Combinatorial teaser

Def: Tuple $\alpha = (\alpha_1, \dots, \alpha_m)$ from $\{0, 1, 2, \dots, m\}$ is an allowable content if

- $\alpha_1 > 0$
- For any j s.t. $\alpha_j > 0$, the next $(\alpha_j) - 1$ entries are 0, and the entry after that is nonzero (if it exists)

Ex:

Tuple	Allowable?
20120	Yes
30040	NO - not enough 0's after 4
11111	Yes
30011	Yes
20113	No
20300	Yes
50000	Yes

Exercise: How many allowable contents of length m are there?

Def: Word w w/ allowable content α means there are α_i i's in w.

$w = 13414$ has content $(2, 0, 1, 2, 0)$
allowabe

Def: A (possibly tied) inversion is

$w = \cdots i \cdots j \cdots$

$i \geq j$

$\text{inv}(13414) = 4$

Key Lemma:

Exercise: Show $\displaystyle\sum_{\substack{w \text{ allowable} \\ \text{content length} \\ m}} (-1)^{\#0\text{'s in content}} q^{\text{inv}(w)} = 1$

Ex: $m=3$, contents $(1,1,1)$, $(2,0,1)$, $(1,2,0)$, $(3,0,0)$

$+$ $-$ $-$ $+$

$123 \to 1$	$113 \to -q$	$111 \to q^3$
$132 \to q$	$131 \to -q^2$	
$213 \to q$	$311 \to -q^3$	
$231 \to q^2$	$122 \to -q$	
$312 \to q^2$	$212 \to -q^2$	
$321 \to q^3$	$221 \to -q^3$	

$$\binom{3}{1,1,1} + \binom{3}{2,1} + \binom{3}{2,1} + \binom{3}{0}$$

$m=1 \to 1$

$m=2 \to 3$

$1, 3, 13$

Overview of lectures on skewing

Geometric interpretation of Rise Delta – Affine Δ-Springer fibers

Sean Griffin, June 12, 17, 24

$$\sum_{\pi \text{ "stack"}} t^{\mathrm{area}(\pi)} q^{\mathrm{hdinv}(\pi)} x^{\pi}$$

$$\sum_{\pi \text{ WPF}} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi)} x^{\pi}$$

Skewing Formula

$$\underbrace{\Delta'_{e_{k-1}}}_{\text{Delta}} e_n = \underline{s^{\perp}_{\boxplus}} (\underbrace{E_{K,k}}_{\text{Rational Shuffle}} \cdot 1)$$

$K = k(n-k+1)$

$\boxplus$ n-k, k-1

Maria, Jun 3

Maria, Jun 5, 10

Algebraic proof, using work from BHMPS '23 and Mellit '21 ✓

+

Combinatorial proof, using LLT's, dinv, sign reversing involution

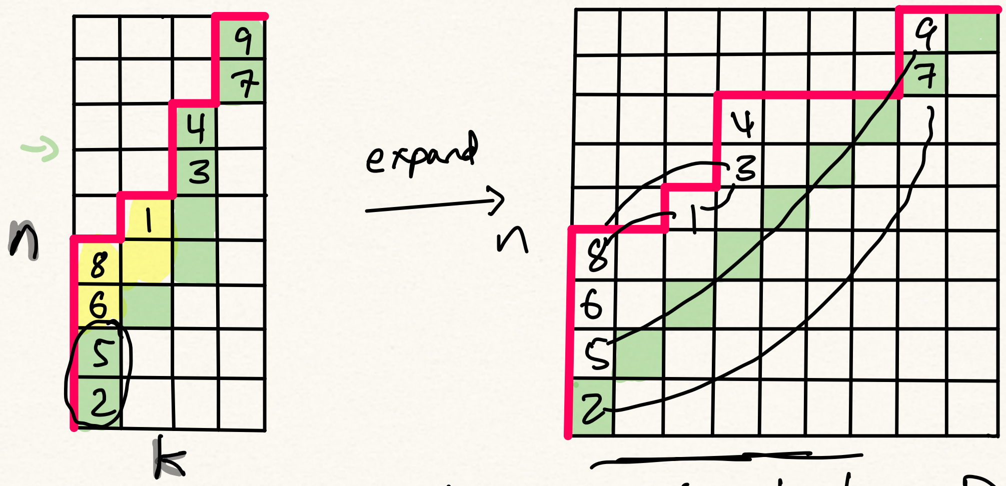
$\Rightarrow$ $\boxed{\text{New comb/alg proof of Rise Delta}}$ $\Leftarrow$

Combinatorics definitions

$\Delta'_{e_{k-1}} e_k$ $\quad WLD^{stack}_{n,k}$

① Delta

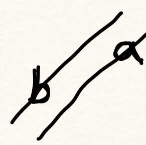
Stacked PF:



area = # boxes above stack, below D
= 4

hdinv = dinv of expansion

= # pairs 

or 

, $a<b$

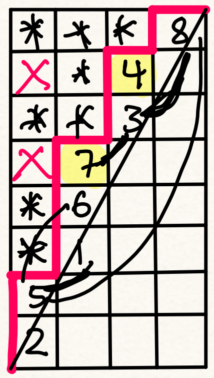
= 5

Rise Delta:

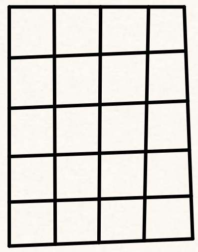
$$\Delta'_{e_{k-1}} e_n = \sum_{\pi \in WLD^{stack}_{n,k}} q^{area(\pi)} t^{hdinv(\pi)} x^{\pi}$$

② Rational Shuffle

$K=8$



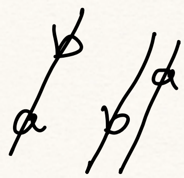
$k=4$



$$\frac{K}{k} = n-k+1 = 2 \Rightarrow n=5$$

area = # boxes whose interior is fully above diag

$= 2$



dinv = pathdinv + tdinv − maxtdinv

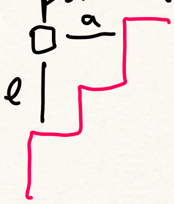
tdinv = # dinv's but on slope $\frac{K}{k}$ diagonals

$= 6$

maxtdinv = max dinv of any PF of same Dyck path $= 12$

pathdinv = # squares above path s.t. if $s = \frac{K}{k}$

leg $\in \{s\cdot arm-1,\ s\cdot arm,\ s\cdot arm+1, \ldots, s\cdot arm+s-1\}$



$= 8$

Ex: $s=2$,

$arm=0 \to leg = 0, 1$

$arm=1 \to leg = 1, 2, 3$

$arm=2 \to leg = 3, 4\ 5$

In ex, pathdinv = 8

so dinv = pathdinv + tdinv − maxtdinv
$= 8+6-12 = 2$

Connection:

Make a stack by removing labels $> n$

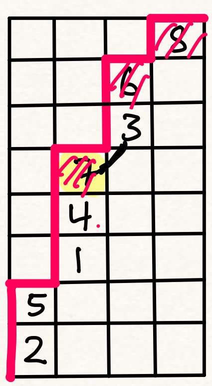
stack heights: If b_1, b_2, b_3, b_4 = # big labels in cols, 0, 1, 1, 1

admissible if $b_i \le n-k = s-1$ $\forall i$, set $h_i = s - b_i$

Ex:

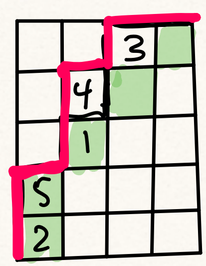
$K=8$

$n=5$



$k=4$

remove >5 "big"



$\boxed{\text{Need to generalize to } \underline{\text{word}} \text{ parking functions!}}$

<u>Def:</u> D – (k,k) Dyck path

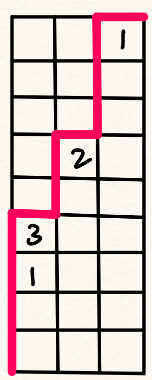
$\underline{b}$ – seq w/ $b_i \leq$ # labels in col i

"big letters"

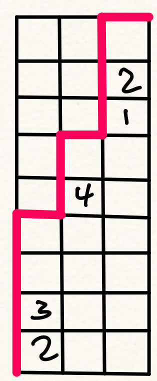
$$\boxed{\sum b_i = k-n}$$

Define $\underline{WPF_b(D,\underline{b})} = \left\{ \begin{array}{l} \text{col strict labelings of} \\ \text{top } b_i \text{ boxes in } i\text{'th col} \end{array} \right\}$

$\underline{WPF_s(D,\underline{s})}$ = labelings of other (small) boxes



$\in WPF_b(D,(2,1,1))$



$\in WPF_s(D,(2,1,2))$

$\underline{b}$ <u>admissible</u> if $b_i \leq n-k \ \forall i$

$\underline{s}$ <u>admissible</u> if its complement is.

<u>Def:</u> $F : \bigcup_{\substack{D,\underline{s} \\ \text{admissible}}} \underline{WPF_s(D,\underline{s})} \longrightarrow \underline{WLD^{\text{stack}}_{n,k}}$

- <u>Delete unlabeled boxes</u>
- <u>Stack heights $h_i = s - b_i$.</u>

Facts (GGG)

1. F is a well-defined bijection
2. $\mathrm{area}(\pi) = \mathrm{area}(F(\pi))$
3. $\mathrm{hdinv}(F(\pi)) = \mathrm{tdinv}_{small}(\pi)$

Now define LLT's for big and small:

Def:

$$f_{D,\underline{b}} = \sum_{\pi_{big} \in WPF(D,\underline{b})} q^{\mathrm{tdinv}_{big}(\pi_{big})} x^{\pi_{big}}$$

$$f_{D,\underline{s}} = \sum_{\pi_{small} \in WPF(D,\underline{s})} q^{\mathrm{tdinv}_{small}(\pi_{small})} x^{\pi_{small}}$$

Def: $d(\underline{s},\underline{b})$ = # attacking tdinv pairs btwn big and small boxes $(b>s)$

so for a standard PF π,

$$\mathrm{tdinv}(\pi) = \mathrm{tdinv}_{small}(\pi) + d(\underline{s},\underline{b}) + \mathrm{tdinv}_{big}(\pi)$$

Def: $\boxed{c_{D,\underline{b}}} = \mathrm{maxtdinv}(D) - \mathrm{pathdinv}(D) - d(\underline{s},\underline{b})$

Key Thm (GGG)

$$\langle f_{D,\underline{b}}, s_{(k-1)^{n-k}} \rangle = \begin{cases} q^{c_{D,\underline{b}}} & \text{if } \underline{b} \text{ admissible} \\ 0 & \text{else} \end{cases}$$

$\boxed{\text{Will prove next time!}}$

How to finish the proof from Key Thm:

Main Thm (GGG)

$$\boxed{s^{\perp}_{(k-1)^{n-k}}\left(E_{k,k}\cdot 1\right) = \Delta'_{e_{k-1}} e_n}$$

holds "combinatorially," i.e.

$$s^{\perp}_{(k-1)^{n-k}} \underbrace{\sum_{\pi\in WPF_{k,k}} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi)} x^{\pi}}$$

$$= \underbrace{\sum_{P\in WLD^{\mathrm{stack}}_{n,k}} t^{\mathrm{area}(P)} q^{\mathrm{hdinv}(P)} x^{P}}$$

Pf: Write $f_D = \sum_{w\in WPF_{k,k}(D)} q^{\mathrm{tdinv}(\pi)} x^{\pi}$

Then $f_D[X+Y] = \sum_{\underline{b},\underline{s}} q^{d(\underline{s},\underline{b})} f_{D,\underline{s}}[X]\cdot f_{D,\underline{b}}[Y]$.

Plethystic: $f_D(x_1, x_2, \dots, y_1, y_2, \dots)$

So $\boxed{s^{\perp}_{\square} f_D(X)} = \langle \underbrace{s_{\square}(Y), f_D[X+Y]} \rangle$

$$= \sum_{\underline{b},\underline{s}} q^{d(\underline{s},\underline{b})} f_{D,\underline{s}}(X) \langle \underbrace{s_{\square}(Y), f_{D,\underline{b}}[Y]} \rangle$$

$$\overset{\text{Key}}{=} \sum_{\substack{\underline{b},\underline{s} \\ \text{admissible}}} q^{d(\underline{s},\underline{b}) + c_{D,\underline{b}}} f_{D,\underline{s}}(X)$$

$$= \boxed{\sum_{\substack{b,\underline{s} \\ \text{admissible}}} q^{\boxed{\text{maxtdinv}(D) - \text{pathdinv}(D)}} f_{D,\underline{s}}}$$

So $s^{\perp}_{\square}(E_{K,k} \cdot 1)$

$$= s^{\perp}_{\square}\left(\sum_{\pi \in WPF_{K,k}} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} x^{\pi} \right)$$

$= \text{pathdinv} + \text{tdinv} - \text{maxtdinv}$

$$= s^{\perp}_{\square} \sum_{D} t^{\text{area}(D)} q^{\text{pathdinv} - \text{maxtdinv}} f_D$$

$$= \sum_{D} t^{\text{area}(D)} q^{\boxed{\text{pathdinv}(D) - \text{maxtdinv}(D)}} s^{\perp}_{\square} f_D$$

$$= \sum_{\substack{D,\underline{b} \\ \text{admissible}}} t^{\text{area}(D)} f_{D,\underline{s}}$$

$$= \sum_{\substack{D,\underline{b} \\ \text{adm.}}} \sum_{\pi \in WPF_{\underline{s}}(D,\underline{s})} t^{\text{area}(D)} q^{\text{tdinv}_{\text{small}}(\pi)} x^{\pi}$$

$$\downarrow F \qquad \downarrow F$$

$$= \sum_{F(\pi) \in WLD^{\text{stack}}_{n,k}} t^{\text{area}(F(\pi))} q^{\text{hdinv}(F(\pi))} x^{F(\pi)}$$

$$= \sum_{P \in WLD^{\text{stack}}_{n,k}} t^{\text{area}(P)} q^{\text{hdinv}(P)} x^{P}$$

$$= \Delta'_{e_{k-1}} e_n \qquad \text{QED}$$

How did we notice skewing?

Prior work: (GG): skewing formula for $\left(\Delta'_{e_{k-1}} e_n\right)\big|_{t=0}$ = Schur expansion in terms of "battery powered tableaux"

Generalized a skewing formula for $\tilde{H}_\mu(x;q) = \mathrm{Frob}_q(R_\mu)$

$e_n^\perp$ $\qquad$ $e_n =$  $= s$

Δ-springer story at $t=0$

$$\begin{tikzcd}
 & R_{n,\mu,k} \arrow[dl, shift right] \arrow[dr] & \\
R_\mu \arrow[ur, shift right] \arrow[d] \arrow[dr] & & R_{n,k} \arrow[dl] \\
\tilde{H}_\mu & R_n &
\end{tikzcd}$$

Schur exp

$R_{n,k} \to t=0$ Δ conjecture

$$R_n = \mathbb{C}[x_1, \ldots, x_n] / (e_1, \ldots, e_n)$$

<u>Cor</u>: Any Schur expansion for $E_{k,k} \cdot 1$ will automatically give a Schur expansion for $\Delta'_{e_{k-1}} e_n$.

Pf: $s^{\lambda}_{\Box}$(schur-y) = skew all Schurs.