Sagan-Swanson basis of SR

(Angarone-Commins-Karn-Murai-R)

Last Time

$$f_J = \prod_{j \in J} x_j \left(\prod_{j < k \leq n} (x_j - x_k) \right) \in S$$

Basis Transfer

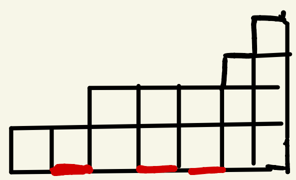
Suppose $\forall \; J \subseteq [n]$, $B_J \subseteq S$ is a set of polynomials.

Let $\mathcal{B} = \bigsqcup_J B_J \theta_J \subseteq \Omega$.

B_J is a basis of $S/(I : f_J)$ for all $J \subseteq [n]$ $\Rightarrow$ $\mathcal{B}$ is a basis of SR

Recall $\quad \mathrm{Hilb}(S/(I : f_J) ; q) = \prod_{i=1}^{n} [st_i(J)]_q$

$n = 7 \qquad J = \{2\,4\,5\}$



$st(J) = 1\; 1\; 2\; 2\; 2\; 3\; 4$

Conj: Sagan-Swanson

Thm (ACKMR) Let $M_J = \{x_1^{a_1} \cdots x_n^{a_n} : a_i < st_i(J) \text{ for all } i\}$,

$M := \bigsqcup_J M_J \cdot \theta_J$. Then M descends to a basis of SR.

⌐ we'll show M_J is linearly indep. in $S/(I : f_J)$ ⌐

$\mathcal{A}$: arrangement in $\mathbb{C}^n$.

$S = \mathbb{C}[x_1, \dots, x_n] = \{\text{polynomial fcns } \mathbb{C}^n \to \mathbb{C}\}$

$\mathrm{Der}(S) = \bigoplus_{i=1}^{n} S\,\partial_i = \{\text{poly. vector fields on } \mathbb{C}^n\}$

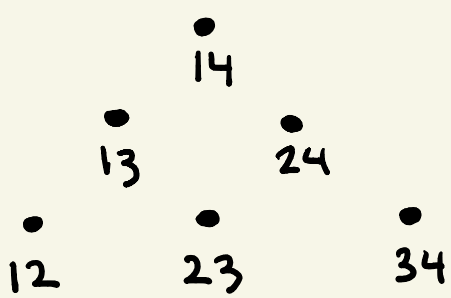
$\cup$

$\mathrm{Der}(\mathcal{A}) = \{\psi \in \mathrm{Der}(S) : \alpha_H \mid \psi(\alpha_H) \text{ for all } H \in \mathcal{A}\}$

S-module $= \{\text{vector fields parallel to } \mathcal{A}\}$

$\mathcal{A}$ is <u>free</u> if $\mathrm{Der}(\mathcal{A})$ is a free S-module

$\mathcal{B} = \{x_i = x_j : 1 \leq i < j \leq n\}$ BRAID ARRANGEMENT



$\mathcal{B}$ is free, basis

$\{x_1^{k-1}\partial_1 + \cdots + x_n^{k-1}\partial_n\}_{k=1}^{n}$

<u>Def</u> The **exponents** $\exp(\mathcal{A}) = \{e_1, \dots, e_n\}$ of a free arrangement $\mathcal{A}$ are the degree multiset of a homog. basis $\psi_1, \dots, \psi_n$ of $\mathrm{Der}(\mathcal{A})$.

<u>Ex</u> $\exp(\mathcal{B}) = \{0, 1, \dots, n-1\}$

$\mathcal{A}$: arr in $\mathbb{C}^n$ $\qquad \mathrm{Der}(\mathcal{A}) \subseteq \mathrm{Der}(S) = \bigoplus_{i=1}^{n} S\cdot\partial_i$

Def (≈ Abe - Maeno - Murai - Numata)

Let $d \geq 0$ and $\mathfrak{a}(\partial_1), \dots, \mathfrak{a}(\partial_n) \in S_d$. Let $\mathfrak{a}: \mathrm{Der}(S) \to S$, $\partial_i \mapsto \mathfrak{a}(\partial_i)$

The **Solomon-Terao ideal** is $\mathfrak{a}_{\mathcal{A}} := \mathfrak{a}(\mathrm{Der}(\mathcal{A})) \subseteq S$.

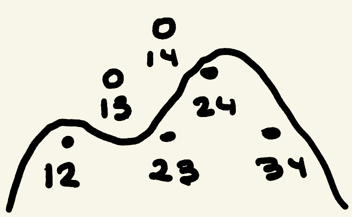
The **Solomon-Terao algebra** is $ST(\mathcal{A};\mathfrak{a}) = S/\mathfrak{a}_{\mathcal{A}}$.
(graded)

Ex $\mathcal{B}$ = braid arr. $\qquad \mathfrak{a}: \mathrm{Der}(S) \to S$, $\partial_i \mapsto x_i$.

$\mathrm{Der}(\mathcal{B})$ basis $x_1^{k-1}\partial_1 + \dots + x_n^{k-1}\partial_n \quad (k = 1, \dots, n)$

$\mathfrak{a}_{\mathcal{B}} = (p_1, p_2, \dots, p_n) = I$. $\qquad ST(\mathcal{B};\mathfrak{a}) = R$.

Ex $\mathcal{A}_h$ = ideal subarr of $\mathcal{B}$ $\qquad \mathfrak{a}: \mathrm{Der}(S) \to S$, $\partial_i \mapsto x_i$

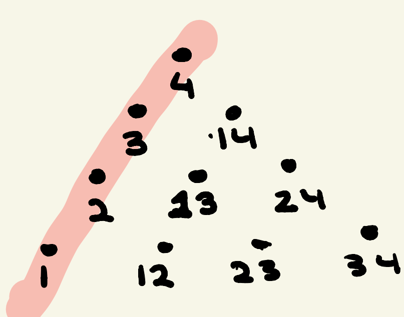


Abe - Horiguchi - Masuda - Murai - Sato

$ST(\mathcal{A}_h;\mathfrak{a}) = H^\bullet$ (regular nilp. Hessenberg variety for h).

$\tilde{\mathcal{B}}$ = extended braid arrangement $= \mathcal{B} \cup \{x_i = 0\}_{i=1}^{n}$.

$\tilde{\mathcal{B}}$ free, basis $\{x_1^k\partial_1 + \dots + x_n^k\partial_n\}_{k=1}^{n}$



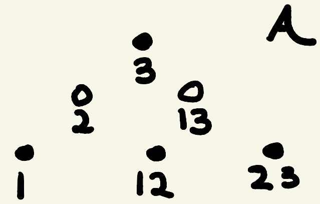
$\mathfrak{i}: \mathrm{Der}(S) \to S$, $\partial_i \mapsto 1$ $\qquad \mathfrak{i}_{\tilde{\mathcal{B}}} = (p_1, \dots, p_n) = I$

$ST(\tilde{\mathcal{B}};\mathfrak{i}) = R$.

FACT Suppose $\mathcal{A} \subseteq \tilde{\mathcal{B}}$ is any free subarrangement.

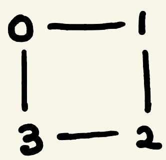
Let $f := \prod_{H \in \tilde{\mathcal{B}} \setminus \mathcal{A}} \alpha_H$. Then $i_{\mathcal{A}} = (i_{\tilde{\mathcal{B}}} : f) = (I : f)$.

Q When is $\mathcal{A}$ free?



not free

$\mathcal{A}$



Γ

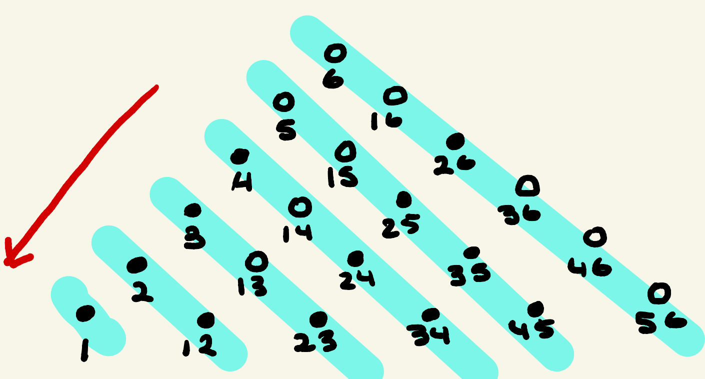
STANLEY

$\mathcal{A}$ is free $\Leftrightarrow$

Γ is CHORDAL

SOUTHWEST SUBARRANGEMENTS $\mathcal{A} \subseteq \tilde{\mathcal{B}}$

FACT Every SW arr. is free with exponents $\eta(\mathcal{A})$.



$\eta(\mathcal{A}) = (1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 1)$

If $\mathcal{A}$ is an arr. & $H \in \mathcal{A}$,

$\mathcal{A} \setminus H$ = DELETION

$\mathcal{A}/H$ = RESTRICTION $= \{ H' \cap H : H' \in \mathcal{A},\ H' \neq H \}$.

FACT Let $\mathcal{A} \subseteq \tilde{\mathcal{B}}$ & $H \in \mathcal{A}$. Assume

1. $(\mathcal{A},\ \mathcal{A} \setminus H,\ \mathcal{A}/H)$ are all free
2. $$\left.\begin{aligned} \exp \mathcal{A} &= \{e_1, \dots, e_k, \dots, e_n\} \\ \exp \mathcal{A}\setminus H &= \{e_1, \dots, e_k - 1, \dots, e_n\} \\ \exp \mathcal{A}/H &= \{e_1, \dots, \hat{e}_k, \dots, e_n\} \end{aligned}\right\} \text{good exponents}$$

We have a SES

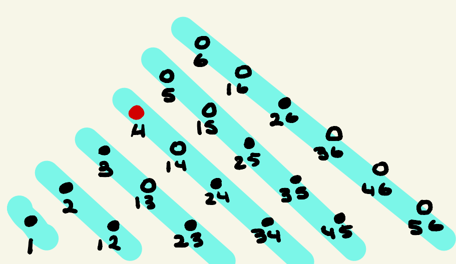
$$0 \to ST(\mathcal{A} \setminus H; i) \xrightarrow{\cdot \alpha_H} ST(\mathcal{A}; i) \xrightarrow{\text{can.}} ST(\mathcal{A}/H; i) \longrightarrow 0$$

IDEA We always have exact sequence

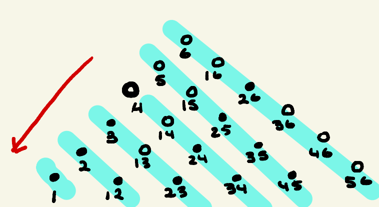
$$0 \to \mathrm{Der}(\mathcal{A} \setminus H) \xrightarrow{\cdot \alpha_H} \mathrm{Der}(\mathcal{A}) \xrightarrow{\text{can.}} \mathrm{Der}(\mathcal{A}/H)$$

Free + good exps implies last map is onto.

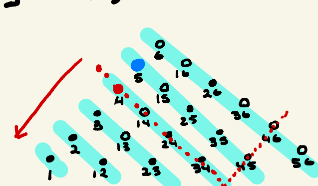
Apply i; use $i_{\mathcal{A}} = I : \left(\prod_{H' \in \tilde{\mathcal{B}}/\mathcal{A}} \alpha_{H'} \right)$.



$h(\mathcal{A}) = (1\ \ 2\ \ 2\ \ 3\ \ 3\ \ 1)$



$h(\mathcal{A}) = (1\ \ 2\ \ 2\ \ 2\ \ 3\ \ 1)$



$h(\mathcal{A}) = (1\ \ 2\ \ 2\ \ \times\ \ 3\ \ 1)$

THM Let $\mathcal{A}$ be a SW arr w/ $h(\mathcal{A}) = (h_1(\mathcal{A}), \dots, h_n(\mathcal{A}))$.

Then $ST(\mathcal{A}; i)$ has $\mathbb{C}$-basis $\{x_1^{a_1} \cdots x_n^{a_n} : a_i < h_i(\mathcal{A})\}$.

Pf If $\mathcal{A}$ has no coord. hyps, then $h_1(\mathcal{A}) = 0$ & WTS $ST(\mathcal{A}; i) = 0$. But $\partial_1 + \cdots + \partial_n \in Der(\mathcal{A})$ so $i_{\mathcal{A}} = S$ & $ST(\mathcal{A}; i) = 0$.

Else, let p be max'l s.t. $\underset{\overset{\parallel}{H_0}}{\{x_p = 0\}}$ is in $\mathcal{A}$.

Have a SES

$$0 \longrightarrow ST(\mathcal{A} \setminus H_0; i) \xrightarrow{\cdot x_p} ST(\mathcal{A}; i) \xrightarrow{x_p \to 0} ST(\mathcal{A}/H_0; i) \longrightarrow 0$$

Inductively get bases on either side. Exactness gives correct basis in the middle. //

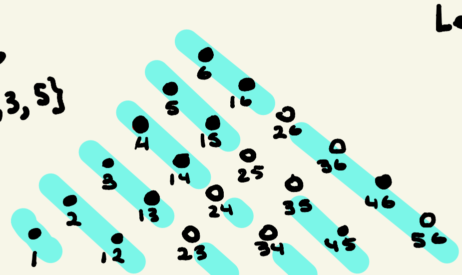
THM Let $J \subseteq [n]$. Then $M_J = \{x_1^{a_1} \cdots x_n^{a_n} : a_i < st_i J\}$ is a basis of $S/(I : f_J)$. ($\Rightarrow$ SAGAN-SWANSON BASIS OF SZ)

Pf If $J \subseteq [n]$, get a SW arr. $\mathcal{A}_J$.

Let $\tilde{f}_J = \prod_{j \in J} \left(\prod_{j < k} (x_j - x_k) \right)$, so $f_J = \tilde{f}_J \cdot \left(\prod_{j \in J} x_j \right)$.

$n = 6$

$J = \{2, 3, 5\}$



$h(\mathcal{A}_J) = (1 \ \underline{2} \ \underline{2} \ 2 \ \underline{3} \ 3)$

$st(J) = (1 \ 1 \ 1 \ 2 \ 2 \ 3)$

We get an injection

$$\varphi : S/(I : f_J) \xrightarrow{\cdot \prod_{j \in J} x_j} S/(I : \tilde{f}_J)$$

M_J maps under φ to a lin. indep subset of $S/(I : \tilde{f}_J)$.

$\Rightarrow$ M_J is lin indep. $\xRightarrow{\text{Hilb. of } S/(I:f_J)}$ M_J basis. ∎