A combinatorial skewing formula for the Rise Delta Theorem

All joint with Sean Griffin and Eugene Gorsky

Thanks to NSF

Shuffle Theorem (Carlsson, Mellit)

$$\nabla e_n = \sum_{\substack{\pi \text{ word parking} \\ \text{function}}} q^{\operatorname{area}(\pi)} t^{\operatorname{dinv}(\pi)} x^{\pi}$$

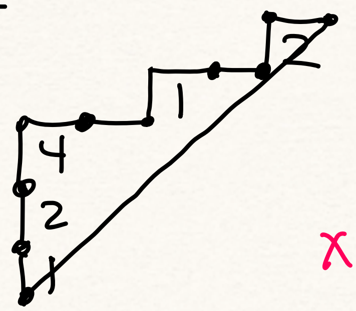
∇ - Macdonald eigenoperator

e_n - symmetric function in $x_1, x_2, x_3, \ldots$

Ex: $e_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots + x_{17} x_{100} x_{107} + \cdots$

Parking function: Column labeling of

Dyck path:

$\pi =$ 

$\leftarrow$ stays above $y = x$

$x^{\pi} = x_1^2 x_2^2 x_4$

area, dinv - combinatorial statistics

Geometric connections: To affine Springers (Hikita)

Hilbert schemes (Haiman)

Two generalizations of Shuffle:

Haglund, Remmel, Wilson

① Delta: Δ_f generalizes ∇,

$\nabla e_n = \Delta_{e_{n-1}} e_n$, $\Delta_{e_k} e_n = ?$

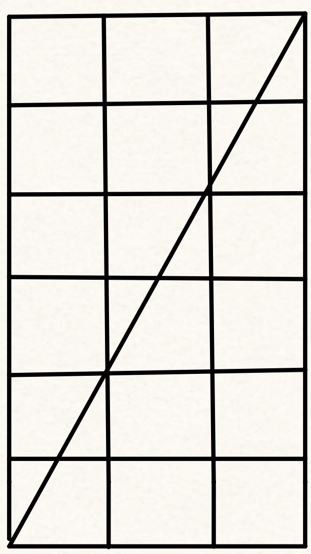
$\Delta'_{e_k} e_n + \Delta'_{e_{k-1}} e_n$

→ Rise, Valley: two combinatorial formulas in terms of parking function generalizations for $\Delta_{e_k} e_n$

→ No known geometric interpretation previously

→ Rise proven in BHMPS '21

② Rational Shuffle: (Bergeron, Garsia, Leven, Xin)



Operators from Elliptic Hall Algebra

$E_{k,k} \cdot 1$

(Mellit)

Overview of next 6 lectures

Geometric interpretation of Rise Delta – Affine Δ-Springer fibers

Sean Griffin, June 12, 17, 24

$$\sum_{\pi \text{ "stack"}} t^{\mathrm{area}(\pi)} q^{\mathrm{hdinv}(\pi)} x^{\pi}$$

$$\sum_{\pi \text{ WPF}} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi)} x^{\pi}$$

Skewing Formula

$$\underbrace{\Delta'_{e_{k-1}} e_n}_{\text{Delta}} = s^{\perp}_{\boxplus}\big(\underbrace{E_{K,k} \cdot 1}_{\text{Rational Shuffle}}\big)$$

$K = k(n-k+1)$

$\boxplus$ $n-k$, $k-1$

Maria, Jun 3

Maria, Jun 5, 10

Algebraic proof, using work from BHMPS '23 and Mellit '21

+

Combinatorial proof, using LLT's, dinv, sign reversing involution

⇓

$\boxed{\text{New comb/alg proof of Rise Delta}}$

Algebraic proof strategy: Reduce to k vars

Background on skewing:

Λ - ring of symmetric functions in $x_1, x_2, x_3, \ldots$ over $\mathbb{Q}(q,t)$

Λ_k - symmetric polynomials in $x_1, \ldots, x_k$

$\pi_k : \Lambda \longrightarrow \Lambda_k$ by

$f(x_1, x_2, \ldots) \longmapsto f(x_1, \ldots, x_k, 0, 0, 0, \ldots)$

Bases: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ partition

- Elementary: $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r}$

  Ex: $e_{(2,1)} = e_2 e_1 = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3)$ in Λ_3

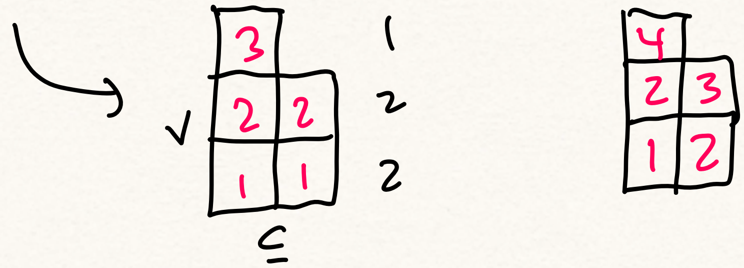
- Homogeneous: $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_r}$

  Ex: $h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$

- Monomial: $m_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$

• **Schur:** $s_\lambda = \sum_{\substack{T \text{ semistandard} \\ \text{shape } \lambda}} x^T$

$s_{(2,2,1)} = x_1^2 x_2^2 x_3 + x_1 x_2^2 x_3 x_4 + \dots$

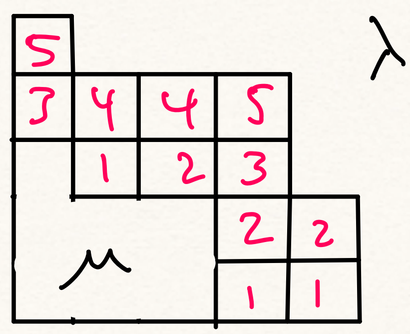


In Λ_k: s_λ for $\ell(\lambda) \leq k$

length; "height"

Skew Schurs

$s_{\lambda/\mu}$:



$x_1^3 x_2^3 x_3^2 x_4^2 x_5^2 + \dots$

Hall Inner Product: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

$$= \begin{cases} 0 & \lambda \neq \mu \\ 1 & \lambda = \mu \end{cases}$$

On Λ_k:

$$\langle s_\lambda, s_\mu \rangle_k = \delta_{\lambda\mu} \quad \text{if } \ell(\lambda), \ell(\mu) \leq k.$$

Extend by linearity.

Adjoint/skewing: $s_\lambda^\perp : \Lambda \to \Lambda$

$s_\lambda^{\perp,k} : \Lambda_k \to \Lambda_k$

defined by

$$\langle f, s_\lambda g \rangle = \langle s_\lambda^\perp f, g \rangle,$$

$$\langle f, s_\lambda g \rangle_k = \langle s_\lambda^{\perp,k} f, g \rangle$$

Fact: $\langle s_\lambda, s_\mu s_\nu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle$

so $s_\mu^\perp s_\lambda = s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\lambda$ ← LR coeff

$$s_\mu^\perp \left(\sum_\lambda c_\lambda s_\lambda \right) = \sum_\lambda c_\lambda s_{\lambda/\mu}$$

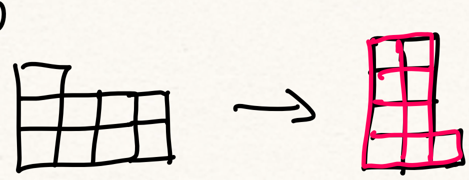
Want: $s_{\square}^{\perp}(E_{k,k}\cdot 1) = \Delta'_{e_{k-1}} e_n$

$\boxed{\text{Snag: Don't have Schur expansions for Rational Shuffle or Delta!}}$

Tricks:

① Apply ω involution

$\omega s_\lambda = s_{\lambda'}$



$\omega e_\lambda = h_\lambda$ $\qquad \langle \omega f, \omega g \rangle = \langle f, g \rangle$

② Note from BHMPS and Mellit that $\omega E_{k,k}\cdot 1$ and $\omega \Delta(n,k)$ only have Schurs with $\ell(\lambda) \le k$

③ Reduce to k vars to prove it

④ Use GL_k rep theory to get a handle on $s_\lambda^{\perp, k}$

Want: $\quad s^{\perp}_{(k-1)^{n-k}}\left(E_{k,k}(1)\right) = \Delta'_{e_{k-1}} e_n$

ω: $\quad s^{\perp}_{(n-k)^{k-1}}\left(\omega E_{k,k}(1)\right) = \omega \Delta'_{e_{k-1}} e_n$

π_k: $\quad s^{\perp,k}_{(n-k)^{k-1}}\left(\pi_k \omega E_{k,k}(1)\right) = \pi_k \omega \Delta'_{e_{k-1}} e_n$

__Ex:__ $\quad E_{(4,2)}(1) = s_{(2,2)} + (t+q)\, s_{(2,1,1)} + (t^2+tq+q^2)\, s_{(1,1,1,1)}$

$$\omega E_{(4,2)}(1) = s_{(2,2)} + (t+q)\, s_{(3,1)} + (t^2+tq+q^2)\, s_{(4)}$$

$$s^{\perp}_{(1)}\, \omega E_{(4,2)}(1) = s_{(2,2)/(1)} + (t+q)\, s_{(3,1)/(1)} + (t^2+tq+q^2)\, s_{(4)/(1)}$$

$s_{(2,2)/(1)} \to s_{(2,1)}$

$s_{(3,1)/(1)} \to s_{(3)} + s_{(2,1)}$

$\Longrightarrow$

$$\Delta'_{e_{k-1}} e_n = (1+t+q)\, s_{(2,1)} + (t+q+t^2+tq+q^2)\, s_{(3)}$$

$\boxed{\text{To prove: need to go "outside" } \Lambda_k \ldots}$

Def: $\Lambda_k^{\pm}$ = symmetric Laurent polynomials!

Ex: $f = 2x_1 + 2x_2 + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_1} \in \Lambda_2^{\pm}$

Basis: $\{(x_1 \cdots x_k)^{-d} s_\lambda : d > 0,\ \ell(\lambda) < k\}$
$\cup \{s_\lambda : \ell(\lambda) \leq k\}$

Note: $(x_1 \cdots x_k)^{-d} s_\lambda = ch(\underbrace{det^{-s} \otimes V_\lambda}_{GL_k\text{-rep}})$

Def: If $f = \sum_{\substack{d>0 \\ \ell(\lambda)<k}} c_{d,\lambda} (x_1 \cdots x_k)^{-d} s_\lambda + \sum_{\substack{\lambda \\ \ell(\lambda)<k}} c_\lambda s_\lambda$

then $f_{pol} = \sum_\lambda c_\lambda s_\lambda$

Ex: $f = 2x_1 + 2x_2 + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}$. Is $f_{pol} = 2x_1 + 2x_2$?

NO!

$$\frac{s_{\square\square\square}}{x_1 x_2} = \frac{\overset{\boxed{1|1|1}}{x_1^3} + \overset{\boxed{1|1|2}}{x_1^2 x_2} + \overset{\boxed{1|2|2}}{x_1 x_2^2} + \overset{\boxed{2|2|2}}{x_2^3}}{x_1 x_2}$$

$$= \frac{x_1^2}{x_2} + \frac{x_2^2}{x_1} + x_1 + x_2$$

$$f = \frac{s_{\square\square\square}}{x_1 x_2} + (x_1 + x_2) = \frac{s_{\square\square\square}}{x_1 x_2} + s_{\square}$$

So $\quad f_{pol} = s_{\square}$

<u>Prop</u>: For $f \in \Lambda_k$, $g \in \Lambda_k^{\pm}$:

$$\langle f, g_{pol} \rangle_k = \langle x^0 \rangle f(x_1^{-1}, \ldots, x_k^{-1})\, g(x_1, \ldots, x_k)$$

<u>Pf sketch</u>: From GL_k rep theory,

$$\langle ch(U), ch(V) \rangle = \dim(U^* \otimes V) \qquad \square$$

<u>Cor</u>: $s_\lambda^{\perp, k} f = \left(s_\lambda(x_1^{-1}, \ldots, x_k^{-1})\, f(x_1, \ldots, x_k) \right)_{pol}$

(Pf: show both sides have same inner product with any g)

Pf outline (notation from BHMPS)

Def: $f(x_1,\dots,x_k)$ Laurent series,

$$\sigma(f) = \sum_{w\in S_k} w\left(\frac{f}{\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)}\right)$$

$k=2$

Ex: $\sigma(x_1) = \dfrac{x_1}{1-\frac{x_2}{x_1}} + \dfrac{x_2}{1-\frac{x_1}{x_2}}$

$$= x_1\left(1+\frac{x_2}{x_1}+\frac{x_2^2}{x_1^2}+\cdots\right) + x_2\left(1+\frac{x_1}{x_2}+\frac{x_1^2}{x_2^2}+\cdots\right)$$

$$= 2x_1 + 2x_2 + \frac{x_2^2}{x_1} + \frac{x_1^2}{x_2} + \cdots$$

Def: $$H^k_{q,t}(f) = \sigma\left(\frac{f\prod_{i<j}\left(1-qt\frac{x_i}{x_j}\right)}{\prod_{i<j}\left(1-q\frac{x_i}{x_j}\right)\left(1-t\frac{x_i}{x_j}\right)}\right)$$

$$\boxed{\text{Both Delta and Rational Shuffle are } H^k_{qt}\text{'s of things for } k \text{ vars}\ldots}$$

<u>Lemma</u> (follows immediately from BHMPS)

$$\pi_k(\omega \Delta'_{e_{k-1}} e_n) = H^k_{q,t}\left(\frac{x_1\cdots x_k\, h_{n-k}(x_1,\dots,x_k)}{\prod_{i\le k-1}\left(1-qt\frac{x_i}{x_{i+1}}\right)}\right)_{pol}$$

<u>Ex</u>: $k=2$, $n=3$

$$\pi_2(\omega \Delta'_{e_1} e_3) = H^2_{q,t}\left(\frac{x_1x_2(x_1+x_2)}{1-qt\frac{x_1}{x_2}}\right)_{pol}$$

$$= \sigma\left(\frac{x_1x_2(x_1+x_2)}{\left(1-q\frac{x_1}{x_2}\right)\left(1-\frac{tx_1}{x_2}\right)}\right)_{pol}$$

Sanity check: at $q=t=0$ should be $s_{(2,1)}$:

$$\underset{q=t=0}{=} \sigma(x_1x_2(x_1+x_2))_{pol}$$

$$= \frac{x_1x_2(x_1+x_2)}{1-\frac{x_1}{x_2}} + \frac{x_1x_2(x_1+x_2)}{1-\frac{x_2}{x_1}}$$

$$= x_1x_2(x_1+x_2)\left(\begin{array}{l}1+\frac{x_1}{x_2}+\frac{x_1^2}{x_2^2}+\cdots \\ +1+\frac{x_2}{x_1}+\frac{x_2^2}{x_1^2}+\cdots\end{array}\right)$$

$$= 3x_1^2x_2 + 3x_2^2x_1 + 2x_1^3 + 2x_2^3 + 2\frac{x_1^4}{x_2} + 2\frac{x_2^4}{x_1} + \frac{2x_1^5}{x_2^2} + \cdots$$

Basis of Schurs times $(x_1x_2)^{-d}$:

$$s_{\boxplus} = x_1x_2\, s_{\square} = x_1^2x_2 + x_2x_1^2$$

$$s_{\square\square\square} = x_1^3 + x_1^2x_2 + x_2x_1^2 + x_1^3$$

$$(x_1x_2)^{-1} s_{\square\square\square\square\square} = \frac{x_1^4}{x_2} + x_1^3 + x_1^2x_2 + x_1x_2^2 + \frac{x_2^4}{x_1}$$

$$(x_1x_2)^{-2} s_7 = \frac{x_1^5}{x_2^2} + \quad '' \quad '' \quad '' \quad '' \quad + \frac{x_2^5}{x_1^2}$$

Note: $$(x_1x_2)^{-2}s_7 - (x_1x_2)^{-1}s_5 = \frac{x_1^5}{x_2^2} + \frac{x_2^5}{x_1^2}$$

So our sum is:

$$3s_{\boxplus} + 2\left(s_{\square\square\square} - s_{\boxplus}\right) + 2\left(\frac{s_5}{x_1x_2} - s_3\right) + 2\left(\frac{s_7}{(x_1x_2)^2} - \frac{s_5}{x_1x_2}\right) + \cdots$$

$$= s_{\boxplus}$$

Lemma: From work of Mellit, 2021 on elliptic Hall alg and Shuffle alg:

$$\pi_k(\omega E_{K,k} \cdot 1) = H^k_{q,t}\left(\frac{x_1^{n-k+1} \cdots x_k^{n-k+1}}{\prod\left(1-qt\frac{x_i}{x_{i+1}}\right)}\right)_{pol}$$

Ex: $k=2$, $n=3$, $\cancel{n}-k+1=2$ so $K=4$.

$$\pi_2(\omega E_{4,2} \cdot 1) = H^2_{q,t}\left(\frac{x_1^2x_2^2}{1-qt\frac{x_1}{x_2}}\right)_{pol}$$

$$= \sigma\left(\frac{x_1^2x_2^2}{\left(1-q\frac{x_1}{x_2}\right)\left(1-t\frac{x_1}{x_2}\right)}\right)_{pol}$$

at $q=t=0$:

$$= 2x_1^2x_2^2 + x_2^3x_1 + x_1^3x_2 + x_2^4 + x_1^4 + \frac{x_2^5}{x_1} + \frac{x_1^5}{x_2} + \cdots$$

$$= 2s_{22} + (s_{31} - s_{22}) + (s_4 - s_{31}) + \left(\frac{s_5}{x_1x_2} - s_4\right) + \cdots$$

$$= \boxed{s_{22}}$$

Delta: $\pi_k(\omega \Delta'_{e_{k-1}} e_n) = H^k_{q,t}\left(\dfrac{x_1\cdots x_k\, h_{n-k}(x_1,\dots,x_k)}{\prod_{i\le k-1}\left(1-q+\frac{x_i}{x_{i+1}}\right)}\right)_{pol}$

Rational: $\pi_k(\omega E_{k,k}\cdot 1) = H^k_{q,t}\left(\dfrac{x_1^{n-k+1}\cdots x_k^{n-k+1}}{\prod\left(1-q+\frac{x_i}{x_{i+1}}\right)}\right)_{pol}$

Connection Lemma:

$$s_{(n-k)^{k-1}}(x_1^{-1},\dots,x_k^{-1}) = \frac{h_{n-k}(x_1,\dots,x_k)}{x_1^{n-k}\cdots x_k^{n-k}}$$

Pf ex:

$$\begin{array}{c|c|c|c|c|c}
 & 4 & 2 & 2 & 2 & 1 \\
\hline
 & 3 & 4 & 4 & 4 & 4 \\
\hline
k-1\left\{\right. & 2 & 3 & 3 & 3 & 3 \\
\hline
 & 1 & 1 & 1 & 1 & 2 \\
\hline
 & \multicolumn{5}{c}{\underbrace{\qquad\qquad\qquad}_{n-k}}
\end{array}
\longrightarrow x_1^{-4}x_2^{-2}x_3^{-5}x_4^{-4} \text{ from schur}$$

$$\cdot\, x_1^5 x_2^5 x_3^5 x_4^5$$

$$= x_1 x_2^3 x_4 \quad \text{from } h_5$$

Pf of skewing: $\varphi = (x_1\cdots x_k)^{n-k+1}/\prod\left(1-q+\frac{x_i}{x_{i+1}}\right)$

$$\left\langle s_\lambda, s^{\perp,k}_{(n-k)^{k-1}}\left(H^k_{q,t}(\varphi)_{pol}\right)\right\rangle = \left\langle s_\lambda s_{(n-k)^{k-1}}, H^k_{q,t}(\varphi)_{pol}\right\rangle$$

$$= \langle x^0\rangle\, s_\lambda(\underline{x}^{-1})\cdot \frac{h_{n-k}}{(x_1\cdots x_k)^{n-k}} H^k_{q,t}(\varphi)_{pol}$$

$$= \left\langle s_\lambda, H^k_{q,t}\left(\frac{\varphi\, h_{n-k}}{(x_1\cdots x_k)^{n-k}}\right)\right\rangle \qquad \square$$