Geometric Interpretation for the Delta Thm (II)

Goal: Explain the identity

$$s^{\perp}_{(k-1)^{n-k}}\left(E_{k(n-k+1),k}\cdot 1\right) = \Delta'_{e_{k-1}} e_n$$

using affine Springer fibers.

Still $t=0$ specialization

Last Time

- $PR_{n,k} := \{(L_1,\dots,L_n) \mid L_i \text{ a line in } \mathbb{C}^k,\ L_1+\dots+L_n=\mathbb{C}^k\}$

  $\underset{\text{open}}{\subseteq} \mathbb{P}^{k-1}\times\dots\times\mathbb{P}^{k-1}$ smooth, noncmpt

- $\Delta Sp_{n,k} := \{V_\bullet = (V_1\subset\dots\subset V_n \subseteq \mathbb{C}^{k(n-k+1)}) \mid NV_i\subseteq V_i,\ \ker N\subseteq V_n\}$

  $\underset{\text{closed}}{\subseteq} Fl(1,1,\dots,1,(n-k)(k+1))$

  where N is nilpotent, $JT(N) = {}_{k}\boxed{}^{\,n-k+1} = (n-k+1)^k$

$$\mathrm{Poin}(PR_{n,k};q) \overset{PR}{=} \underline{\mathrm{rev}_q}\left(\sum_{\substack{P\in WLD^{\text{stack}}_{n,k}\\ \mathrm{area}(P)=0\\ P \text{ standard}}} q^{\mathrm{dinv}(P)}\right) \overset{GLW}{\underset{(*)}{=}} \mathrm{Poin}(\Delta Sp_{n,k};q)$$

$[m_{i}]\ \Delta'_{e_{k-1}} e_n\big|_{t=0}$

An <u>affine paving</u> of X is a filtration of X by closed subvarieties

$$X = X_n \supseteq X_{n-1}\supseteq X_{n-2}\supseteq\dots\supseteq X_1\supseteq X_0=\emptyset$$

such that $\underbrace{X_i\setminus X_{i-1}}_{\text{"cells"}} \cong \mathbb{C}^{m_i}$ for some m_i

__Fact__ If X is compact and has an affine paving, then

$$\mathrm{Poin}(X;q) = \sum_i q^{m_i}.$$

__Thm__ (G-Levinson-Woo)

$\Delta Sp_{n,k}$ has an affine paving in which each cell C contains one torus-fixed point corresponding to some $P \in WLD^{\text{stack}}_{n,k}$ with $\mathrm{area}(P)=0$, and

$$\mathrm{codim}_{\Delta Sp_{n,k}}(C) = \mathrm{dinv}(P).$$

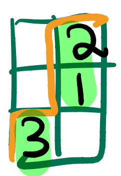
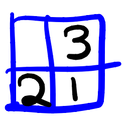
(*) follows immediately as a corollary

Cells are given by intersecting $\Delta Sp_{n,k}$ with Schubert cells.

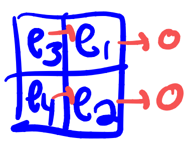
[* Why care?

We can use different decompositions of $\Delta Sp_{n,k}$ to get different combinatorial formulas for $\Delta'_{e_{k-1}} e_n \big|_{t=0}$]

__Ex of thm__ $n=3,\ k=2 \to \dim(\Delta Sp_{3,2}) = 2$

$P=$  $\leftrightarrow$  $\leftrightarrow$ $\begin{bmatrix} 0&0&1\\ 1&0&0\\ 0&0&0\\ 0&1&0 \end{bmatrix}$ $\overset{\longleftrightarrow}{\text{cell}}$ $C = \left\{ \begin{bmatrix} a&b&1\\ 1&0&0\\ 0&c&0\\ 0&1&0 \end{bmatrix} : a,b,c \in \mathbb{C} \right\} \cap \Delta Sp_{3,2}$

$\cap\!\!\!\!\;$ $RSL_{3,2}$ $\qquad$ $\cap$ $(Sp_{3,2})^T$ $\qquad$ Schubert cell $\qquad$ $NV_i \subseteq V_i \Rightarrow a=c$

$N=$ 

Exercise $= \left\{ \begin{bmatrix} a&b&1\\ 1&0&0\\ 0&a&0\\ 0&1&0 \end{bmatrix} : a,b \in \mathbb{C} \right\}$

$\cong \mathbb{C}^2$

$\mathrm{dinv}(P) = 0$ $\qquad$ $\mathrm{codim}(C) = 2 - \dim(C) = 2-2 = 0$ ✓

Next Goal Use Springer Theory to construct an S_n action on $H^*(\Delta Sp_{n,k})$ and upgrade the Poincaré polynomial to a Frobenius series.

$\mathcal{N} := \{\text{nilpotent } n\times n \text{ matrices } N\}$ nilpotent cone

$\underbrace{Fl(1,\dots,1)}_{(1^n)} := \{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim V_i/V_{i-1} = 1 \ \forall i\}$

$\tilde{\mathcal{N}} := T^*(Fl(1^n)) \cong \{(V_\bullet, \mathcal{N}) \in Fl(1^n) \times \mathcal{N} \mid N V_i \subseteq V_i \ \forall i\}$

$$\begin{tikzcd} & \tilde{\mathcal{N}} \text{ smooth} \arrow[dl, "\text{vector bundle}"'] \arrow[d, "\rho"] & (V_\bullet, N) \arrow[d, mapsto] \\ Fl(1^n) & \mathcal{N} & N \end{tikzcd}$$

ρ is called the Springer resolution of $\mathcal{N}$

- Given $N \in \mathcal{N}$, the Springer fiber is

$$Sp_N := \rho^{-1}(N) \cong \{V_\bullet \in Fl(1^n) \mid N V_i \subseteq V_i \ \forall i\} \underset{\text{closed}}{\subseteq} Fl(1^n)$$

- Springer ('76) defined an action of S_n on $H^*(Sp_N)$.

$\widetilde{\mathfrak{gl}}_n = \{(V_\bullet, x) \in Fl(\mathbb{C}^n) \times \mathfrak{gl}_n \mid xV_i \subseteq V_i\}$ $\qquad \mathfrak{gl}_n = \{n\times n \text{ matrices}\}$

$\mathfrak{gl}_n^{rs} = \{x \in \mathfrak{gl}_n \mid x$ regular semisimple$\}$ $\longrightarrow$ $x \sim \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$, $\lambda_i \neq \lambda_j$

$\widetilde{\mathfrak{gl}}_n^{rs} = \{(V_\bullet, x) \in Fl(\mathbb{C}^n) \times \mathfrak{gl}_n^{rs} \mid xV_i \subseteq V_i\}$

Covering space with deck transforms S_n

$$\begin{tikzcd}
\widetilde{\mathfrak{gl}}_n^{rs} \arrow[r, hook] \arrow[d, "p^{rs}"'] & \widetilde{\mathfrak{gl}}_n \arrow[d, "p'"] & \widetilde{\mathcal{N}} \arrow[l, hook'] \arrow[d, "p"] & (V_\bullet, N) \arrow[d, mapsto] \\
\mathfrak{gl}_n^{rs} \arrow[r, hook, "\text{open dense}"'] & \mathfrak{gl}_n & \mathcal{N} \arrow[l, hook', "\text{closed}"] & N
\end{tikzcd}$$

- Given $x \in \mathfrak{gl}_n^{rs}$, $V_\bullet \in (p^{rs})^{-1}(x)$,

  $xV_i \subseteq V_i \quad \forall i$.

- Since $x \sim \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$ $\lambda_i \neq \lambda_j$, the eigenspaces of x are all 1-dim'l

  $L_1, \dots, L_n$ eigenlines ($xL_i = L_i$)

- Since $xV_i \subseteq V_i$, $xV_i = V_i$, and V_i is a direct sum of eigenlines, so

  $\exists\, \sigma \in S_n$ s.t. $V_i = L_{\sigma_1} \oplus \cdots \oplus L_{\sigma_i} \quad \forall i$

Ex $x = \begin{bmatrix}3 & & \\ & -1 & \\ & & 4\end{bmatrix} \in \mathfrak{gl}_3^{rs}$, eigenlines $\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle$

$(p^{rs})^{-1}(x)$ is the set of $3!$ "permutation flags"

$$\left\{\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1\end{bmatrix}, \dots\right\}$$

Permuting eigenlines defines an action of S_n on each fiber $(p^{rs})^{-1}(x)$, which extends to a fiber-wise action of $S_n \curvearrowright \widetilde{\mathfrak{gl}}_n^{rs}$.

In fact, $\mathbb{C}S_n \cong \mathrm{End}(Rp^{rs}_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{gl}}_n^{rs}})$

Using the theory of perverse sheaves, this action transfers to an action of S_n on the cohomology of each fiber of ρ', so in particular for each fiber of ρ:

$$S_n \circlearrowright H^*(\rho^{-1}(N)) = H^*(Sp_N).$$

☆ Cool, but where is the combinatorics?! ☆

Let $\mu = JT(N)$

$$\mu = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array}\!\!\begin{array}{l}\\ \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}\end{array}$$

$$\mathrm{Poin}(Sp_N; q) = \mathrm{rev}_q\Big(\sum_{\substack{P \text{ row-strict} \\ \text{tableaux} \\ \text{of shape } \mu}} q^{\mathrm{inv}(P)}\Big)$$

$$P = \begin{array}{|c|c|c|}\hline 3 & 1 & \\ \hline 5 & 4 & 2 \\ \hline \end{array}$$

$$= \langle \tilde{H}_\mu(x;q), h_{(1^n)} \rangle \qquad (m_{(1^n)} \text{ coeff.})$$

Special property:

For each $(\alpha_1, \ldots, \alpha_\ell) \vdash n$, there is a partial flag variety

$$\begin{array}{ccc} Sp_N & \subseteq & Fl(1^n) \\ \downarrow \pi & & \downarrow \pi \\ \pi(Sp_N) & \subseteq & Fl(\alpha_1, \ldots, \alpha_\ell) := \{ V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_\ell) \mid V_i \subseteq \mathbb{C}^n, \dim(V_i/V_{i-1}) = \alpha_i \} \end{array}$$

$N V_i \subseteq V_i$

$$\mathrm{Poin}(\pi(Sp_N); q) = \mathrm{Hilb}\left(H^*(Sp_N)^{S_{\alpha_1} \times \cdots \times S_{\alpha_\ell}} ; q \right)$$

$$= \langle \tilde{H}_\mu(x;q), h_\alpha \rangle \qquad (m_\alpha \text{ coeff})$$

$$= \mathrm{rev}_q\Big(\sum_{\substack{P \text{ row strict} \\ \text{shape } \mu \\ \text{content } \alpha}} q^{\mathrm{inv}(P)} \Big).$$

Projecting to partial flag varieties $\longleftrightarrow$ Taking invariants.

<u>Next Time</u> Relate $\Delta Sp_{n,k}$ to the "partial resolutions" of Borho–MacPherson to obtain the "$t=0$" skewing formula.

<u>Borho and MacPherson:</u>

For each α, there is a "partial resolution" of $\mathcal{N}$

$$\tilde{\mathcal{N}}^{\alpha} = \{ (V_\bullet, N) \in Fl(\alpha) \times \mathcal{N} \mid N V_i \subseteq V_i \}$$

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[d] \arrow[dd, bend right=60, "\rho"'] \\
\tilde{\mathcal{N}}^{\alpha} \arrow[d, "\xi"] \\
\mathcal{N}
\end{tikzcd}$$

has a stratification $\bigsqcup_{\vec{\lambda}} \mathcal{O}_{\vec{\lambda}}$ $\xrightarrow{\xi_{\vec{\lambda}}} \mathcal{N}$

$\Delta Sp_{n,k}$ is a fiber of $\xi_{\vec{\lambda}}$ over $JT(\mathcal{N}) = k\,\overset{n-k+1}{\boxed{}}$

$\rightsquigarrow$ Use BM's theory to derive the $\underline{t=0}$ skewing formula.