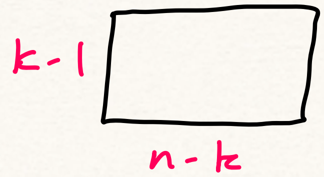
Recall goal:

$$\boxed{s^{\perp}_{\square}\left(\text{Rational Shuffle}\right) = \text{Delta}}$$



$K = k(n-k+1)$

$$s^{\perp}_{\square}\left(E_{K,k}\cdot 1\right) = \Delta'_{e_{k-1}} e_n$$

$$s^{\perp}_{\square}\left(\sum_{P\in WPF_{K,k}} q^{\mathrm{area}(P)}\, t^{\mathrm{dinv}(P)}\, x^{P}\right)$$

$$= \sum_{\pi\in WLD^{\mathrm{stack}}_{n,k}} q^{\mathrm{area}(\pi)}\, t^{\mathrm{dinv}(\pi)}\, x^{\pi}$$

Boiled it down last time to

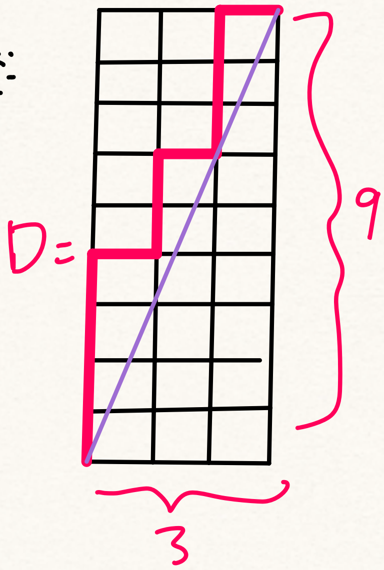
<u>Key Thm</u>: if $f_{D,\underline{b}} = \sum_{\pi_{big}\in WPF_b(D,\underline{b})} q^{\mathrm{tdinv}_{big}(\pi_{big})}\, x^{\pi_{big}}$

then $\langle f_{D,\underline{b}}, s_{\square}\rangle = \begin{cases} q^{c_{D,\underline{b}}} & \underline{b} \text{ admissible} \\ 0 & \text{else} \end{cases}$

Admissible: each $b_i \le n-k$ = slope -1

$$\sum b_i = K - n$$

Ex:



$k = 3$ $\quad K = 9$ $\quad n - k + 1 = 3$

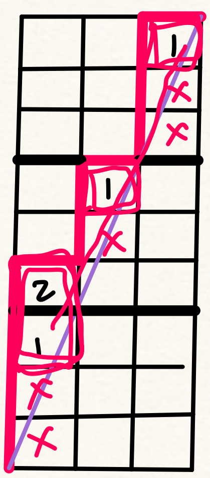
$n = 5$ $\quad \frac{K}{k}$

$\underline{b} = (2,1,1)$
$(1,2,1)$
$(2,0,2)$
⋮

are admissible

Ex for $\underline{b} = (2,1,1)$:

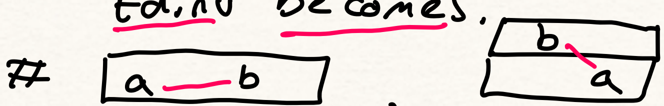
$\Pi_{big} =$



turn diagonals into rows:

tdinv = 2

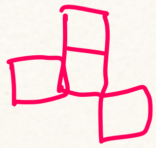
tdinv becomes:



$a < b$

Computing

$$\boxed{f_{D,\underline{b}} = \sum_{\pi_{big} \in WPF_b(D,\underline{b})} q^{tdinv_{big}(\pi_{big})} x^{\pi_{big}}}$$



for $\underline{D \text{ as above}}$, $\underline{\underline{b} = (2,1,1)}$:

$\begin{array}{lll} 2 & & \\ 1 & 1 & 1 \end{array}$ $\quad \underline{x_1^3 x_2}$

tdinv $= 2$

$\underline{q^2 m_{31}}$

$\begin{array}{lll} 2 & & \\ 1 & 2 & 2 \end{array}$ $\quad \underline{x_1 x_2^3}$

tdinv $= 2$

$\begin{array}{lll} 2 & & \\ 1 & 1 & 2 \end{array}$ $\qquad \begin{array}{lll} 2 & & \\ 1 & 2 & 1 \end{array}$

$(q^3 + q^2) m_{22}$

$\begin{array}{lll} 3 & & \\ 2 & 1 & 1 \end{array}$ $\quad \begin{array}{lll} 3 & & \\ 1 & 2 & 1 \end{array}$ $\quad \begin{array}{lll} 3 & & \\ 1 & 1 & 2 \end{array}$

$2 \qquad 3 \qquad 4$

$\begin{array}{lll} 2 & & \\ 1 & 3 & 1 \end{array}$ $\quad \begin{array}{lll} 2 & & \\ 1 & 1 & 3 \end{array}$

$2 \qquad 3$

$\underline{(2q^2 + 2q^3 + q^4) m_{211}}$

~~for content (1,1,1,1):~~

$\cancel{(q^5 + 3q^4 + 8q^3 + 8q^2 + 3q + 1) m_{1111}}$

So $f_{D,\underline{b}} = q^2 m_{31} + (q^3 + q^2) m_{22} + (2q^2 + 2q^3 + q^4) m_{211} + \cdots$

What is $\langle f_{D,\underline{b}}, \boxed{s_{\boxplus}} \rangle$?

→ Have f in terms of m_λ's, not s_λ's

→ $\boxed{\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}}$

→ $\boxed{\text{Write } s_{\boxplus} \text{ in terms of } h\text{'s!}}$

Jacobi – Trudi: $s_{\boxplus} = \boxed{\det \begin{pmatrix} h_2 & h_3 \\ h_1 & h_2 \end{pmatrix}}$

$$s_{(2,2,1)} = \det \begin{pmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & h_0 & h_1 \end{pmatrix}$$

$$= h_{2,2} - h_{3,1}$$

so $\langle s_{\boxplus}, f_{D,\underline{b}} \rangle = \langle h_{22} - h_{31}, q^2 m_{31} + (q^3 + q^2) m_{22} + \dots \rangle$

$$= (q^3 + q^2) - q^2$$

$$= q^3$$

$$c_{\underline{D},\underline{b}} = \text{maxtdinv}(D) - \text{pathdinv}(D) - d(\underline{s}, \underline{b})$$

$$= 13 - 8 - 2 = 3$$

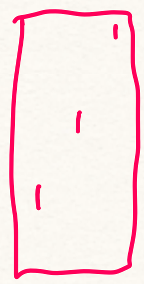
In general: write

$$\underline{s_{(k-1)^{n-k}}} = \det \underbrace{\begin{pmatrix} h_{k-1} & h_k & h_{k+1} & \cdots & \\ h_{k-2} & h_{k-1} & & & \\ \vdots & h_{k-2} & \ddots & & \\ & & \ddots & \ddots & \\ h_{2k-n} & & & h_{k-2} & h_{k-1} \end{pmatrix}}_{n-k}$$

<u>Claim:</u> We can set $\underline{h_{k+1}, h_{k+2}, \ldots = 0}$.

Indeed, no m_λ appears in $f_{D,b}$ with any $\lambda_i \geq k+1$ since each letter in P has at most multiplicity $k = \#$ cols.

So can rewrite matrix:



$$\begin{pmatrix} h_{k-1} & h_k & 0 & 0 & 0 & 0 \\ h_{k-2} & h_{k-1} & \ddots & 0 & 0 & 0 \\ & h_{k-2} & \ddots & \ddots & 0 & 0 \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & h_k \\ h_{2k-n} & & & & h_{k-2} & h_{k-1} \end{pmatrix}$$

$$h_\alpha \to h_{k-\alpha}$$

Complement matrix: take $k-\alpha \overset{\tilde{\alpha}}{=}$ for each subscript α above:

$$\begin{pmatrix} h_1 & h_0 & & 0 \\ \vdots & \ddots & \ddots & \\ & & \ddots & h_0 \\ h_{n-k} & \cdots & & h_1 \end{pmatrix} \quad k-(2k-n)$$

Ex: $n-k=3$:

$$\det\begin{pmatrix} h_1 & h_0 & 0 \\ h_2 & h_1 & h_0 \\ h_3 & h_2 & h_1 \end{pmatrix} = \underbrace{h_{111} - h_{201} - h_{120} + h_{300}}_{\text{allowable contents!}}$$

Jacobi-Trudi mod $h_{k+1}, \ldots = 0$

Lem: (GGG) $$s_{(k-1)^{n-k}} \text{"="} \sum_{\alpha \text{ allowable}} (-1)^{\operatorname{sgn}(\alpha)} \boxed{h_{\tilde{\alpha}}}$$

$\operatorname{sgn}(\alpha)$ = # 0's in α

complement

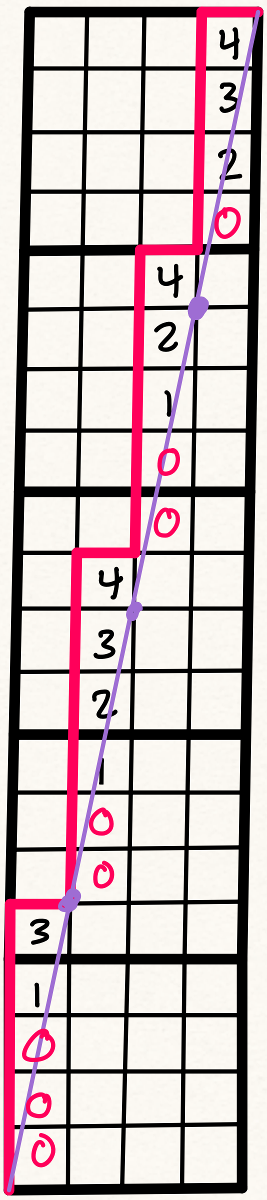
Cor: (GGG) $$\langle s_{(k-1)^{n-k}}, f_{D,\underline{b}} \rangle = \sum_{\alpha \text{ allowable}} \sum_{P \in P(D,\underline{b},\tilde{\alpha})} (-1)^{\operatorname{sgn}(\alpha)} q^{\operatorname{tdinv}_{big}(P)}$$

Big parking fns w/ content $\tilde{\alpha}$

Need: A bijective way to cancel + and − terms to get $q^{c_{D,\underline{b}}}$.

(Sign reversing involution!)

Setup:

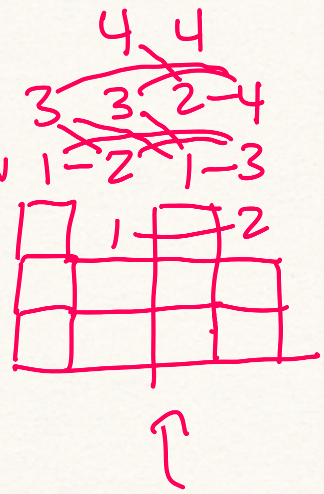


$K=20$

$k=4$

shrink $\longrightarrow$

move ith col down $\frac{K}{k}\cdot i$ steps



tdinv = 12

$b=(2,4,3,3)$

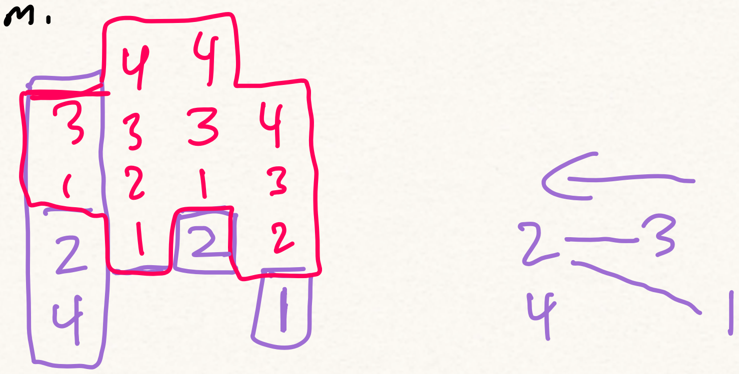
slope = 5

to go from $\tilde{\alpha}$ to α:

slope -1

Def: Complement col of G has complementary set from $\{1, 2, \ldots, n-k\}$ written below it, increasing top to bottom.

Ex:



Reading word: of complement is

3214

$\overline{\mathrm{Inv}}(\bar{P}) =$ # "tied" inversions of complement $\bar{P}$: count tied pairs.

(GGG)
Thm: $\mathrm{Inv}(P) + \overline{\mathrm{Inv}}(\bar{P})$ only depends on D, $\underline{b}$ (shape of shrink(P)).

Pf: consider 2 cols at a time, changing one letter in P, lots of casework $\square$

So suffices to find $\overline{\text{Inv}}$ preserving SRI on complements.

★ Reading words with allowable contents are called Fubini Words, A000670 on OEIS

IPAC 2025 audience!

Ex: Fubini words of length 3, and pairing:

$\alpha=(1,1,1) \to \alpha=(1,2,0)$

$\boxed{\circlearrowleft \; 123}$

132 ⟺ 122

213 ⟺ 113

231 ⟺ 131

312 ⟺ 212

321 ⟺ 221

311 ⟺ 111

Def: φ by:

→ Let i = max entry s.t. there is only one i
and if $j < i$ is next largest,
i left of every j.

→ Let m = largest repeated letter

Running ex: $\alpha = (2,0,1,3,0,0,1,1)$

$w = 3\,7\,4\,4\,1\,8\,4\,1$

$i = 7$
$m = 4$

φ

Case 1: If $i > m$ (or m does not exist)

replace i with j

Ex: $3\,4\,4\,4\,1\,8\,4\,1$ $\qquad \alpha = (2,0,1,4,0,0,0,1)$

$m = 4$
$i = 3$

φ

Case 2: If $i < m$ (or i does not exist)
replace first m w/ $t-1$ where
t = smallest letter $> m$ (or $t = n-k+1$)

Ex: $3\,7\,4\,4\,1\,8\,4\,1$

no repeats

Case 3: If no i or m, do nothing.

Ex: 12345678

Lemma: (GGG)

1. φ is a SRI
2. Preserves tied inversions
3. Unique fixed pt $123\ldots n\text{-}k$

Lem (GGG) φ restricts to a sign reversing involution on the complementary parking functions of a given shape (and therefore on the $D, \underline{b}$ parking function by taking the complement again) and preserves inversions.

Unique fixed pt for any $D, \underline{b}$, (as long as $\underline{b}$ admissible) $\boxed{Q_{D,\underline{b}}}$

Ex:

$$\begin{array}{cccc} & 4 & 4 & \\ 3 & 3 & 2 & 4 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 2 \\ 4 & & & 1 \end{array} \xrightarrow{\text{RW}} \underset{i=3}{3}214$$

$$\updownarrow \qquad\qquad \updownarrow \varphi$$

$$\begin{array}{cccc} & 4 & 4 & \\ 3 & 3 & 3 & 4 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 2 \\ 4 & & & 1 \end{array} \longleftarrow 2214$$

Fixed:

Q:

$$\begin{array}{cccc} & 4 & 4 & \\ 3 & 3 & 3 & 4 \\ 1 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 4 & & & 3 \end{array}$$

tdinv

$$\boxed{\text{Cor: } \langle s_{(k-1)^{n-k}}, f_{D,\underline{b}} \rangle = q^{\mathrm{tdinv}_{big}(Q_{D,\underline{b}})} \text{ if } \underline{b} \text{ admissible, } 0 \text{ else.}}$$

So we only need to show

$$\mathrm{tdinv}_{big}(Q_{D,\underline{b}}) = \boxed{c_{D,\underline{b}}}$$

Recall:

$$c_{D,\underline{b}} = \text{maxtdinv}(D) - \boxed{\text{pathdinv}(D)} - d(\underline{s},\underline{b})$$

<u>Simplification:</u> fill Q w/ 0's below in each col, then

$$\text{tdinv}(Q') = \text{tdinv}_{big}(Q) + d(\underline{s},\underline{b}):$$

Ex:

$$Q' = \begin{array}{cccc} & 4 & 4 & \\ 3 & 3 & 3 & 4 \\ 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & & \\ \hline \end{array} \rightarrow \text{tdinv}(Q') = \text{tdinv}_{big} + d(\underline{s},\underline{b})$$

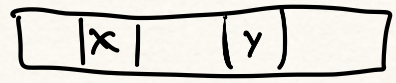
Want:

$$\text{tdinv}_{big}(Q_{D,\underline{b}}) = c_{D,\underline{b}}$$

$$\text{tdinv}_{big}(Q) = \text{maxtdinv}(D) - \text{pathdinv}(D) - d(\underline{s},\underline{b})$$

$$(*) \quad \boxed{\text{tdinv}(Q') - \text{maxtdinv}(D) + \text{pathdinv}(D) = 0}$$

Reinterpret pathdinv on shrunken diagram:

\# pairs of boxes 

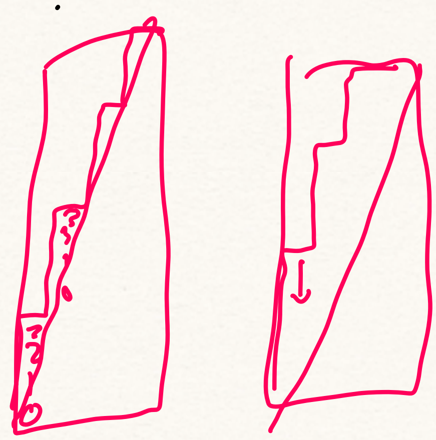
in same row s.t.

- y contains a label (possibly 0)
- x at most $n-k+1$ steps from top box of its col

Proof strategy: for $(*)$:

Show it holds for "base case":

4	4	4	4	4	4
3	3	3	3	3	3
2	2	2	2	2	2
1	1	1	1	1	1
0	0	0	0	0	0



Then reduce to this case from any other w/ simple moves, show that

tdinv − maxtdinv + pathdinv is unchanged

(see paper).