Def: For \( n \) odd, define \( F(n) := \{ (b,f) : b = (b_1, b_2, \ldots, b_n) \} \), \( f = (f_1, f_2, \ldots, f_n) \). 

For \( (b,f) \in \mathcal{P}(n,m) \), define \( S(b,f) \) to be the filled diagram \( (D', f') \) obtained by interchanging column 1 and column 2, with \( f_1 \) (now in column 2, i.e. cell \( \text{col}\{n,1\} \)) to \( f_2 \), and \( f_2 \) (now in column 1, i.e. cell \( \text{col}\{1,2\} \)) to \( f_1 \).

For any filled diagram \( (D,f) \), if \( [D] > [D'] \), then define \( S^2(D,f) \) to be the filled diagram obtained by applying \( S \) on the subdiagram \( [D'] \) of \( D \) (and keep all other columns).

Def: For \( n \) even, define \( \text{DES}(n) := \{ (b,f) : b = (b_1, b_2, \ldots, b_n) \} \).

\( \text{DES}(n) = \text{DES}(n) \setminus \{(1, n-1)\} \) if \( n = 4k \).

\( \text{DES}(n) = \text{DES}(n) \setminus \{(1, n-1)\} \) otherwise.

e.g., \( \text{DES}(6) = \{ (1, 6), (2, 5), (3, 4) \} \).

Since \( (5,4) \) contains a descent, we ignore it when counting \( \text{DES}(6) \).
Lemma: \( \text{Let } (a, b, c) \in \mathbb{R}_+^3, \text{ and } a_i, b_i, c_i \geq 0. \) \( \text{If } \text{Det}(a_{11} - b_{12} - c_{13}) \neq 0, \text{ then } \text{Det}(a_{21} - b_{22} - c_{23}) \neq 0. \) \( \text{and } (a_{31} - b_{32} - c_{33}) \neq 0. \) \( \text{Then } \text{Det}(a_{11} - b_{12} - c_{13}) = \text{Det}(a_{21} - b_{22} - c_{23}) = \text{Det}(a_{31} - b_{32} - c_{33}). \)

(a) \( \text{Det}(a_{11}) = \text{Det}(a_{22}) = \text{Det}(a_{33}) \)
(b) \( \text{Det}(a_{11}) \neq \text{Det}(a_{22}) \neq \text{Det}(a_{33}) \)
(c) \( \text{Det}(a_{11}) = \text{Det}(a_{22}) = \text{Det}(a_{33}) = 0 \)

Scale:

\( a = 83.1179a + 66.1 \), \( \sigma = 5.1452.16 \Rightarrow \text{Det}(a_{11}) = 0 \neq \text{Det}(a_{22}) = \text{Det}(a_{33}) = 0 \Rightarrow \text{Det}(a_{11}) = 0 \neq \text{Det}(a_{22}) = \text{Det}(a_{33}) = 0 \)

\( \text{std}(a) = 0 \Rightarrow \text{std}(a_{11}) = \text{std}(a_{22}) = \text{std}(a_{33}) = 0 \Rightarrow \text{Det}(a_{11}) = \text{Det}(a_{22}) = \text{Det}(a_{33}) = 0 \)

\( \text{std}(a_{11}) = \text{std}(a_{22}) = \text{std}(a_{33}) = 0 \Rightarrow \text{Det}(a_{11}) = \text{Det}(a_{22}) = \text{Det}(a_{33}) = 0 \)

Prop. For positive integers \( m, n \) and \( m \neq n \), there exists a bijection \( \phi : \mathbb{Z}_m \rightarrow \mathbb{Z}_n \) satisfying:

(1) \( |\text{Det}(\phi)| = |\text{Det}(e)| \) for any \( e \in \mathbb{Z}_m \)
(2) \( \text{Det}(\phi(e)) = \text{Det}(\theta(e)) \) for any \( e \in \mathbb{Z}_m \)
(3) \( |\text{Det}(\phi)| = |\text{Det}(e)| \) for any \( e \in \mathbb{Z}_n \)

\( \phi \) is called a “local” bijection.

By (1) and (2), we have

\( \text{Det}(\phi) = \text{Det}(\theta) \) for any \( (a, b, c) \in \mathbb{R}_+^3 \)

By (3), (2) and (1), we have

\( \text{Det}(\phi) = \text{Det}(\theta) \) for any \( (a, b, c) \in \mathbb{R}_+^3 \)

That's why (3) is "local".