Operator Theorem for $SR/SI^{\perp}$

(CONJ: SWANSON-WALLACH ; FIELDS GROUP)

LAST WEEK

- $S = \mathbb{C}[x_1 \dots x_n]$

  $I = (S_+^{\mathfrak{S}_n}) \subset S$

  $R = S/I$ CLASSICAL COINVARIANTS

- $\Omega = \mathbb{C}[x_1 .. x_n] \otimes \Lambda\{\theta_1 \dots \theta_n\} = S \otimes E$

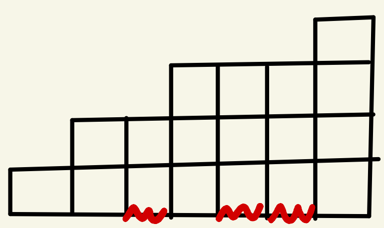
  $SI = (\Omega_+^{\mathfrak{S}_n}) \subset \Omega$

  $SR = \Omega / SI$ SUPERSPACE COINVARIANTS

LAZY STAIRCASES

$J \subseteq [n] \qquad st(J) = (st_1 J, \dots, st_n J)$

$n = 7 \qquad J = \{3, 5, 6\}$



$st\, J = 1\ 2\ 2\ 3\ 3\ 3\ 4$

$$f_J := \prod_{j \in J} x_j \left(\prod_{i > j} (x_j - x_i) \right) \in S$$

Complete intersection (if $1 \notin J$), degrees $st_1 J, \dots, st_n J$

We showed as ideals in S,

$$in_J(SI) \supseteq (I : f_J)$$

$$= (e_1, \dots, e_{n-|J|}) + \Big(\prod_{\substack{i<j \\ i \notin J}} (x_j - x_i) : j \in J \Big)$$

Complete intersection (if $1 \notin J$), degrees $st_1 J, \ldots, st_n J$

We showed

$$\mathrm{in}_J(SI) \supseteq (I : f_J)$$

$$= (e_1, \ldots, e_{n-|J|}) + \Big(\prod_{\substack{i<j \\ i \notin J}} (x_j - x_i) : j \in J \Big)$$

$\Rightarrow$

$$\mathrm{Hilb}(SR; q, z) \leq \sum_{J \subseteq [n]} z^{|J|} [st_1(J)]_q \cdots [st_n(J)]_q .$$

INVERSE SYSTEMS

$f = f(x_1, \ldots, x_n) \in S \quad \rightsquigarrow \quad \partial f = f(\partial_1, \ldots, \partial_n) : S \to S$

$\odot : S \times S \longrightarrow S$

$f \odot g = \partial f(g)$

$\langle -, - \rangle : S \times S \to \mathbb{C}$

$\langle f, g \rangle =$ const. term of $f \odot g$

$\mathfrak{a} \subset S$ homogeneous ideal

$\Rightarrow \quad \mathfrak{a}^\perp =$ INVERSE SYSTEM $= \{ g \in S : f \odot g = 0 \text{ for all } f \in \mathfrak{a} \}$

$$\mathrm{Hilb}(S/\mathfrak{a}; q) = \mathrm{Hilb}(\mathfrak{a}^\perp; q)$$

STEINBERG If $I = (S_+^{\mathfrak{S}_n}) = (e_1, \ldots, e_n)$,

$$I^\perp = S \odot \delta \qquad \delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Inverse systems in Ω

$\partial_i : \Omega \to \Omega$ $\qquad \partial_i^\theta : \Omega \to \Omega$

$\partial_i = \partial/\partial x_i$

$\partial_1^\theta(\cancel{\theta_1}\theta_3\theta_4) = \theta_3\theta_4$

$\partial_2^\theta(\theta_1\theta_3\theta_4) = 0$

$\partial_3^\theta(\theta_1\theta_3\theta_4) = -\theta_1\theta_4$

$\partial_4^\theta(\theta_1\theta_3\theta_4) = \theta_1\theta_3$

$\partial_i\partial_j = \partial_j\partial_i \qquad \partial_i\partial_j^\theta = \partial_j^\theta\partial_i \qquad \partial_i^\theta\partial_j^\theta = -\partial_j^\theta\partial_i^\theta$

for $1 \le i,j \le n$

$\Rightarrow$ Given $f = f(x_1,\dots,x_n,\theta_1,\dots,\theta_n) \in \Omega$, have a well-defined $\partial f : \Omega \to \Omega$

$$\partial f = (\partial_1,\dots,\partial_n,\partial_1^\theta,\dots,\partial_n^\theta).$$

$\Rightarrow$ Module structure $\quad \odot : \Omega \times \Omega \to \Omega \quad f \odot g := \partial f(g)$

Bilinear form $\quad \langle -,- \rangle : \Omega \times \Omega \to \mathbb{C} \quad \langle f,g \rangle =$ const. term of $f \odot g$

* $\Omega = \bigoplus_{i,j} \Omega_{ij}$ is orthogonal wrt $\langle -,- \rangle$
* $\langle -,- \rangle$ is a perfect pairing on each Ω_{ij}

$\Rightarrow$ If $\mathfrak{a} \subset \Omega$ is a bigraded ideal then

$$\mathrm{Hilb}(\Omega/\mathfrak{a}; q,z) = \mathrm{Hilb}(\mathfrak{a}^\perp; q,z)$$

where $\mathfrak{a}^\perp =$ INVERSE SYSTEM

$= \{ g \in \Omega : f \odot g = 0 \text{ for all } f \in \mathfrak{a} \}.$

Also, if $\mathfrak{a}$ is $\mathfrak{S}_n$-stable, so is $\mathfrak{a}^\perp$ and

$\Omega/\mathfrak{a} \cong \mathfrak{a}^\perp$ as bigraded $\mathfrak{S}_n$-mods

Q What does $SI^{\perp}$ look like?

Suppose for any $J \subseteq [n]$, we could find $g_J \in SI^{\perp}$ of form...

$$\circledast \quad g_J = (f_J \odot \delta)\, \theta_J + \left(\begin{matrix} \text{S-linear comb. of } \theta_K \\ \text{for } K \underset{\text{lex}}{>} J \end{matrix} \right).$$

Cor 1 $\operatorname{in}_J(SI) = (I : f_J)$.

⌜ $\supseteq$ last week ✓

$\subseteq$ Let $h_J \in \operatorname{in}_J(SI)$. Then $\exists$ $h \in SI$ with

$$h = h_J \theta_J + \left(\text{S-lin. comb. of } \theta_L \text{ for } L \underset{\text{lex}}{<} J \right)$$

Then $$0 = \underset{SI}{h} \odot \underset{SI^{\perp}}{g_J} = \pm\left(h_J \odot (f_J \odot \delta)\right) = \pm (h_J \cdot f_J) \odot \delta$$

$$\underset{\text{STEINBERG}}{\Rightarrow} \quad h_J \in (I : f_J).$$ ⌟

Cor 2 $$\operatorname{Hilb}(SR; q, z) = \sum_{J \subseteq [n]} z^{|J|} \overbrace{[st_1(J)]_q \cdots [st_n(J)]_q}^{\operatorname{Hilb}(S/\operatorname{in}_J(SI);\, q)}.$$

Cor 3 ("Basis Transfer") Suppose for all $J \subseteq [n]$, have a set B_J of polys in S. Let

$$\mathcal{B} := \bigsqcup_J B_J \theta_J.$$

If B_J descends to a basis of $S/(I : f_J)$ $\forall$ $J \subseteq [n]$, then $\mathcal{B}$ descends to a basis of SR.

⌜ $\operatorname{in}_J(SI) = (I : f_J)$ ⌟

FACT $SI^{\perp}$ contains $\delta = \prod_{i<j} (x_i - x_j)$,

is closed under $\partial_1, \dots, \partial_n$,

and is closed under $d_j : \Omega \to \Omega$ $\quad d_j f = \sum_{i=1}^{n} \frac{\partial^j f}{\partial x_i^j} \theta_i$
$(j \geq 1)$

Higher Euler operators

$(d_i d_j = -d_i d_j)$

Ex Let $f \in S$. Then

$$d_{24} f := d_2 d_4 f = \sum_{1 \leq i,j \leq n} \underset{\in}{\overset{S}{f_{ij}}} \, \theta_{ij}$$

$$f_{35} = (\partial_3^4 \partial_5^2 - \partial_3^2 \partial_5^4) f = \det \begin{pmatrix} x_3^4 & x_3^2 \\ x_5^4 & x_5^2 \end{pmatrix} \odot f$$

Let $y_1, y_2, \cdots$ be commuting variables.

For $r \le n$, $P_{nr}(\underline{y}) =$ power matrix $r \times n$

$$P_{53}(\underline{y}) = \begin{bmatrix} y_1^5 & y_1^4 & y_1^3 & y_1^2 & y_1 \\ y_2^5 & y_2^4 & y_2^3 & y_2^2 & y_2 \\ y_3^5 & y_3^4 & y_3^3 & y_3^2 & y_3 \end{bmatrix}.$$

$y_2(y_2-x_3)(y_2-x_4)(y_2-x_5)$

$F_{nr}(\underline{x}, \underline{y}) =$ factor matrix $r \times n$

$$f_j(\underline{x}, y) = y \cdot (y - x_{j+1}) \cdots (y - x_{n-1})(y - x_n)$$

$$F_{53}(\underline{x}, \underline{y}) = \begin{bmatrix} f_1(\underline{x}, y_1) & f_2(\underline{x}, y_1) & \cdots & f_5(\underline{x}, y_1) \\ f_1(\underline{x}, y_2) & f_2(\underline{x}, y_2) & \cdots & f_5(\underline{x}, y_2) \\ f_1(\underline{x}, y_3) & f_2(\underline{x}, y_3) & \cdots & f_5(\underline{x}, y_3) \end{bmatrix}$$

<u>OBS</u> There is an $n \times n$ lower Δ'lar S-matrix C w/ 1s on main diagonal with

$$P_{nr}(\underline{y}) \cdot C = F_{nr}(\underline{x}, \underline{y})$$

$\Rightarrow$ $\det C = 1$, C^{-1} exists over S

If $J \subseteq [n]$ and $|J| = r$, E_J $(n-r) \times n$

$n = 5$

$J = \{2, 3, 5\}$

$$\begin{array}{c} \begin{matrix} 1 & \cancel{2} & \cancel{3} & 4 & \cancel{5} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Let $H_J := E_J \cdot C^{-1}$.

For $J \subseteq [n]$ w/ $|J| = r$, define $D_J : \Omega \to \Omega$

by

$$D_J(f) = \sum_{\substack{U \subseteq [n] \\ |U| = n-r}} (-1)^{\Sigma U} \Delta_U(H_J) \odot d_{U^*} f$$

where $\Sigma U = \sum_{u \in U} u$, $\Delta_U(H_J)$ = maxl minor of H_J, col set U

$$U^* = \{ n-i+1 : i \in [n] \setminus U \}.$$

<u>FACT</u> If $f \in S$, then for $J \subseteq [n]$ w/ $|J| = r$ &

$\{a_1 < \cdots < a_r\} \leq_{\text{Gale}} \{b_1 < \cdots < b_r\} \iff a_i \leq b_i$

given $|K| = r$,

① $[\theta_K] D_J(f) = 0$ unless $K \leq_{\text{Gale}} J$.

② $[\theta_J] D_J(f) = \pm f_J \odot \delta$

$\Rightarrow$ "OPERATOR THM" $SI^\perp$ is smallest subsp. of Ω
① containing δ, ② closed under $\partial_1, \dots, \partial_n$,
③ closed under d_j for $j \geq 1$.

$$[\theta_K] D_J(f) \doteq \det \left[\frac{P_{nr}(\underline{x}_K)}{H_J} \right] \odot \delta$$

$\det C = 1$

$$= \det \left[\frac{F_{nr}(\underline{x}, \underline{x}_K)}{E_J} \right] \odot \delta$$

$$\det\left[\frac{F_{nr}(\underline{x},\underline{x}_K)}{E_J}\right] \doteq \det \bar{F}_{nr}(\underline{x},\underline{x}_K)$$

r×r submatrix in cols J

$J = \{2\,3\,5\}$ $\quad K = \{k_1 < k_2 < k_3\}$

$$\det \bar{F}_{nr}(\underline{x},\underline{x}_K) = \det\begin{bmatrix} f_2(\underline{x},x_{k_1}) & f_3(\underline{x},x_{k_1}) & f_5(\underline{x},x_{k_1}) \\ f_2(\underline{x}\ x_{k_2}) & f_3(\underline{x},x_{k_2}) & f_5(\underline{x},x_{k_2}) \\ f_2(\underline{x},x_{k_3}) & f_3(\underline{x},x_{k_3}) & f_5(\underline{x},x_{k_3}) \end{bmatrix} = (*)$$

$= 0$ unless $k_1 \le 2$, $k_2 \le 3$, $k_3 \le 5$

Also if $K = J$, $(*) = f_2(\underline{x},x_2)\, f_3(\underline{x},x_3)\, f_5(\underline{x},x_5)$

$= f_{235}$ //