Maria's Talks

joint w/ Maria Gillespie + Eugene Gorsky

$$\boxed{s^{\perp}_{(k-1)^{n-k}}\left(E_{k,k}\cdot 1\right) = \Delta'_{e_{k-1}} e_n}$$

Goal: A geometric interpretation of this identity in terms of affine Springer fibers.

- Outline of talks

Talk 1 Combinatorics of $\Delta'_{e_{k-1}} e_n$ at $t=0$

$\rightsquigarrow$ Geometry of
- Spanning line configurations
- Δ-Springer fibers

Talk 2 Crash course in Springer Theory + Borho-MacPherson Partial Resolutions

Talk 3 Affine Springer theory + the Delta Theorem

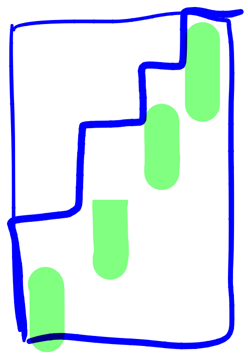
Thm (Rise Delta Thm, D'A-M, BHMPS)

$$\Delta'_{e_{k-1}} e_n = \sum_{P \in WLD^{stack}_{n,k}} q^{dinv(P)} t^{area(P)} x_P$$

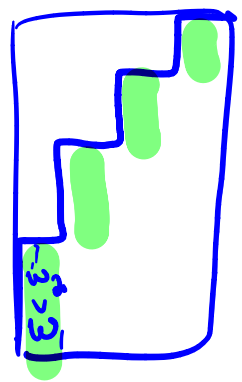
Start with two simplifying assumptions:

① Area 0 :

When $area(P)=0$,



area 0



P

② P is standard : Labels $1,2,\dots,n$ are used exactly once in P.

$$\boxed{\{P \in WLD^{stack}_{n,k} \mid area(P)=0,\ P \text{ is standard}\}}$$

$\wr$ $\quad\quad$ $\wr$

$RSL_{n,k}$

{Partial row-strict labelings of $k\times(n-k+1)$ rect}
s.t. labels decrease, each row is nonempty

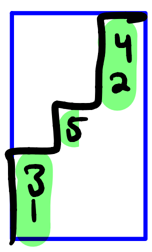
$\simeq$

$FW_{n,k}$

Fubini words on $1,\dots,k$
$\{w = w_1 \cdots w_n \mid \{w_1,\dots,w_n\} = \{1,\dots,k\}\}$

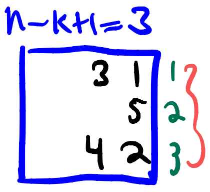
$n=5,\ k=3$

k cols



$n-k+1=3$

$k=3$



k rows

(1, 3, 1, 3, 2)

letters $1,\dots,k$

Both $RSL_{n,k}$ and $FW_{n,k}$ are in bijection with the set of torus-fixed points in two particular varieties:

- (Pawlowski–Rhoades) Spanning configuration spaces

  Given $n \geq k$,

$$PR_{n,k} := \{ (L_1, \ldots, L_n) \mid L_i \text{ a line in } \mathbb{C}^k,\ L_1 + \cdots + L_n = \mathbb{C}^k \}$$

$$\underset{\text{open subvariety}}{\subseteq} \quad \mathbb{P}^{k-1} \times \cdots \times \mathbb{P}^{k-1}$$

A point $(L_1, \ldots, L_n) \in PR_{n,k}$ can be represented by

$$\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_k \end{bmatrix} k \overset{n}{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}} \quad \text{where } L_i = \mathrm{Span}\{v_i\} \quad \text{s.t.}$$

- Each column is nonzero
- The matrix unique up to scaling columns
- Some $k \times k$ minor is nonzero.

The torus $\underset{\substack{\rotatebox{90}{$\in$} \\ t = (t_1, \ldots, t_k)}}{T} = (\mathbb{C}^*)^k$ acts on $PR_{n,k}$ on the left

$$t \cdot (L_1, \ldots, L_n) = (tL_1, \ldots, tL_n)$$

The torus-fixed points $(PR_{n,k})^T$ are represented by matrices with exactly one nonzero entry in each column:

<u>Ex</u> $(PR_{3,2})^T = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right\}$

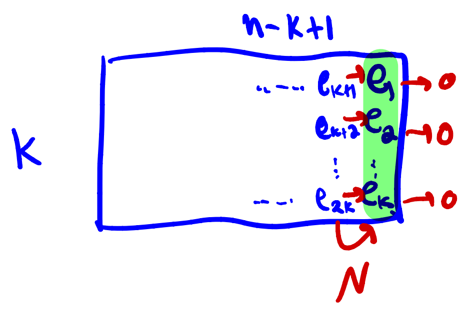
$$\updownarrow$$

$$\begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & t_1 & t_1 \\ t_2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$FW_{3,2} = \{ 112, 122, 121, 211, 221, 212 \}$

• (G-Levinson-Woo) Delta-Springer fibers

Define a nilpotent matrix N acting on $\mathbb{C}^{k(n-k+1)}$:



$N e_i = e_{i-k}$

$N e_1 = \cdots = N e_k = 0$

$$\Delta Sp_{n,k} := \{ V_\bullet = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \mathbb{C}^{k(n-k+1)}) \mid \dim V_i = i,\ NV_i \subseteq V_i\ \forall i,\ V_n \supseteq \ker N \} \underset{\text{closed}}{\subseteq} Fl(\underbrace{1,\dots,1}_{n};\ k(n-k+1)-n)$$

$V_\bullet \in \Delta Sp_{n,k}$ can be represented by a $k(n-k+1) \times n$ matrix

$$\begin{array}{cc} & n \\ k(n-k+1) & \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \end{array}, \quad V_i = \operatorname{span}\{v_1, \dots, v_i\}$$

s.t.

$$\operatorname{rk} \begin{bmatrix} v_1 & \cdots & v_i & Nv_i \end{bmatrix} \leq i$$

$T = (\mathbb{C}^*)^k$ acts on $\Delta Sp_{n,k}$ by $t \cdot e_{i+km} = t_i\, e_{i+km}$

$\ \ \psi$

$t = (t_1, \dots, t_k)$

Ex $n=3$, $k=2$

Nilpotent: $\begin{array}{|cc|}\hline e_3 \to & e_1 \\ e_4 \to & e_2 \\ \hline\end{array} \begin{matrix}\to 0 \\ \to 0\end{matrix}$ $k=2$

$n-k+1=2$

$N\underline{e_3} = \underline{e_1}$, $N\underline{e_1} = 0$

$N\underline{e_4} = \underline{e_2}$, $N\underline{e_2} = 0$

$$N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of $V_\bullet \in \Delta Sp_{3,2}$

$V_1 = \langle e_1 \rangle$ $\quad NV_1 = 0 \subseteq V_1$ ✓

$V_2 = \langle e_1, e_2 \rangle$ $\quad NV_2 = 0 \subseteq V_2$ ✓

$V_3 = \langle e_1, e_2, e_3 \rangle$ $\quad NV_3 = \langle e_1 \rangle \subseteq V_3$ ✓

Rule: $NV_i \subseteq V_i \iff$ Build up the flag from right to left in each row

Conditions:

- $NV_i \subseteq V_i \quad \forall i$ ✓
- $V_3 \supseteq \langle e_1, e_2 \rangle$ ✓

$\begin{array}{|cc|}\hline e_3 \to & e_1 \\ e_4 \to & e_2 \\ \hline\end{array} \begin{matrix}\to 0 \\ \to 0\end{matrix}$ $\quad (\Delta Sp_{3,2})^T = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

$RSL_{3,2} = \left\{ \begin{array}{|c|c|}\hline 3 & 1 \\ \hline & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline & 1 \\ \hline 3 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline & 3 \\ \hline 2 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline & 2 \\ \hline 3 & 1 \\ \hline\end{array} \right\}$

Poincaré Series

Given a complex variety X, its Poincaré series is

$$\mathrm{Poin}(X;q) = \sum_{i \geq 0} (\dim H^i(X))\, q^{i/2}$$

An affine paving of X is a filtration by closed subvar's

$$X = X_n \supseteq X_{n-1} \supseteq X_{n-2} \supseteq \cdots \supseteq X_1 \supseteq \emptyset$$

Such that $\underbrace{X_i \setminus X_{i-1}}_{\text{"cells"}} \cong \mathbb{C}^{m_i}$ for some m_i.

Fact If X is compact and has an affine paving,

$$\mathrm{Poin}(X;q) = \sum_i q^{m_i}.$$

Thm (G-Levinson-Woo)

$\Delta Sp_{n,k}$ has an affine paving in which each cell C contains one torus-fixed point corresponding to some $P \in \mathcal{LD}_{n,k}$, and

$$\mathrm{codim}_{\Delta Sp_{n,k}}(C) = \mathrm{dinv}(P).$$

Therefore,

$$\mathrm{Poin}(\Delta Sp_{n,k};q) = \mathrm{rev}_q\left(\sum_{\substack{P \in \mathcal{LD}_{n,k} \\ \mathrm{area}(P)=0 \\ P \text{ standard}}} q^{\mathrm{dinv}(P)} \right).$$

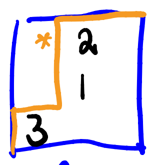
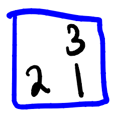
$$\mathrm{rev}_q(a_0 + a_1 q + \cdots + a_d q^d) = a_d + a_{d-1} q + \cdots + a_0 q^d$$

* Why care?

We can use different decompositions of $\Delta Sp_{n,k}$ to get different combinatorial formulas for $\Delta'_{e_{k-1}} e_n \big|_{t=0}$

Ex of thm

$n=3,\ k=2$

$P=$  $\leftrightarrow$  $\leftrightarrow$ $\begin{bmatrix} 1 & & 1 \\ & & \\ & 1 & \end{bmatrix}$ $\xrightarrow{\text{cell}}$ $C=\left\{\begin{bmatrix} a & b & 1 \\ 1 & & \\ & c & \\ & 1 & \end{bmatrix} : a,b,c\in\mathbb{C}\right\} \cap \Delta Sp_{3,2}$

$\mathcal{LD}_{3,2}$ $\qquad$ $RSL_{3,2}$ $\qquad$ $(Sp_{3,2})^T$

$\operatorname{dinv}(P)=0$

Schubert cell

$= \left\{ \begin{bmatrix} a & b & 1 \\ 1 & & \\ & a & \\ & 1 & \end{bmatrix} : a,b\in\mathbb{C}\right\}$

$\dim(C)=2 = 2-\operatorname{dinv}(P)$

Pawlowski–Rhoades proved $PR_{n,k}$ also has the same Poincaré poly:

$$\operatorname{Poin}(PR_{n,k};q) = \operatorname{rev}_q\left(\sum_{\substack{P\in\mathcal{LD}_{n,k}\\ \operatorname{area}(P)=0\\ P \text{ standard}}} q^{\operatorname{dinv}(P)}\right) = \operatorname{Poin}(\Delta Sp_{n,k};q)$$

$PR_{n,k}$: spanning line configs

Poincaré duality: If a complex variety X is smooth and compact, $\operatorname{Poin}(X,q) = \operatorname{rev}_q \operatorname{Poin}(X;q)$.

- $PR_{n,k}$ is smooth but noncompact
- $\Delta Sp_{n,k}$ is compact but not smooth.

Next Time: Use Springer Theory to construct an S_n action on $H^*(\Delta Sp_{n,k})$, give new formulas for $\Delta'_{e_{k-1}} e_n \big|_{t=0}$.