{perms of the parts of } \alpha \text{ of equal size}\}$$

$$= S_{\alpha'_1} \times \cdots \times S_{\alpha'_\ell} \quad \text{where } \alpha'_i = \#\text{ of parts of } \alpha \text{ of size } i$$

$$= N(L_\alpha)/L_\alpha$$

$$\Rightarrow W_\alpha \circlearrowright H^*(\xi^{-1}(N)) \quad \forall N$$

$\tilde{\mathcal{N}}^\alpha$ has a stratification, $\tilde{\mathcal{N}}^\alpha = \bigcup_{\vec{\lambda}} \mathcal{O}_{\vec{\lambda}}$

where $\vec{\lambda} = (\lambda^1, \dots, \lambda^\ell)$, $\lambda^i \vdash \alpha_i$,

$$\mathcal{O}_{\vec{\lambda}} = \{ (V_\bullet, N) \in Fl(\alpha) \times \mathcal{N} \mid NV_i \subseteq V_i \text{ and } JT(N \circlearrowright V_i/V_{i-1}) = \lambda^i \}$$

$$\overline{\mathcal{O}}_{\vec{\lambda}} = \text{- - - - -} \qquad JT(N \circlearrowright V_i/V_{i-1}) \underset{\uparrow}{\leq} \lambda^i$$

dominance order on partitions

The <u>Borho-MacPherson fibers</u> are

$$BM_{\vec{\lambda},N} := \xi_{\vec{\lambda}}^{-1}(N) = \{ V_\bullet \in Fl(\alpha) \mid NV_i \subseteq V_i,\ JT(N \circlearrowright V_i/V_{i-1}) \leq \lambda^i \ \forall i \}$$

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[d, "\pi_\alpha"] \arrow[dd, bend right, "p"'] & \\
\tilde{\mathcal{N}}^\alpha \arrow[d, "\xi"] & \overline{\mathcal{O}}_{\vec{\lambda}} \arrow[l, hook'] \arrow[dl, "\xi_{\vec{\lambda}}"] \\
\mathcal{N} &
\end{tikzcd}$$

$$\pi_\alpha^{-1}((\underset{\underset{Fl(\alpha)}{\uparrow}}{V_\bullet}, N)) = \{ W_\bullet \in Fl(1^n) \mid W_\bullet \text{ extends } V_\bullet \text{ and } NW_i \subseteq W_i \}$$

$W_1 \subset W_2 \subset \cdots \quad W_\bullet \subset \cdots$

$$0 \underset{\alpha_1}{\subset} V_1 \underset{\alpha_2}{\subset} V_2 \underset{\alpha_3}{\subset} V_3 \subset \cdots$$

$$\cong \prod_i Sp_{(N \circlearrowright V_i/V_{i-1})}$$

<u>Thm</u> (Borho-MacPherson 83) Let $d = \sum \underline{n}(\lambda^i)$.

If $\overline{\mathcal{O}}_{\vec{\lambda}}$ is rationally smooth at every pt of $BM_{\vec{\lambda},N}$, then

$$H^i(BM_{\vec{\lambda},N}) \otimes H^{2d}(\pi_\alpha^{-1}(V_\bullet, N)) \underset{W_\alpha}{\cong} H^{i+2d}(Sp_N)^{V^{\lambda^1} \otimes \cdots \otimes V^{\lambda^\ell}}$$

for a fixed $(V_\bullet, N) \in \mathcal{O}_{\vec{\lambda}}$

Given $S_n \circlearrowright W$, $S_{\alpha_1} \times \cdots \times S_{\alpha_\ell} \leq S_n$, $W^{V^{\lambda^1} \otimes \cdots \otimes V^{\lambda^\ell}} := (V^{\lambda^1} \otimes \cdots \otimes V^{\lambda^\ell})$-isotypic cmpt of $\left(W \downarrow^{S_n}_{S_{\lambda^1} \times \cdots \times S_{\lambda^\ell}}\right)$

$\lambda^i \vdash \alpha_i$

<u>Corollary</u> W_α acts on both sides, and the isomorphism is as W_α-modules in the following case that we care about....

① Connect this to $\Delta Sp_{n,k}$

② Check rational smoothness condition in that case

③ Translate B-M's isomorphism into symmetric functions.

Prop $\Delta Sp_{n,k} = BM_{((1),(1),\ldots,(n-k)^{k-1}),\, N}$

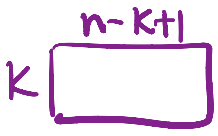
where $JT(N) = (n-k+1)^k$.

Pf Recall

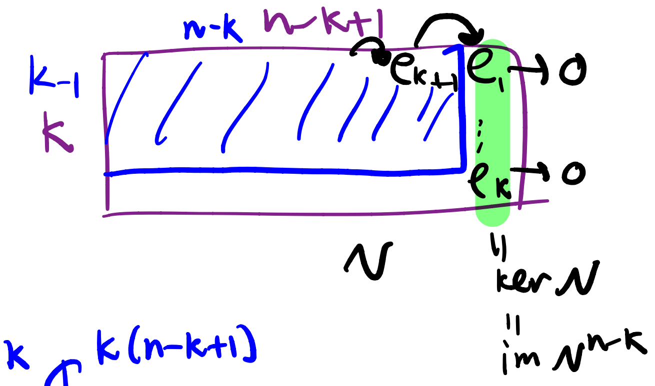
$\Delta Sp_{n,k} := \{ V_\bullet = (V_1 \subset \cdots \subset V_n \subseteq \mathbb{C}^{k(n-k+1)}) \mid NV_i \subseteq V_i,\ \underline{\ker N \subseteq V_n} \}$

$\alpha = (\overbrace{1,1,\ldots,1}^{n},(n-k)(k-1))$ type of partial flag

$JT(N) = (n-k+1)^k$ by construction



Since $JT(N)$ is rectangular, $\ker N = \operatorname{im} N^{n-k}$



Let $V_\bullet$ s.t. $N V_i \subseteq V_i$.

$V_\bullet \in \Delta Sp_{n,k}$

$\iff V_n \supseteq \ker N = \operatorname{im} N^{n-k} = N^{n-k}\mathbb{C}^{k(n-k+1)}$

$\iff N^{n-k}\left(\mathbb{C}^{k(n-k+1)}/V_n\right) = 0$

$\iff$ All parts of $JT\left(N \circlearrowright \underbrace{\mathbb{C}^{k(n-k+1)}}_{V_{n+1}}/V_n\right) \leq n-k$

$\iff JT(N \circlearrowright V_{n+1}/V_n) \leq (n-k)^{k-1}$

$\iff V_\bullet \in BM_{((1),(1),\ldots,(1),\,(n-k)^{k-1})}$ ☐

② Why is $\overline{\mathcal{O}}_{\vec{\lambda}}$ rationally smooth on $BM_{\vec{\lambda},N}$

when $\vec{\lambda} = ((1), \dots, (1), (n-k)^{k-1})$,

$JT(N) = (n-k+1)^k$?

Checking rational smoothness of X in a nbhd of x

$\Leftrightarrow$ Check $IH_{x'}(X)$ are trivial in a nbhd of x

Thm (Lusztig)

$$\mathrm{Hilb}_q(IH^\bullet_N(\overline{\mathcal{O}}_\lambda)) = q^{-\underline{n}(\lambda)} \tilde{K}_{\lambda,\mu}(q)$$

shape λ

content μ

$$\overline{\mathcal{O}}_\lambda = \{N \in \mathcal{N} \mid JT(N) \le \lambda\}$$

$$JT(N) = \mu \le \lambda$$

$\mu \le (n-k)^{k-1}$ and

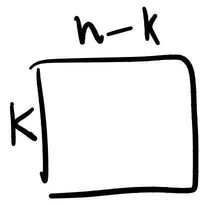
$\ell(\mu) \le k$.

$\Updownarrow$

Prop (Gillespie-G)

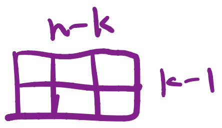
For any $\mu \vdash (n-k)(k-1)$ s.t. $\mu \subseteq (n-k)^k$

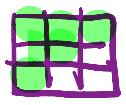
then
$$\tilde{K}_{(n-k)^{k-1},\mu}(q) = q^{\underline{n}((n-k)^{k-1})}$$

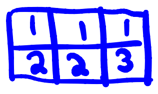


Here, μ is the possible $JT(N \circlearrowright \mathbb{C}^{k(n-k+1)}/V_n)$ for $V_\bullet \in \Delta Sp_{n,k}$

Ex $n=6$, $k=3$

Shape 

content μ $\le$ 



unique

Exercise Prove it.

③ Translation to skewing formula:

By B-M's thm + rational smoothness:

$\vec{\lambda} = ((1), \dots, (1), (n-k)^{k-1})$, $\alpha = (1, 1, \dots, 1, (n-k)(k-1))$

$JT(N) = (n-k+1)^k$

$$H^i(BM_{((1),\dots,(1),(n-k)^{k-1})}) \otimes H^{2d}(Sp_{(n-k)^{k-1}}) \cong H^{i+2d}(Sp_N)^{V^{(n-k)^{k-1}}}$$

Exercise If $S_m \curvearrowright W$, and $|\mu| \leq m$,

$$\mathrm{Frob}(W^{V^\mu}) = \dim(V^\mu) \cdot s_\mu^\perp \mathrm{Frob}(W).$$

Combining,

Next Time: Affine springer fibers and the full skewing thm