A proof of the Fields Conjectures

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IPAC 2026

Classical coinvariants

- $S := \mathbb{C}[\vec{x}_n] := \mathbb{C}[x_1, x_2, \dots, x_n]$
- $\mathfrak{S}_n \circlearrowright S$ by $w \cdot x_i = x_{w(i)}$.
- $I := (S)_+^{\mathfrak{S}_n} = (f \in S : f(0) = 0,\ w \cdot f = f\ \forall w)$

  $= (p_i / e_i / h_i : 1 \leq i \leq n)$
- $R := S/I$ (coinvariant ring)
- $R = \bigoplus_i R_i$ and $\mathfrak{S}_n \circlearrowright R_i$ (graded $\mathfrak{S}_n$-module)

Selected results for R

$$\dim R = n!$$

$$R \cong_{\mathfrak{S}_n} \mathbb{C}[\mathfrak{S}_n]$$

(Chevalley)

$$\sum_i \dim R_i \, q^i = [n]_q!$$

$$R_i\big|_{V_\lambda} = \#\, T \in SYT(\lambda) \text{ w/ } \mathrm{maj}(T) = i$$

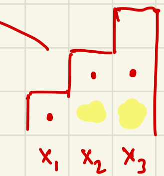
(Lusztig, Stanley)

$\{x_1^{a_1} \cdots x_n^{a_n} : a_i < i\}$ descends to a basis for R. (Artin)

Example : n = 3

- $\dim R = 3!$

- $\{1, x_2, x_3, x_2 x_3, x_3^2, x_2 x_3^2\}$ is a basis.



- $1 + q + q + q^2 + q^2 + q^3 = [3]_q! = \sum_i \dim R_i \, q^i$

- $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$ $\quad$ $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$

$V_3 \qquad q V_{21} \qquad q^2 V_{21} \qquad q^{1+2} V_{111}$

Other types

- $W :=$ irreducible Weyl group of rank n
- $W \curvearrowright V$ as a reflection representation $\Rightarrow W \curvearrowright \mathbb{C}[V]$
- $R := \mathbb{C}[V] / (\mathbb{C}[V])^W_+$
- Similar results hold, e.g. $R \cong_W \mathbb{C}[W]$. (Chevalley)

Diagonal coinvariants

- $\mathfrak{S}_n \curvearrowright \mathbb{C}[\vec{x}_n^{(1)}, \vec{x}_n^{(2)}, \ldots, \vec{x}_n^{(k)}]$, $w \cdot x_i^{(j)} = x_{w(i)}^{(j)}$
- $R^{(k)} := \mathbb{C}[\vec{x}_n^{(1)}, \ldots, \vec{x}_n^{(k)}] \,/\, (\mathbb{C}[\vec{x}_n^{(1)}, \ldots, \vec{x}_n^{(k)}])_+^{\mathfrak{S}_n}$
- Results at $k = 2$:
  - $\dim R^{(2)} = (n+1)^{n-1}$ (Haiman)
  - $R^{(2)} \cong_{\mathfrak{S}_n} \mathbb{C}[\text{parking functions}] \otimes \text{sign}$
  - Shuffle Theorem (Carlsson, Mellit)
- Few results for $k > 2$ (Lentfer)

Superspace

$$E := \Lambda\{\theta_1, \ldots, \theta_n\}, \quad \Omega := S \otimes E$$

$$x_i x_j = x_j x_i \qquad x_i \theta_j = \theta_j x_i \qquad \theta_i \theta_j = -\theta_j \theta_i$$

$$\Downarrow$$

$$\theta_i^2 = 0$$

- $\mathfrak{S}_n \circlearrowright \Omega$ by $w \cdot x_i = x_{w(i)}$, $w \cdot \theta_i = \theta_{w(i)}$
- $SI := (\Omega)_+^{\mathfrak{S}_n} = (p_i, dp_i : 1 \leq i \leq n)$ (Solomon) $df = \sum_{i=1}^n \theta_i \frac{\partial f}{\partial x_i}$
- $SR := \Omega / SI = \bigoplus_{i,j} SR_{i,j} = \bigoplus_j SR_{*,j}$

x-deg $\quad$ θ-deg

But ... why superspace?

- $\cong$ "regular" (polynomial-valued) differential forms $\quad\quad \theta_i \leftrightarrow dx_i$

- Well-studied in math. physics

  $x_i \leftrightsquigarrow$ boson, $\quad \Theta_i \leftrightsquigarrow$ fermion

- Easier than diagonal coinvariants?
  - Only 2^n monomials in $\Theta_1, \dots, \Theta_n$. $\quad\quad (\Theta_J := \Theta_{j_1}\Theta_{j_2}\cdots\Theta_{j_k})$

- Connections to Delta Theorem

- It's beautiful!

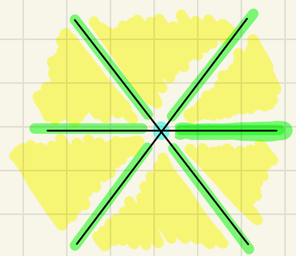
Ordered set partitions

- $OP_{n,k} := \left\{ \pi = (\pi_1, \ldots, \pi_k) : \bigsqcup_i \pi_i = \{1, 2, \ldots, n\} \right\}$

- $OP_n := \bigsqcup_k OP_{n,k}$

- $OP_{n,k} \longleftrightarrow$ codimension k faces in G_n Coxeter complex

$(\{3\}, \{1,2\}) \longleftrightarrow x_3 < x_1 = x_2$



$x_1 = x_2 < x_3 \longleftrightarrow (\{1,2\}, \{3\})$

$(\{1,2,3\})$

$(\{1,2\}, \{3\})$ $(\{3\}, \{1,2\})$

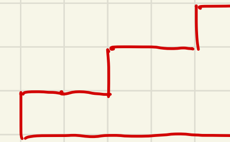
$(\{1,3\}, \{2\})$ $(\{2\}, \{1,3\})$

$(\{2,3\}, \{1\})$ $(\{1\}, \{2,3\})$

G_3

Lazy staircases

- $$|OP_{n,k}| = k!\,\mathrm{Stir}(n,k) = \sum_{\substack{1=b_1\le \dots \le b_n = k \\ b_{i+1}-b_i \in \{0,1\}}} b_1 \cdot b_2 \cdot \ldots \cdot b_n$$



"lazy staircase"

$b = 1\,2\,2\,2$

1

$1|2 \qquad 2|1$

$13|2 \qquad 1|23 \qquad 23|1 \qquad 2|13$

$134|2 \qquad 13|24 \qquad 14|23 \qquad 1|234 \qquad 234|1 \qquad 23|14 \qquad 24|13 \qquad 2|134$

Ordered set partitions "govern" SR like

permutations "govern" R.

Fields Conjectures

(Bergeron, Colmenarejo, Li, Machacek, Sulzgruber, Zabrocki '19)

$$\dim SR = |OP_n|$$

$$SR \cong_{\mathfrak{S}_n} \mathbb{Q}[OP_n] \otimes \mathrm{sign}$$

$$\sum_i \dim SR_{i,j}\, q^i = \sum_{\substack{1 = b_1 \le \dots \le b_n = n-j \\ b_{i+1} - b_i \in \{0,1\}}} [b_1]_q \cdots [b_n]_q$$

$$\left\{ x_1^{a_1} \cdots x_n^{a_n} \Theta_J \;:\; \exists\, \vec{b} \text{ w/ } a_i < b_i,\; J = \{ i+1 : b_{i+1} = b_i \} \right\}$$

$$SR_{i,j}\Big|_{V_\lambda} = \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} \prod_{\ell=1}^{\mathrm{des}(T)} \left(1 + z q^{-\ell}\right) \Bigg|_{q^i z^j}$$

Fields Theorems!

$$\dim SR = |OP_n|$$

$$SR \cong_{\mathfrak{S}_n} \mathbb{Q}[OP_n] \otimes \mathrm{sign}$$

(Rhoades, W. '23)

$$\sum_i \dim SR_{i,j}\, q^i = \sum_{\substack{1 = b_1 \leq \cdots \leq b_n = n-j \\ b_{i+1} - b_i \in \{0,1\}}} [b_1]_q \cdots [b_n]_q$$

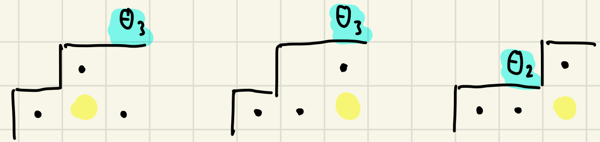
$$\left\{ x_1^{a_1} \cdots x_n^{a_n} \theta_J \;:\; \exists\, \vec{b} \text{ w/ } a_i < b_i,\; J = \{ i+1 : b_{i+1} = b_i \} \right\}$$

(Angarone, Commins, Karn, Murai, Rhoades '24)

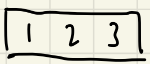
(Murai, Rhoades, W. '25)

$$SR_{i,j} \Big|_{V^\lambda} = \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} \prod_{\ell=1}^{\mathrm{des}(T)} \left(1 + z q^{-\ell}\right) \Bigg|_{q^i z^j}$$

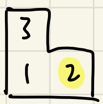
Example : $SR_{1,1}$ at $n=3$



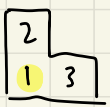
Basis : $\{ x_2 \theta_3 , \quad x_3 \theta_3 , \quad x_3 \theta_2 \}$



$V^3 : q^0 z^0$



$V^{21} : q^2 (1 + z/q)$



$V^{21} : q (1 + z/q)$

$V^{111} : q^{1+2} (1 + z/q)(1 + z/q^2)$

$$\Big|_{q^1 z^1} = V^{21} \oplus V^{111}$$

Schedule

1. Decompose SR into polynomial quotients
(leading terms)

?

2. Show the resulting ideals are nice
(commutative algebra, orbit harmonics)

3. Sketch one direction of the decomposition
(hyperplane arrangements, derivation modules)

Then Brendon takes over! (harmonics, bases, ...)